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ON REMOVABLE SETS FOR DEGENERATED ELLIPTIC EQUATIONS

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In the paper the necessary and sufficient condition of compact removability is obtained.

Key words: *elliptic equations, removable sets, Green operator, Hilbert space.*

The questions of compact removability for Laplace equation is studied by Carleson [1]. The uniform elliptic equation of the second order of divergent structure is studied by E.I.Moiseev [2]. The compact removability for elliptic and parabolic equations of nondivergent structure is considered by E.M.Landis [3]. T.S.Gadjiev, V.A.Mamedova [4]. The removability condition of compact in the space of continuous functions is constructed in the papers Harvey, Polking [5], T.Kilpelainen [6]. The different questions of qualitative properties of solutions of uniformly degenerated elliptic equations are studied by S.Chanillo, R.Z. Wreeden [7]. The uniform elliptic operator of the second order of divergent structure is considered in the paper [8].

Let E_n be n dimensional Euclidean space of the points $x=(x_1,...,x_n)$. Denote by $R>0$ for $B_R(x_R^0)$ the ball $\{x:|x-x^0|<R\}$, and by $Q_T^R(x_R^0)$ the cylinder $B_R(x^0)\cup(0,T)$. Further let for $x^0\in E_n$, $R>0$ and $k>0$ $\varepsilon_{r,k}(x^0)$ be an ellipsoid $\left\{x:\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{R^{\alpha_i}}<(kR)^2\right\}$. Let D be an bounded domain E_n with the domain $\partial D, 0\in D$. ε is a such kind of ellipsoid that $\overline{D}\subset\varepsilon, \mathrm{B}(\varepsilon)$ is a set of all functions, satisfying in $\overline{\varepsilon}$ the uniform Lipschitz condition and having zero near the $\partial\varepsilon$.

Denote by α and $(\alpha_1,...,\alpha_n)$ the vector $\langle\alpha\rangle=\alpha_1,...,\alpha_n$.

Denote by $W_{2,\alpha}^1(D)$ the Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W^1_{2,\alpha}(D)} = \left(\int_D \left(u^2 + \sum_{i=1}^{n} \lambda_i(x) u_i^2 \right) dx \right)^{1/2},$$

where

$$u_i = \frac{\partial u}{\partial x_i}, i = 1,..,n. \quad \lambda_i(x) = \left(|x|_\lambda\right)^{\alpha_i}, |x|_\alpha = \sum_{i=1}^{n} |x_i| \frac{2}{2+\alpha_i},$$

$$0 \le \alpha_i < \frac{2}{n-1} \tag{1}$$

Further, let $\overset{\circ}{W}{}^1_{2,\alpha}(D)$ be a degenerated set of all functions from $C_0^\infty(D)$ by the norm of the space $W^1_{2,\alpha}(D)$. Denote by $\mathrm{M}(D)$ the set of all bounded in D functions.

Let $E \subset D$ be some compact. Denote by $A_E(D)$ the totality of all functions $u(x) \in C^\infty(\overline{D})$, each of which there exists some neighbourhood of the compact E, in which $u(x) = 0$.

The compact E is called the removable relative to the first boundary value problem for the operator L in the space $\mathrm{M}(D)$, if all generalized solution of the equation $\mathrm{L}u = 0$ in ∂/E formed in zero on ∂D and belonging to the space $\mathrm{M}(D)$, identically equal to zero. We'll say that the function $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ is non-negative on the set $H \subset \varepsilon$, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, if there exists the sequence of the functions $\{u_{(m)}(x)\}, m = 1,2,..$, such that $u_m(x) \in \mathrm{B}(\varepsilon)$, $u_m(x) \ge 0$ for $x \in H$ and $\lim_{m\to\infty} \|u_{(m)} - u\|_{W^1_{2,\alpha}(\varepsilon)} = 0$.

The function $u(x) \in W^1_{2,\alpha}(D)$ is non-negative and ∂D in sense $W^1_{2,\alpha}(D)$, if there exists the sequence of the functions $\{u_m(x)\}, m = 1,2,..$, such, that $u_{(m)}(x) \in C^1(D), u_m(x) \ge 0$ for $x \in \partial D$ and $\lim_{m\to\infty} \|u_{(m)} - u\|_{W^1_{2,\alpha}(\varepsilon)} = 0$. It is easy to determine the inequalities $u(x) \ge const, u(x) \ge v(x), u(x) \le 0$, and also equality $u(x) = 1$ on the set H in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, if at the same time $u(x) \ge 1$ and $u(x) \le 1$ on H, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

Let $\omega(x)$ be measurable function in D, finite and positive for a.e. $x \in D$. Denote by $\mathrm{L}_{p,\omega}(D)$ the Banach space of the functions given on D, with the norm

$$\|u\|_{L_{p,\omega}(D)} = \left(\int_D (\omega(x))^{p/2} |u|^p \, dx \right)^{1/p}, \ 1 < p < \infty.$$

Let $W^1_{p,\alpha}(D)$ be a Banach space of the functions given on $u(x)$, with the finite norm D.

$$\|u\|_{W^1_{p,\alpha}(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n (\lambda_i(x))^{p/2} |u_i|^p \right) dx \right)^{1/p}, \ 1 < p < \infty$$

Analogously to $\overset{\circ}{W}{}^1_{2,\alpha}(D)$, it is introduced the subspace $\overset{\circ}{W}{}^1_{p,\alpha}(D)$ for $1 < p < \infty$. The space, conjugated to $\overset{\circ}{W}{}^1_{p,\alpha}(D)$ we'll denote by $\overset{*}{W}{}^1_{p,\alpha}(D)$.

We'll consider the elliptic operator in the bounded domain $D \subset E_n$

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

In assumption, that $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover for all $\xi \in E_n$ and a.e. $x \in D$ the condition

$$\gamma \sum_{i=1}^n \lambda_i(x) \xi_i^2 \le \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \le \gamma^{-1} \sum_{i=1}^n \lambda_i(x) \xi_i^2 \qquad (2)$$

Here $\gamma \in (0,1]$ is a constant.

The function $u(x) \in W^1_{2,\alpha}(D)$ is called the generalized solution of the equation $Lu = f(x)$ in D, if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_{x_i} \eta_{x_j} \, dx = \int_D f \eta dx \qquad (3)$$

be fulfilled.

Here $f(x)$ is a given function from $L_2(D)$.

Let $E \subset D$ be some compact. The function $u(x) \in W^1_{2,\alpha}(D \setminus E)$ is called generalized solution of the equation $Lu = f(x)$ in $D \setminus E$, vanishing on ∂D, if integral identity (3) is fulfilled for any function $\eta(x) \in A_E(D)$.

We'll assume that the coefficients of the operator L continued in $E_n \setminus D$ with saving condition (1), (2). For this, it is sufficient, for example, let's assume $a_{ij}(x) = \delta_{ij} \lambda_i(x)$ for $x \in E_n \setminus D$, $i, j = 1,..,n$, where δ_{ij} is a Croneker symbol.

Let $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in L_{2,\lambda^{-1}}(D)$, $i = 1,2,..,n$, are a given functions. Let's consider the first boundary value problem

$$\mathrm{L}u = f^0(x)+\sum_{i=1}^{n}\frac{\partial f^i(x)}{\partial x_i},\ \ x\in D \tag{4}$$

$$(u(x)-h(x))\in \overset{\circ}{W}{}^1_{2,\alpha}(D) \tag{5}$$

The function $u(x)\in W^1_{2,\alpha}(D)$ we'll call generalized solution of problem (4)- (5) if for any function $\eta(x)\in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}\eta_{x_j}dx=\int_D\left(-f^0\eta+\sum_{i=1}^{n}f^i\eta_{x_i}\right)dx$$

is fulfilled.

Our aim to get the necessary and sufficient condition of compact removability E in the class of bounded functions.

Preliminaries statements.

At first, we introduce some auxiliary statements.

Lemma 1. *If relative to the coefficients of the operator* L, *condition (1), (2) be fulfilled, then the first boundary value problem (4)-(5) has a unique generalized solution* $u(x)$ *at any* $h(x)\in W^1_{2,\alpha}(D)$, $f^0(x)\in h_2(D)$, $f^i(x)\in L_{2,\lambda_i^{-1}}(D)$, $i=1,2,..,n,$. *At this there exists* $P_0(\alpha,n)$ *such that, if* $p>p_0$, $h(x)\in W^1_{p,\alpha}(D)$, $f^0(x)\in h_p(D)$, $f^i(x)\in L_{2,\lambda_i^{-1}}(D)$, $i=1,2,..,n,$ $\partial D\in C^1$, *then solution* $u(x)$ *is continuous in* $\overline{D}$.

Lemma 2. *Let relative to the coefficients of the operator* L *conditions (1), (2) be fulfilled. Then any generalized solution of the equation* $\mathrm{L}u=0$ *in* D *is continuous by Holder at each strictly internal domain* ∂.

Lemma 3. *Let relative to the coefficients of the operator* L, *conditions (1), (2) be fulfilled and* $\overline{\varepsilon_{R,1}}<D$. *Then for any positive generalized solution* $u(x)$ *the equation* $\mathrm{L}u=0$ *in* D *the Harnack inequality is true*

$$\sup_{\varepsilon_{R,1}(0)}u\le C_1(\gamma,\alpha,n)\inf_{\varepsilon_{R,1}(0)}u \tag{6}$$

If at this $y\in\partial\varepsilon_{R,2}(0)$ *and* $\overline{\varepsilon_{R,1}}(0)\subset D$, *then the inequality of form (6) is true in ellipsoid* $\varepsilon_{R,1}(y)$.

Lemma 4. *Let relative to the coefficients of the operator* L *conditions (1), (2) be fulfilled, and* $u(x)$ *be generalized solution of the first boundary-value problem (4), (5) at* $f^i(x)\equiv 0, i=0,..,n$. *Then if* $h(x)$ *is bounded on* ∂D *in sense* $W^1_{2,\alpha}(D)$, *then for solution* $u(x)$ *the following maximum principle is* *true*

$$\inf_{\partial D} h \le \inf_{D} u \le \sup_{D} \le \sup_{\partial D} h,$$

where $\inf_{\partial D} h\left(\sup_{\partial D} h\right)$ *is an exact lower (upper) bound those numbers* a *, for which* $h(x) \ge a$ $(h(x) \le a)$ *on* ∂D *in sense* $W^1_{2,\alpha}(D)$.

These lemmas are proved analogously to paper [7]. Therefore, we don't give the proof of these lemmas.

Let $H \subset \varepsilon$ be some compact, V_H be a set of all functions $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, such that $\varphi(x) \ge 1$ on H, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Let's consider the functional

$$J_\theta(\varphi) = \int_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) \varphi_i \varphi_j dx, \ \varphi(x) \in V_H$$

L is a H compact capacity relative to ellipsoid ε is called the value $\inf_{\varphi \in V_H} J_\theta(u)$ and denoted by $cap^{(\varepsilon)}_{L}(H)$. In case $\varepsilon = E_n$, the corresponding value is called L capacity of the compact H and denoted by $cap_{L}(H)$.

Lemma 5. *There exists the unique function* $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ *such that* $u(x) \ge 1$ *on* H *in sense* $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ *and* $cap^{(\varepsilon)}_{L}(H) = J_{L}(u)$. *Moreover,* $u(x) = 1$ *on* H *in sense* $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

Proof. It is easy to see that V_H is convex closed set in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. From the fact that $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ is a Hilbert space, it follows the existence of unique function $u(x) \in V_H$, which achieved an exact lower bound of the functional $J_{L}(\varphi)$. Let's next $\{u(x)\}^1 = \begin{cases} u(x) \text{ if } u(x) \le 1 \\ 1 \text{ if } u(x) > 1 \end{cases}$

It is clear, that $\{u(x)\}^1 \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Moreover, $\{u(x)\}^1 \in V_H$. Denote by $A^+ = \{x : x \in \varepsilon, u(x) > 1\}$. We have

$$J_{L}\left\{u(x)^1\right\} = \left(\int_{A^+} + \int_{\varepsilon \backslash A^+}\right) \sum_{i,j=1}^{n} a_{ij}(x) \{u\}^1_i \{u\}^1_j dx = \int_{\varepsilon \backslash A^+} \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j dx \qquad (7)$$

On the other side, according to (1)

$$\int_{A^+} \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j dx \ge 0 \qquad (8)$$

From (7) and (8) we conclude

$$J_{\mathrm{L}}\{u(x)^1\} \le J_{\mathrm{L}}(u) = \inf_{\varphi\in V_H} J_{\mathrm{L}}(\varphi)$$

i.e., $J_{\mathrm{L}}\{u(x)^1\} = J_{\mathrm{L}}(u)$. From uniqueness extreme function it follows, that $\{u(x)\}^1 = u(x)$, and lemma is proved.

The function $u(x)$, which achieved an exact lower bound of the functional $J_{\mathrm{L}}(\varphi)$ on the set V_H is called L capacity of the compact potential H relative to the ellipsoid ε.

Lemma 6. L *be a capacity potential* $u(x)$ *of the compact* H *relative to* ε *is a generalized solution of the equation* $\mathrm{L}u = 0$ *in* $\varepsilon \setminus H$ *,vanishing on* 0 *and* $\partial\varepsilon$ *in* 1 *on* ∂H *sense* $W^1_{2,\alpha}(\varepsilon)$.

Proof. It is sufficient to show the truthiness of the first part of assertion of lemma. Let $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ and $\eta(x) \ge 0$ on H in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Then for any $\varepsilon > 0$ $(u(x) + \varepsilon\eta(x)) \in V_H$. Therefore

$$J_{\mathrm{L}}(u + \varepsilon\eta) \ge J_{\mathrm{L}}(u).$$

Thus

$$J_{\mathrm{L}}(u) + \varepsilon^2 J_{\mathrm{L}}(\eta) + 2\varepsilon \int_\varepsilon \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \ge J_{\mathrm{L}}(u),$$

i.e.

$$J_{\mathrm{L}}(u) + 2\varepsilon \int_\varepsilon \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \ge 0.$$

Tending ε to zero, we conclude

$$\int_\varepsilon \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \ge 0. \qquad (9)$$

It is easy to see as $\eta(x)$ in (9) we can take any function from $C^1(\overline{\varepsilon})$ with compact support in $\varepsilon \setminus H$. Then

$$\int_{\varepsilon\setminus H} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \ge 0.$$

Substituting $\eta(x)$ on -$\eta(x)$, we arrive to the equality

$$\int_{\varepsilon\setminus H} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx = 0$$

Lemma is proved.

Let μ be a charge of bounded variation, given on ε. We'll say, that the function $u(x) \in L_1(\varepsilon)$ is a weak solution of the equation $\mathrm{L}u = -\mu$, equaling to zero on $\partial\varepsilon$, if for any function $\varphi(x)\in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)\cap C(\overline{\varepsilon})$ the integral identity

$$\int_{\varepsilon} u\mathrm{L}\varphi dx = \int_{\varepsilon}\varphi d\mu.$$

is fulfilled.

According to lemma 1 (at $h=0$) there exists the continuous linear operator H from $\overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$ in $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, such that for any functional $T\in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$, the function $u=H(T)$ is unique in $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ generalized solution of the equation $\mathrm{L}u=T$.

The operator H is called Green operator.

By lemma 1 this operator at $p>p_0$ we transform $\overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$ to $C(\overline{\varepsilon})$. It is easy to see, that the function $u(x)$ is weak solution of the equation $\mathrm{L}u=-\mu$, equaling to zero on $\partial\varepsilon$, iff for any function $\psi(x)\in C(\overline{\varepsilon})$ the integral identity

$$\int_{\varepsilon} u\psi dx = \int_{\varepsilon} H(\psi)d\mu. \tag{10}$$

is fulfilled.

By analogy with [8] we can show that for each measure μ on ε there exists the unique weak solution of the equation $\mathrm{L}u=-\mu$ equaling to zero on $\partial\varepsilon$.

Let's say, that the charge $\mu\in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$ if there exists the vector $\overline{f}(x)=\left(f^{\circ}(x),f^{1}(x),...,f^{n}(x)\right)$ $f^{0}(x)\in h_2(\varepsilon)$, $f^{i}(x)\in L_{2,\lambda_i}(\varepsilon)$, $i=1,2,..,n$, for any function $\varphi(x)\in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)\cap C(\overline{\varepsilon})$ the integral identity

$$\mu(\varphi)=\int_{\varepsilon}\varphi d\mu = \int_{\varepsilon}\left(f^{\circ}\varphi-\sum_{i=1}^{n} f^{i}\varphi_i\right)dx.$$

Is true.

At this, it is evident that

$$\left|\int_{\varepsilon}\varphi d\mu\right| \le C_2\left(\overline{f}\right)\|\varphi\|_{W^{1}_{2,\alpha}(\varepsilon)}.$$

Lemma 7. *The weak solution $u(x)$ of the equation $\mathrm{L}u=-\mu$, equaling to zero on $\partial\varepsilon$, belongs to $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, iff $\mu\in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$.*

Proof. At first, we'll show that if the function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ satisfies the integral identity

$$\int_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx = -\int_{\varepsilon} \varphi d\mu \qquad (11)$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon) \cap C(\overline{\varepsilon})$, then it is weak solution of the equation $\mathrm{L}u = -\mu$, equaling to zero on $\partial\varepsilon$. Really, assuming $\varphi = H(\psi)$, $\psi(x) \in C(\overline{\varepsilon})$ we obtain

$$\int_{\varepsilon} H(\psi) d\mu = \int_{\varepsilon} \varphi d\mu = -\int_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx =$$

$$= \int_{\varepsilon} u \sum_{i,j=1}^{n} \left(a_{ij}(x)\varphi_j\right)_i dx = \int_{\varepsilon} u \mathrm{L}\varphi dx = \int_{\varepsilon} u \psi dx,$$

and now it is sufficient to use the identity (10). We'll show that $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$. For this, it is sufficient to prove, that if $f^i(x) = \sum_{i=1}^{n} a_{ij}(x) u_i(x)$, then $f^i(x) \in L_{2,\lambda_i^{-1}}(\varepsilon)$, $i = 1,2,..,n,$. Assume in condition (11) $\xi_1 = ... = \xi_{i-1} = \xi_{i+1} = ... = \xi_n = 0$, $\xi_i = \frac{1}{\sqrt{\lambda_i(x)}}$.

We'll obtain

$$\gamma \le \frac{a_{ii}(x)}{\lambda_i(x)} \le \gamma^{-1}; \; i = 1,..,n. \qquad (12)$$

Let $i \ne j$. Assuming $\xi_k = 0$ at $k \ne j$ and $k \ne i$, $\xi_i = \frac{1}{\sqrt{\lambda_i(x)}}$, $\xi_j = \frac{1}{\sqrt{\lambda_j(x)}}$, we'll obtain

$$2\gamma \le \frac{a_{ii}(x)}{\lambda_i(x)} + \frac{a_{jj}(x)}{\lambda_j(x)} + \frac{2a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le 2\gamma^{-1}$$

Using (12), we conclude

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1} - \gamma; \; i, j = 1,...,n; i \ne j \qquad (13)$$

From (12) and (13) it follows that

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1};\, i, j = 1,...,n; \tag{14}$$

Thus, from (14) take out for $j = 1,...,n$

$$\int_\varepsilon \frac{1}{\lambda_j(x)}\left(f^j\right)^2 dx = \int_\varepsilon \frac{1}{\lambda_j(x)}\left(\sum_{i=1}^n a_{ij}(x)u_i\right)^2 dx \le \gamma^{-2} n \sum_{i=1}^n \int_\varepsilon \lambda_i(x)u_i^2 dx < \alpha$$

So, $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$. Inversely, if $u(x)$ is a weak solution of the equation $\mathrm{L}u = -\mu$, vanishing on $\partial\varepsilon$, then there exists $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$, such that

$$\left(f^\circ \varphi - \sum_{i=1}^n f^i \varphi_i\right)dx = \int_\varepsilon \varphi d\mu = \int_\varepsilon u \mathrm{L}\varphi dx =$$

$$= \int_\varepsilon u \sum_{i,j=1}^n \left(a_{ij}(x)\varphi_j\right)_i dx = -\int_\varepsilon \sum_{i,j=1}^n a_{ij}(x)u_i\varphi_j dx$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \cap C(\bar{\varepsilon}), \mathrm{L}\varphi(x) \in C(\bar{\varepsilon})$.

Then, from lemma 1 we obtain that $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. The lemma is proved.

Let now $\delta(x)$ be Dirac measure, concentrated at the point 0, y is an arbitrary fixed point ε.

The weak solution $g(x, y)$ of the equation $\mathrm{L}y = -\delta(x - y)$, vanishing on $\partial\varepsilon$ is called Green function of the operator L in ε.

In case $\varepsilon = E_n$ the corresponding function is called the fundamental solution of the operator L and denoted by $G(x, y)$.

According to above proved, if $\psi(x)$ is an arbitrary function from $C(\bar{\varepsilon})$, then the generalized solution $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ of the equation $\mathrm{L}\varphi = -\psi$ can be introduced in the following from

$$\varphi(y) = \int_\varepsilon g(x, y)\psi(x)dx.$$

We can show, that $g(x, y)$ is non-negative in $\varepsilon \times \varepsilon$, moreover, $g(x, y) = g(y, x)$.

3. Removability criterion of the compact in the space $M(D)$.

Theorem 1. *Let relative to the coefficients of the operator* L, *conditions (1)-(2) be fulfilled. Then for removability of the compact* $E \subset D$ *relative to the first boundary value problem for the operator* L *in the space*

$\mathsf{M}(D)$ *it is necessary and sufficient, that*

$$cap_{\mathsf{L}}(E)=0 \tag{15}$$

Proof. Let the ellipsoid ε has the same sense, that above. It is easy to see that if condition (15) be fulfilled, then

$$cap_{\mathsf{L}}^{(\varepsilon)}(E)=0$$

Not losing generality, we can limited with case, when the coefficients of the operator L is continuously differentiable in $\overline{\varepsilon}$. Let's fixed an arbitrary $\varepsilon>0$ and $x^0\subset D\setminus E$. By virtue of (15) there exists the neighbourhood H of the compact E, such that

$$cap_{\mathsf{L}}^{(\varepsilon)}\left(\overline{H}\right)<\varepsilon \tag{16}$$

At this, we can assume that ε is such small, that

$$dist\left(x^0,\overline{H}\right)\geq\frac{1}{2}dist\left(x^0,E\right) \tag{17}$$

Denote by $V_H(x)$ and μ_H the L-capacity potential of the compact $\overline{H}$ relative to the ellipsoid ε and $\overline{\mathsf{L}}$-capacity of the distribution $\overline{H}$, respectively. According to above proved

$$V_H(x)=\int_{\varepsilon}g(x,y)d\mu_H(y),$$

moreover the function $V_H(x)$ is generalized solution of the equation $\mathsf{L}V_H=0$ in $\varepsilon\setminus\overline{H}$, vanishing on 0 and in $\partial\varepsilon$ on 1 in ∂H sense $W_{2,\alpha}^1(\varepsilon)$. Let now, $u(x)\in\mathsf{M}(D)$ is an arbitrary solution of the equation $\mathsf{L}u=0$ in $D\setminus E$, vanishing on ∂D, $M=\sup_D|u|$. It is easy to see, that the function $V_H(x)$ is non-negative on ∂D, in sense $W_{2,\alpha}^1(D)$. Hence, it follows, that the function $u(x)-MV_H(x)$ is generalized solution of the equation $\mathsf{L}u=0$ in $D\setminus\overline{H}$, is non-positive on $\partial\left(D\setminus\overline{H}\right)$. According to lemma 4 $u(x)-MV_H(x)\leq 0$ and $D\setminus\overline{H}$ in particular

$$u\left(x^0\right)\leq MV_H\left(x^0\right)\leq M\sup_{y\in\partial H}g\left(x^0,y\right)\mu_H\left(\overline{H}\right)=M\sup_{y\in\partial H}g\left(x^0,y\right)cap_{\mathsf{L}}^{(\varepsilon)}\left(\overline{H}\right) \tag{18}$$

By virtue of continuity of the function $g(x,y)$ at $x\neq y$ and inequality (17) we obtain

$$\sup_{y\in\partial H}g\left(x^0,y\right)\leq C_6\left(\gamma,\alpha,n,x^0,E\right)$$

Thus, from (16) and (18) we conclude

$$u\left(x^0\right)\leq MC_6\varepsilon \tag{19}$$

Using an arbitrariness ε, we lead to the inequality

$$u\left(x^0\right)\le 0 \tag{20}$$

Making analogous considerations with the function $u(x)+MV_H(x)$, we obtain

$$u\left(x^0\right)\ge 0 \tag{21}$$

From (19)-(20) and an arbitrariness of the point x^0 it follows, that $u(x)\equiv 0$ in $D\setminus E$. Thereby, the sufficiency of condition (28) is proved. Let's prove its necessarily. Let's assume that $cap_{\mathrm{L}}(E)>0$. Denote by ε' the ellipsoid, such that $\overline{\varepsilon}' \subset \delta$, $E\subset \varepsilon'$. Assume $D=\varepsilon$. Further, let $u_E(x)$ be V_E-L capacity potential of the compact E relative to the ellipsoid ε' and L-capacity distribution E, respectively. Following to [10], we can give the equivalent definition of Vallee-Poussin type of L-capacity of the compact E, relative to the ellipsoid ε'. Let $g(x,y)$ be a Green function of the operator L in ε'. Let's call the measure μ on E, L-admissible, if $\mu\subset E$ and

$$V_\mu^E(x)=\int_{\varepsilon'} g(x,y)d\mu(y)\le 1 \quad for \quad x\in \sup p\mu \tag{22}$$

The value $\sup\mu(E)=cap_{\mathrm{L}}^{\left(\varepsilon'\right)}(E)$, where an exact upper boundary is taken on all L-admissible measures, is called L-capacity of the compact E, relative to the ellipsoid ε'.

Analogously, the L-capacity $cap_{\mathrm{L}}(E)$ is determined. At this by the standard method we show, that there exists the unique measure, on which an exact upper boundary of the functional $\mu(E)$ is reached, by the set of all L-admissible measures μ. This measure is L-capacity distribution of the compact E.

According to the above proved, the function $u_E(x)$ is generalized solution of the equation $\mathrm{L}u_E=0$ in $\varepsilon'\setminus E$, equaling to zero on $\partial\varepsilon'$. Besides, from (21) and maximum principle it follows that $u_E(x)\in M\left(\varepsilon'\right)$. On the other side $u_E(x)\not\equiv 0$, as $V_H(E)>0$. Theorem is proved.

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CIRLAŞAN ELLİPTİK TƏNLİKLƏRİN ARADAN QALDIRILA BİLƏN ÇOXLUQLARI

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XÜLASƏ

Məqalədə kompaktın aradan qaldırıla bilməsi üçün zəruri və kafi şərt alınmışdır.

Açar sözlər: *elliptik tənliklər, aradan qaldırıla bilən çoxluqlar, Qrin operatoru, Hilbert fəzası.*

УСТРАНИМЫЕ МНОЖЕСТВА ВЫРАЖДАЮЩИХСЯ ЭЛЛИПТИЧЕСКИХ УРАВНЕНИЙ

Г.Ю.МЕХТИЕВА, Т. ГАДЖИЕВ, Н. БАЙРАМОВА

РЕЗЮМЕ

В статье получены необходимые и достаточные условия для устранимости компакта.

Ключевые слова: *эллиптические уравнении, устранимые множества, оператор Грина, гильбертово пространство.*

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