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PROPERTIES OF THE RESOLVENT OF A POLYNOMIAL OPERATOR PENCIL OF THIRD ORDER WITH MULTIPLE CHARACTERISTICS

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In this paper we study the analytic properties of the resolvent of a polynomial operator pencil of third-order. The distinguishing feature of the investigated pencil is that its principal part has multiple characteristics.

Key words: *polynomial operator pencil, resolvent, discrete spectrum, self-adjoint operator, compact operator.*

Let H be a separable Hilbert space and consider a polynomial operator pencil of third order in H

$$P(\lambda)=(-\lambda E+A)(\lambda E+A)^2+\sum_{s=1}^{2}\lambda^{3-s}A_s\,, \qquad (1)$$

where E is the identity operator, A, A_s, $s=1,2$, are linear operators in H. Denote by

$$P_0(\lambda)=(-\lambda E+A)(\lambda E+A)^2, \quad P_1(\lambda)=\sum_{s=1}^{2}\lambda^{3-s}A_s\,.$$

The following theorem holds.

Theorem 1. *Suppose that* A *is* a self-adjoint positive-definite operator, A_sA^{-s}, $s=1,2$, *are bounded operators in* H, *and the following inequality holds:*

$$\alpha=\sum_{s=1}^{2}a_s\left\| A_sA^{-s}\right\|<1,$$

where $a_1 = a_2 = \dfrac{2}{3\sqrt{3}}$. *Then the resolvent of the pencil* $P(\lambda)$ *exists on the imaginary axis and the following estimation holds*

$$\sum_{j=0}^{2} \left\| \lambda^{3-j} A^j P^{-1}(\lambda) \right\| \le const . \tag{2}$$

Proof. Let $\lambda = i\zeta$, $\zeta \in R = (-\infty, +\infty)$. Since by the condition of the theorem, A is a self-adjoint positive-definite operator in H, it is obvious that for $\lambda = i\zeta$ the operator pencil $P_0(\lambda)$ is invertible in H, and

$$P_0^{-1}(\lambda) = (\lambda E + A)^{-2} (-\lambda E + A)^{-1}, \ \lambda = i\zeta, \ \zeta \in R.$$

Therefore for $\lambda = i\zeta$ operator pencil (1) is represented in the form

$$P(\lambda) = \left(E + P_1(\lambda) P_0^{-1}(\lambda)\right) P_0(\lambda). \tag{3}$$

It is clear that $P_1(\lambda) P_0^{-1}(\lambda)$ can be written as

$$P_1(\lambda) P_0^{-1}(\lambda) = \sum_{s=1}^{2} \lambda^{3-s} A_s A^{-s} A^s P_0^{-1}(\lambda).$$

Then by the boundedness of the operators $A_s A^{-s}$, $s = 1, 2$

$$\left\| P_1(\lambda) P_0^{-1}(\lambda) \right\| \le \sum_{s=1}^{2} \left\| A_s A^{-s} \right\| \left\| \lambda^{3-s} A^s P_0^{-1}(\lambda) \right\|. \tag{4}$$

Further we will estimate the norm $\left\| \lambda^{3-s} A^s P_0^{-1}(\lambda) \right\|$, $s = 1, 2$, for $\lambda = i\zeta$, $\zeta \in R$. First consider the case $s = 1$. Taking into account the spectral decomposition of the operator A, we obtain:

$$\left\| \lambda^2 A P_0^{-1}(\lambda) \right\| = \sup_{\sigma \in \sigma(A)} \left| \frac{\sigma (i\zeta)^2}{(i\zeta + \sigma)^2 (-i\zeta + \sigma)} \right| =$$

$$= \sup_{\sigma \in \sigma(A)} \frac{\sigma \zeta^2}{(\zeta^2 + \sigma^2)^{3/2}} = \sup_{\sigma \in \sigma(A)} \left(\frac{\sigma^2 \zeta^4}{(\zeta^2 + \sigma^2)^3} \right)^{1/2},$$

where $\sigma(A)$ is the spectrum of the operator A. Denoting $\eta = \dfrac{\zeta^2}{\sigma^2}$, we have

$$\left\| \lambda^2 A P_0^{-1}(\lambda) \right\| \le \sup_{\eta \ge 0} \left(\frac{\eta^2}{(\eta + 1)^3} \right)^{1/2}.$$

Note that the function $f(\eta)=\dfrac{\eta^2}{(\eta+1)^3}$ takes its maximum value at the point $\eta_0=2$, i.e. $f_{\max}=f(\eta_0)=\dfrac{4}{27}$. Therefore

$$\left\| \lambda^2 A P_0^{-1}(\lambda) \right\| \le \frac{2}{3\sqrt{3}} = a_1 . \qquad (5)$$

Proceeding in a similar way, in the case $s=2$ we have

$$\left\| \lambda A^2 P_0^{-1}(\lambda) \right\| = \sup_{\sigma\in\sigma(A)} \left| \frac{i\zeta\sigma^2}{(i\zeta+\sigma)^2(-i\zeta+\sigma)} \right| =$$

$$= \sup_{\sigma\in\sigma(A)} \frac{|\zeta|\sigma^2}{(\zeta^2+\sigma^2)^{3/2}} = \sup_{\sigma\in\sigma(A)} \left(\frac{\zeta^2\sigma^4}{(\zeta^2+\sigma^2)^3} \right)^{1/2} .$$

Using the notation $\eta=\dfrac{\zeta^2}{\sigma^2}$, we have

$$\left\| \lambda A^2 P_0^{-1}(\lambda) \right\| \le \sup_{\eta\ge 0} \left(\frac{\eta}{(\eta+1)^3} \right)^{1/2} .$$

The function $g(\eta)=\dfrac{\eta}{(\eta+1)^3}$ takes its maximum value at the point $\eta_0=\dfrac{1}{2}$, i.e. $g_{\max}=g(\eta_0)=\dfrac{4}{27}$. Therefore

$$\left\| \lambda A^2 P_0^{-1}(\lambda) \right\| \le \frac{2}{3\sqrt{3}} = a_2 . \qquad (6)$$

Thus, if we take into account (5) and (6) in (4), we get:

$$\left\| P_1(\lambda) P_0^{-1}(\lambda) \right\| \le \sum_{s=1}^{2} a_s \left\| A_s A^{-s} \right\| = \alpha < 1 .$$

Then for $\lambda=i\zeta$, $\zeta\in R$, the operator $E+P_1(\lambda)P_0^{-1}(\lambda)$ is invertible and from (3) we obtain that on the imaginary axis there exists the resolvent $P^{-1}(\lambda)$:

$$P^{-1}(\lambda) = P_0^{-1}(\lambda)\left(E + P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1} .$$

Now we prove inequality (2). Since

$$\sum_{j=0}^{2} \left\| \lambda^{3-j} A^j P^{-1}(\lambda) \right\| =$$

$$=\sum_{j=0}^{2}\left\|\lambda^{3-j}A^{j}P_0^{-1}(\lambda)\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\|\leq$$

$$\leq\sum_{j=0}^{2}\left\|\lambda^{3-j}A^{j}P_0^{-1}(\lambda)\right\|\left\|\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)^{-1}\right\|$$

and as above, it was proved that

$$\left\|\lambda^{3-s}A^{s}P_0^{-1}(\lambda)\right\|\leq a_s=\frac{2}{3\sqrt{3}},\ s=1,2,$$

it is necessary to estimate the norm $\left\|\lambda^{3}P_0^{-1}(\lambda)\right\|$. For $\lambda=i\zeta$, $\zeta\in R$,

$$\left\|\lambda^{3}P_0^{-1}(\lambda)\right\|=\sup_{\sigma\in\sigma(A)}\left|\frac{(i\zeta)^{3}}{(i\zeta+\sigma)^{2}\,(-i\zeta+\sigma)}\right|=$$

$$=\sup_{\sigma\in\sigma(A)}\frac{|\zeta|^{3}}{\left(\zeta^{2}+\sigma^{2}\right)^{3/2}}\leq 1.$$

Consequently, for $\lambda=i\zeta$, $\zeta\in R$, we have

$$\sum_{j=0}^{2}\left\|\lambda^{3-j}A^{j}P^{-1}(\lambda)\right\|\leq\left(1+a_1+a_2\right)\cdot\frac{1}{1-\alpha}=const\,.$$

The theorem is proved.

Remark 1. *From the proof of Theorem 1, it is clear that for* $\lambda=i\zeta$, $\zeta\in R$, *we have*

$$\left\|A^{q}P^{-1}(\lambda)\right\|\leq const\,|\lambda|^{q-3},\ \ 0<q<3,\ \ \lambda\neq 0. \qquad (7)$$

We now estimate the resolvent of pencil (1) on some sectors adjoining to the imaginary axis.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then for sufficiently small* $\varphi>0$ *on the sectors*

$$\Gamma_{\frac{\pi}{2}\pm\varphi}=\left\{\lambda:\lambda=re^{i\left(\frac{\pi}{2}\pm\varphi\right)},\,r>0\right\},\quad \Gamma_{-\frac{\pi}{2}\pm\varphi}=\left\{\lambda:\lambda=re^{-i\left(\frac{\pi}{2}\pm\varphi\right)},\ r>0\right\}$$

the operator pencil $P(\lambda)$ *is invertible, and estimation (2) holds.*

Proof. Let $\lambda\in\Gamma_{\frac{\pi}{2}+\varphi}$. Then for $\lambda=\zeta e^{i\left(\frac{\pi}{2}+\beta\right)}$, $0<\beta\leq\varphi$, we have

$$P(\lambda)=P\left(i\zeta e^{i\beta}\right)=P(i\zeta)-(i\zeta)^{3}\left(e^{3i\beta}-1\right)E-$$
$$-(i\zeta)^{2}\left(e^{2i\beta}-1\right)A+i\zeta\left(e^{i\beta}-1\right)A^{2}+$$
$$+(i\zeta)^{2}\left(e^{2i\beta}-1\right)A_1+i\zeta\left(e^{i\beta}-1\right)A_2\,. \qquad (8)$$

By Theorem 1 operator pencil (1) is invertible on the imaginary axis, and estimation (2) holds. Then (8) can be written as

$$P(\lambda)=P\left(i\zeta e^{i\beta}\right)=\left(E+Q(\beta;\zeta)\right)P(i\zeta),$$

where

$$Q(\beta;\zeta)=-(i\zeta)^3\left(e^{3i\beta}-1\right)P^{-1}(i\zeta)-(i\zeta)^2\left(e^{2i\beta}-1\right)AP^{-1}(i\zeta)+$$
$$+i\zeta\left(e^{i\beta}-1\right)A^2P^{-1}(i\zeta)+(i\zeta)^2\left(e^{2i\beta}-1\right)A_1P^{-1}(i\zeta)+i\zeta\left(e^{i\beta}-1\right)A_2P^{-1}(i\zeta)$$

.

It is easy to note that

$$\left\|Q(\beta;\zeta)\right\|\le\left|e^{3i\beta}-1\right|\left\|(i\zeta)^3P^{-1}(i\zeta)\right\|+\left|e^{2i\beta}-1\right|\left\|(i\zeta)^2AP^{-1}(i\zeta)\right\|+$$
$$+\left|e^{i\beta}-1\right|\left\|i\zeta A^2P^{-1}(i\zeta)\right\|+\left|e^{2i\beta}-1\right|\left\|A_1A^{-1}\right\|\left\|(i\zeta)^2AP^{-1}(i\zeta)\right\|+$$
$$+\left|e^{i\beta}-1\right|\left\|A_2A^{-2}\right\|\left\|i\zeta A^2P^{-1}(i\zeta)\right\|.$$

Since $0<\beta\le\varphi$ and if we take into account Theorem 1, then we have:

$$\left\|(i\zeta)^3P^{-1}(i\zeta)\right\|\le\frac{1}{1-\alpha},$$

$$\left\|(i\zeta)^{3-j}A^jP^{-1}(i\zeta)\right\|\le\frac{1}{1-\alpha}a_j,\quad j=1,2,$$

$$\left|e^{ik\beta}-1\right|\le 2\sin\frac{k\varphi}{2},\quad k=1,2,3.$$

Therefore, for sufficiently small φ and for $\zeta\in R$ we obtain that

$$\left\|Q(\beta;\zeta)\right\|\le const\max_{k=1,2,3}\left|e^{ik\beta}-1\right|\le$$
$$\le const\max_{k=1,2,3}\left(2\sin\frac{k\varphi}{2}\right)\le\alpha_1<1.$$

Then $E+Q(\beta;\zeta)$ is invertible for small φ $(0<\beta\le\varphi)$ and $\zeta\in R$. Consequently,

$$P^{-1}(\lambda)=P^{-1}(i\zeta)\left(E+Q(\beta;\zeta)\right)^{-1}$$

and

$$\sum_{j=0}^{2}\left\|\lambda^{3-j}A^jP^{-1}(\lambda)\right\|=$$
$$=\sum_{j=0}^{2}\left\|\left(i\zeta e^{i\beta}\right)^{3-j}A^jP^{-1}(i\zeta)\left(E+Q(\beta;\zeta)\right)^{-1}\right\|\le$$
$$\le\sum_{j=0}^{2}\left\|(i\zeta)^{3-j}A^jP^{-1}(i\zeta)\right\|\left\|\left(E+Q(\beta;\zeta)\right)^{-1}\right\|\le$$

$$\leq (1 + a_1 + a_2) \cdot \frac{1}{1-\alpha} \cdot \frac{1}{1-\alpha_1} = const .$$

The other cases are proved similarly. The theorem is proved.

Remark 2. *Note that on the sectors* $\Gamma_{\frac{\pi}{2}\pm\varphi}$ *and* $\Gamma_{-\frac{\pi}{2}\pm\varphi}$ *estimation (7) also holds.*

Remark 3. *When obtaining the results of theorems 1 and 2 we did not require a compact operator* A^{-1}.

We denote by $\sigma_\infty(H)$ the space of compact operators acting in H .

If besides the above mentioned conditions on the operator coefficients of pencil (1) we assume that $A^{-1} \in \sigma_\infty(H)$, then it is easy to show that this operator pencil has a discrete spectrum with a unique limit point at infinity. Indeed, taking into account the conditions on the operator coefficients, we can rewrite pencil (1) in the form

$$P(\lambda) = -\lambda^3 E - \lambda^2 A + \lambda A^2 + A^3 + \sum_{s=1}^{2} \lambda^{3-s} A_s =$$

$$= \left(-\lambda^3 A^{-3} - \lambda^2 A^{-2} + \lambda A^{-1} + E + \sum_{s=1}^{2} \lambda^{3-s} A_s A^{-s} A^{-3+s} \right) A^3 =$$

$$= (E + K(\lambda)) A^3 ,$$

where

$$K(\lambda) = -\lambda^3 A^{-3} - \lambda^2 A^{-2} + \lambda A^{-1} + \sum_{s=1}^{2} \lambda^{3-s} A_s A^{-s} A^{-3+s} .$$

This shows that $K(\lambda) \in \sigma_\infty(H)$ for any $\lambda \in C$, where C is the complex plane, and $E + K(0) = E$ is invertible. Then by the M.V.Keldysh Lemma [1] $E + K(\lambda)$ is invertible everywhere except at the isolated points, that are the eigen values of the pencil $E + K(\lambda)$ and have a limit point only at infinity. And from the representation $P(\lambda) = (E + K(\lambda))A^3$ it follows that operator pencil (1) also has this property.

Note that the questions involved in this paper, for polynomial operator pencils of fourth order, whose principal part has multiple characteristics were studied, for example, in the papers [2], [3].

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TƏKRARLANAN XARAKTERİSTİKALI ÜÇTƏRTİBLİ POLİNOMİAL OPERATOR DƏSTƏNİN REZOLVENTASININ XASSƏLƏRİ HAQQINDA

A.L.ELBABLİ

XÜLASƏ

İşdə bir üçtərtibli polinomial operator dəstənin rezolventasının analitik xassələri öyrənilmişdir. Tədqiq olunan dəstənin fərqləndirici xüsusiyyəti odur ki, onun baş hissəsi təkrarlanan xarakteristikaya malikdir.

Açar sözlər: polinomial operator dəstə, rezolventa, diskret spektr, öz-özünə qoşma operator, tamam kəsilməz operator.

О СВОЙСТВАХ РЕЗОЛЬВЕНТЫ ПОЛИНОМИАЛЬНОГО ОПЕРАТОРНОГО ПУЧКА ТРЕТЬЕГО ПОРЯДКА С КРАТНОЙ ХАРАКТЕРИСТИКОЙ

А.Л.ЭЛЬБАБЛИ

РЕЗЮМЕ

В работе изучены аналитические свойства резольвенты одного полиномиального операторного пучка третьего порядка. Отличительной чертой исследуемого пучка является то, что его главная часть имеет кратную характеристику.

Ключевые слова: полиномиальный операторный пучок, резольвента, дискретный спектр, самосопряженный оператор, вполне непрерывный оператор.

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