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ON THE CONVERGENCE OF AN ITERATIVE PROCESS FOR AN INFINITE FAMILY OF BOUNDED LINEAR OPERATORS WITH APPLICATION

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In this paper, we consider an iterative process for an infinite family of bounded linear operators on a Banach space. Boundedness and convergence of the considered iterative process are established in the framework of Banach spaces. We also apply the main result presented in this paper for certain problem in summability theory.

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1. Introduction

Throughout this paper, we assume that X is a real or complex Banach space. The symbol $F(T)$ stands for the set of fixed points of T (for a single valued mapping $T : X \to X$, $x \in X$ is called a *fixed point* of T if and only if $T(x) = x$). The Banach algebra of all bounded linear operators on X is denoted by $B(X)$. By N we denote the set of all nonnegative integers, i.e., $N = \{0,1,2,...\}$. We shall write c and c_0 for the spaces of all convergent and null sequences, respectively. Also by l_p and bv_p we denote the space of all p-absolutely summable sequences and p-bounded variation sequences, respectively.

Recall that a linear operator T on a Banach space X is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in X$.

A linear operator T on a Banach space X is called a *contraction* if $\|T\| \le 1$. Obviously, due to its linearity, T is contraction if and only if it is nonexpansive.

A linear operator T on a Banach space X is called a *strictly contractive* if $\|T\|<1$. Obviously, the unique fixed point of a strictly contractive linear operator is 0. Then a bounded linear operator $T\in B(X)$ that has a nonzero fixed point cannot be strictly contractive.

Many authors have investigated some iteration processes for a finite family of nonexpansive operators [12,13,16].

This paper is motivated by some problems in the study of the spectrum and the fine spectrum of some linear operators defined by some particular limitation matrices (*cf.* [4,14,15]). In [4], the authors have established the following Lemma.

Lemma 1.1. *Let* (c_n) *and* (d_n) *be two sequences of complex numbers such that* $\lim_{n\to\infty} c_n = c$ *and* $|c|<1$. *Define the sequence* (z_n) *of complex numbers such that* $z_{n+1}=c_{n+1}z_n+d_{n+1}$. *Then we have:*

(i) If (d_n) *is bounded, then* (z_n) *is bounded,*

(ii) If (d_n) *is convergent, then* (z_n) *is convergent,*

(iii) If (d_n) *is null sequence, then* (z_n) *is null sequence.*

In this paper we consider the following iterative process for an infinite family of bounded linear operators:

$$z_0\in X, \quad z_{n+1}=T_n z_n+y_n, \quad n\in N, \tag{1.1}$$

where $T_n: X\to X$ is a bounded linear operator on *X*, for each $n\in N$, and (y_n) is a sequence in *X*.

The purpose of this paper is to study the boundedness and the convergence of the sequence (z_n) which is generated by the iterative process (1.1) for an infinite family of bounded linear operators on an arbitrary real or complex Banach space. More precisely, we prove under conditions that the sequence (z_n) is bounded if and only if the sequence (y_n) is bounded. Also, we prove under conditions that the sequence (z_n) converges to a unique fixed point of a linear operator W which is defined by $Wz=Tz+y$, $z\in X$, where $y_n\to y$, as $n\to\infty$ and $T_n\to T$ uniformly on *X*. The result presented in this paper may be applied to certain problems in summability theory.

2. Main Results

In this section, we are going to prove the following main result

Theorem 2.1. *Let X be a real or complex Banach space and* $T_n\in B(X)$, *for each* $n\in N$, *and* (y_n) *is any sequence in X. Also,* (z_n) *be a sequence generated by (1.1). If* $T_n\to T$ uniformly on *X and T is a strictly contractive operator, then we have the following:*

(1) The sequence (z_n) *is bounded if and only if* (y_n) *is bounded.*

(2) If (y_n) *is convergent and* $\lim_{n\to\infty} y_n = y$, *then the sequence* (z_n) *converges to unique fixed point* $x \in F(W)$, *where W is given by*

$$Wz = Tz + y, \qquad z \in X$$

and x has the form $x = \sum_{n=0}^{\infty} T^n y$.

In order to prove this result, we need the following lemma.

Lemma 2.2 *Let* $T \in B(X)$ *be a strictly contractive operator on a Banach space X, and* $y \in X$. *Then the sequence* (x_n) *which is generated by* $x_{n+1} = Tx_n + y$, *converges to the limit*

$$x = \sum_{n=0}^{\infty} T^n y$$

and the limit x is the unique fixed point of the operator W which is given by

$$Wz = Tz + y, \qquad z \in X.$$

Proof. It is clear that *W* is a contractive operator. Indeed, if z_1 and z_2 are any elements of *X*, then

$$\|Wz_1 - Wz_2\| = \|Tz_1 - Tz_2\| \le \|T\| \|z_1 - z_2\|,$$

and the operator *W* has unique fixed point $x \in X$, and also $x = \lim_{n\to\infty} x_n$.

Since

$$x_{n+1} = T^{n+1} x_0 + (I + T + \dots + T^n) y,$$

we have

$$x = \sum_{n=0}^{\infty} T^n y.$$

This completes the proof of the lemma.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. (1) Since $\|T\| < 1$, then there exist a real number $n_0 \in N$, such that $\|T_n\| < q < 1$ for all $n \ge n_0$. From (1.1), we have

$$\|z_n\| \le \|T_{n_0}\| \|T_{n_0+1}\| \cdots \|T_{n-1}\| \|z_{n_0}\| +$$
$$+ \|T_{n_0+1}\| \|T_{n_0+2}\| \cdots \|T_{n-1}\| \|y_{n_0}\| + \dots + \|T_{n-1}\| \|y_{n-2}\| + \|y_{n-1}\|,$$

for all $n \ge n_0$. If (y_n) is a bounded sequence, then there exists a real number *K* such that $\|y_n\| \le K$, for all $n \in N$. Then,

$$\|z_n\| \le q^{n-n_0} \|z_{n_0}\| + q^{n-n_0-1} K + \dots + qK + K =$$
$$= q^{n-n_0} \|z_{n_0}\| + \frac{1 - q^{n-n_0}}{1-q} K.$$

for all $n \ge n_0$. This implies that the sequence (z_n) is bounded. The reverse of the assertion is obvious.

(2) Consider the sequence (x_n) which is generated by

$$x_{n+1} = Tx_n + y,$$

where $y = \lim\limits_{n\to\infty} y_n$. In Lemma 2.2, we have shown that the sequence (x_n) converges to the element

$$x = \sum_{n=0}^{\infty} T^n y.$$

In order to show that (z_n) is convergent sequence with the limit x, it is sufficient to prove that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Indeed,

$$\begin{aligned} \|z_{n+1} - x_{n+1}\| &= \|T_n z_n + y_n - Tx_n - y\| = \\ &= \|T_n z_n - Tz_n + Tz_n - Tx_n + y_n - y\| \le \\ &\le \|T_n - T\|\|z_n\| + \|T\|\|z_n - x_n\| + \|y_n - y\| \le \\ &\le M\,\|T_n - T\| + \|T\|\|z_n - x_n\| + \|y_n - y\|, \end{aligned}$$

where $M = \sup\limits_n \left(\|z_n\|\right)$ (the boundedness of (z_n) follows from the first assertion).

From the conditions of the theorem, it follows that

$$\lim_{n\to\infty} \left(M\,\|T_n - T\| + \|y_n - y\|\right) = 0.$$

Then for any $\varepsilon > 0$, there is a number n_ε such that

$$M\,\|T_n - T\| + \|y_n - y\| < \varepsilon,$$

for all $n \ge n_\varepsilon$. Therefore

$$\|z_{n+1} - x_{n+1}\| \le \|T\|\|z_n - x_n\| + \varepsilon,$$

for all $n \ge n_\varepsilon$. Then it is easy to see that

$$\begin{aligned} \|z_{n+1} - x_{n+1}\| &\le \|T\|^{n-n_\varepsilon+1} \|z_{n_\varepsilon} - x_{n_\varepsilon}\| + \varepsilon\left(1 + \|T\| + \|T\|^2 + \ldots + \|T\|^{n-n_0}\right) \le \\ &\le \|T\|^{n-n_\varepsilon+1} \|z_{n_\varepsilon} - x_{n_\varepsilon}\| + \frac{\varepsilon}{1 - \|T\|}, \end{aligned}$$

for all $n \ge n_\varepsilon$. But $\|T\|^{n-n_\varepsilon+1} \to 0$ as $n \to \infty$. So, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0,$$

and so,

$$\lim_{n\to\infty} \|z_n - x\| = 0.$$

This completes the proof of the theorem.

Remark 2.3 The results in Theorem 2.1 are generalization of the results in Lemma 1.1.

3. Application

In this section we apply Theorem 2.1 for certain problem in summability theory.

It is known that the infinite real matrices as operators acting on sequence spaces play main role in summability theory, among them the difference operator-matrices and their generalizations (*cf*. [1-11,14,15]).

Now, let (a_k) be either constant or strictly decreasing sequence of positive real numbers such that

$$\lim_{n\to\infty} a_k = a > 0$$

and

$$\sup_k (a_k) \leq 2a\,.$$

It is easy to see that $\sup_k (a_k) = a_0$.

In [14], the authors introduced new generalized difference operator

$$\Delta_a = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ -a_0 & a_1 & 0 & \cdots \\ 0 & -a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and investigated the spectrum of the operator Δ_a over the sequence space c_0.

In order to determine the spectrum of the operator Δ_a, it is required to study the resolvent operator $(\Delta_a - \lambda I)^{-1}$, where *I* is the identity operator on the Banach space c_0.

If $|\lambda - a| > a$, then $(\Delta_a - \lambda I)$ is triangle and hence $(\Delta_a - \lambda I)^{-1}$ exists. Calculating $(\Delta_a - \lambda I)^{-1}$, we get

$$(\Delta_a - \lambda I)^{-1} = (s_{nk}) = \begin{pmatrix} \dfrac{1}{(a_0-\lambda)} & 0 & 0 & \cdots \\ \dfrac{a_0}{(a_0-\lambda)(a_1-\lambda)} & \dfrac{1}{(a_1-\lambda)} & 0 & \cdots \\ \dfrac{a_0 a_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)} & \dfrac{a_1}{(a_1-\lambda)(a_2-\lambda)} & \dfrac{1}{(a_2-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In order to prove the boundedness of the resolvent $(\Delta_a - \lambda I)^{-1}$ on c_0 it is necessary to establish the boundedness of the sequence

$$S_n = \frac{a_0 a_1 \ldots a_{n-1}}{|a_0-\lambda||a_1-\lambda|\ldots|a_n-\lambda|} + \ldots. + \frac{a_{n-1}}{|a_{n-1}-\lambda||a_n-\lambda|} + \frac{1}{|a_n-\lambda|}, \tag{3.1}$$

[14]. Suppose that

$$\beta = \lim_{n\to\infty} \frac{a_{n-1}}{|a_n - \lambda|} = \frac{a}{|a-\lambda|} < 1. \tag{3.2}$$

It is easy to show that

$$\lim_{n\to\infty} \frac{1}{|a_n - \lambda|} = \frac{\beta}{a} \tag{3.3}$$

and also,

$$S_n = \frac{a_{n-1}}{|a_n - \lambda|} S_{n-1} + \frac{1}{|a_n - \lambda|}. \tag{3.4}$$

Denote

$$f_n = \frac{a_{n-1}}{|a_n - \lambda|}, \quad g_n = \frac{1}{|a_n - \lambda|}, \quad n \in N$$

From (3.2) and (3.3), it follows that the sequences (f_n) and (g_n) belong to *c*.

Let us introduce the following operators

$$T_n x = f_n x, \quad n \in N$$

where $x \in c$. From (3.2), we have

$$\lim_{n\to\infty} T_n = T, \quad \|T\| = \frac{a}{|a-\lambda|} < 1$$

Now, by applying Theorem 2.1, we get that the sequence (S_n) in (3.4) is bounded.

Thus we have proved the following theorem.

Theorem 3.1. *Suppose that* $|\lambda - a| > a$. *Then, the following assertions are true*

(1) The sequence (S_n) *is bounded if and only if the sequence* $\left(\frac{1}{|a_n - \lambda|}\right)$ *is bounded.*

(2) The sequence (S_n) *is convergent if the sequence* $\left(\frac{1}{|a_n - \lambda|}\right)$ *is convergent.*

(3) If $\lim_{n\to\infty} S_n = s$ *, then* $s = \frac{1}{|\lambda - a| - a}$

Remark 3.2. Note that, in [14], the authors used the convergence of the sequence (S_n) without proof.

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XƏTTİ MƏHDUD OPERATORLARIN SONSUZ AİLƏSİ ÜÇÜN BİR İTERATİV PROSESİN YIĞILMASI VƏ TƏTBİQİ HAQQINDA

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XÜLASƏ

Bu məqalədə Banax fəzasında xətti məhdud operatorların sonsuz ailəsi üçün bir iterativ prosesin yığılmasına baxılmışdır. Həmin iterativ prosesin Banax fəzasında məhdudluğu və yığılması öyrənilmişdir. Məqalənin əsas nəticəsi cəmləmə nəzəriyyəsinin bir məsələsinin həllinə tətbiq olunmuşdur.

Açar sözlər: iterativ proses, yığılma, məhdud xətti operator, Banax fəzası

О СХОДИМОСТИ ОДНОГО ИТЕРАТИВНОГО ПРОЦЕССА БЕСКОНЕЧНОГО СЕМЕЙСТВА ОГРАНИЧЕННЫХ ЛИНЕЙНЫХ ОПЕРАТОРОВ С ПРИЛОЖЕНИЕМ

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РЕЗЮМЕ

В работе рассматривается сходимость итеративного процесса одного бесконечного семейства ограниченных линейных операторов. Изучены ограниченность и сходимость рассматриваемого итеративного процесса в банаховом пространстве. Основной результат работы был применен для решения одной задачи из теории суммирования рядов.

Ключевые слова: итеративный процесс, сходимость, ограниченный линейный оператор, пространство Банаха.

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