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ON THE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR $\Delta_{a,b}$ OVER THE SEQUENCE SPACE c_0

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In this paper we determine the spectrum of a new generalized difference operator, denoted by $\Delta_{a,b}$, over the sequence space c_0 . The class of the introduced operator includes some other special cases such as the generalized difference operator Δ_v, the generalized difference operator $B(r, s)$, the difference operator Δ, the right shift and Zweier operators. The boundedness of the operator $\Delta_{a,b}$ on the sequence space c_0 has been proved. Also, the norm of this operator has been found.

1. Introduction

Let X be a nontrivial complex normed space and $T : D(T) \to X$ also be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on the domain of T. If T_λ has an inverse, we denote it by $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it the *resolvent operator* of T.

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. In this paper, $c_0, c, l_p, bv_p, T^*, X^*, B(X), R(T), \sigma(T,X), \sigma_p(T,X), \sigma_r(T,X), \sigma_c(T,X)$ respectively denote null sequences; convergent sequences; p-absolutely summable sequences; p- bounded variation sequences; the adjoint operator of T; the dual of X ; the linear space of all bounded linear operators on X into itself; the range of T; the spectrum of T on X; the point spectrum of T on X; the residual spectrum of T on X; and the continuous spectrum of T on X.

In this paper we introduce the generalized difference operator $\Delta_{a,b}$ on the sequence space c_0 as follows:

$\Delta_{a,b} : c_0 \to c_0$ is defined by, $\Delta_{a,b} x = \Delta_{a,b}(x_n) = (a_n x_n + b_{n-1} x_{n-1})_{n=0}^{\infty}$, with $x_{-1} = 0$ and $b_{-1} = 0$, where (a_n) and (b_n) are two sequences of nonzero real numbers such that:

$$\lim_{n\to\infty} a_n = a, \ \lim_{n\to\infty} b_n = b \neq 0, \text{ and } a_n \neq a + b, \ a_n \neq a - b, \text{ for all } n \in \mathbb{N}.$$

The operator $\Delta_{a,b}$ can be represented by the matrix

$$\Delta_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that the operator $\Delta_{a,b}$ is a straightforward generalization of the difference operator Δ and its generalizations [see[3]-[5] and [9]].

Now, we may give:

Lemma 1.1 [11, *p*. 129]. *The matrix* $A = (a_{nk})$ *gives rise to a bounded linear operator* $T \in B(c_0)$ *from* c_0 *to itself if and only if*

(1) *The rows of A are in* l_1 *and their* l_1 *norms are bounded,*

(2) *The columns of A are in* c_0.

The operator norm of T is the supremum of the l_1 *norms of the rows.*

Lemma 1.2 [7, *p*. 59]. *T has a dense range if and only if* T^* *is one to one.*

We summarize the knowledge in the existing literature concerning with the spectrum of the linear operator defined by some particular limitation matrices over some sequence spaces. The fine spectrum of the difference operator Δ over the sequence space l_p, $(1 \le p < \infty)$ was determined by A.Akhmedov and F.Başar [1] and over the sequence spaces c_0 and c by B. Altay and F. Başar [4]. B. De Malafosse [8] computed the spectrum of the difference operator on the space s_r, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_k \frac{|x_k|}{r^k}$, $r > 0$. A.Akhmedov and F. Başar [2] determined the fine spectrum of the difference operator on the space bv_p, $(1 \le p < \infty)$. Note that the sequence space bv_p was introduced and studied by B. Altay and F. Başar [6]. The continuous dual of bv_p determined by A. Akhmedov in [2]. P. Srivastava and S. Kumar determined the spectrum and fine spectrum of the generalized difference operator Δ_v over the sequence space c_0 in [9] and over the sequence space l_1 in [10]. The operator Δ_v is a special case of our introduced operator $\Delta_{a,b}$ when $b_k = -a_k = -v_k$, for all $k \in \mathbb{N}$. The same problem in the case when the sequence (a_k) is assumed to be constant except for finitely many elements was investigated in [3] by A. Akhmedov.

In this work, our purpose is to study the spectrum of the generalized difference operator $\Delta_{a,b}$ on the sequence space c_0. The main results of the present work are more general than the corresponding results of [4], [5] and [9].

2. On the spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0

In this section, we establish the boundedness of the operator $\Delta_{a,b}$ on c_0. Also, we examine the spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0.

Theorem 2.1. $\Delta_{a,b} \in B(c_0)$ with a norm $\|\Delta_{a,b}\|_{c_0} = \sup_k (|a_k| + |b_{k-1}|)$.

Proof. The matrix $\Delta_{a,b}$ satisfies the conditions in Lemma 1.1, and so $\Delta_{a,b} \in B(c_0)$. Now, let us take any $x = (x_k) \in c_0$. Then, it is clear that

$$\|\Delta_{a,b}\|_{c_0} \le \sup_k (|a_k| + |b_{k-1}|).$$

Conversely, let $x = (1, 0, 0, ...)$. Then

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \frac{\left\|\Delta_{a,b}x\right\|_{c_0}}{\left\|x\right\|_{c_0}} = \max\left\{|a_0|, |b_0|\right\}. \tag{2.1}$$

On the other hand, for each $k \in \mathbb{N}$, let $y = (y_n)$ be the sequence such that $y_k = 1$, $y_{k+1} = 1$ and $y_n = 0$ for all $n \in \mathbb{N} \setminus \{k, k+1\}$. Then we can see that

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \frac{\left\|\Delta_{a,b}y\right\|_{c_0}}{\left\|y\right\|_{c_0}} = \max\left\{|a_k|, |a_{k+1} + b_k|, |b_{k+1}|\right\}. \tag{2.2}$$

Combining (2.1) and (2.2), we then have

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \max\left\{|a_{k-1}|, |a_k + b_{k-1}|, |b_k|\right\}, \text{ for all } k \in \mathbb{N}. \tag{2.3}$$

Similarly, we can also show that

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \max\left\{|a_{k-1}|, |a_k - b_{k-1}|, |b_k|\right\}. \tag{2.4}$$

Consequently, (2.3) and (2.4) imply that

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \max\left\{|a_k + b_{k-1}|, |a_k - b_{k-1}|\right\}, \text{ for all } k \in \mathbb{N},$$

and so

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq |a_k| + |b_{k-1}|, \text{ for all } k \in \mathbb{N}.$$

Thus

$$\left\|\Delta_{a,b}\right\|_{c_0} \geq \sup_k (|a_k| + |b_{k-1}|).$$

This completes the proof.

Theorem 2.2. Denote the set $\{\lambda \in \mathbb{C} : |a - \lambda| \leq |b|\}$ by *D* and the set $\{a_k : a_k \notin D\}$ by *E*. Then the set *E* is finite and $\sigma(\Delta_{a,b}, c_0) = D \cup E$.

Proof. It is easy to see that *E* is a finite set and $\{a_k : k \in \mathbb{N}\} \subseteq D \cup E$. Now, we prove that $\sigma(\Delta_{a,b}, c_0) \subseteq D \cup E$.

Let $\lambda \notin D \cup E$. Then $|a - \lambda| > |b|$ and $\lambda \neq a_k$, for all $k \in \mathbb{N}$. So, $(\Delta_{a,b} - \lambda I)$ is triangle and hence $(\Delta_{a,b} - \lambda I)^{-1}$ exists. Let $y = (y_k) \in c_0$ and solving the equation $(\Delta_{a,b} - \lambda I)x = y$, for $x = (x_k)$ in terms of *y*, we get

$$x_k = \frac{(-1)^k b_0 b_1 \ldots b_{k-1}}{(a_0 - \lambda)(a_1 - \lambda)\ldots(a_k - \lambda)} y_0 + \ldots - \frac{b_{k-1}}{(a_{k-1} - \lambda)(a_k - \lambda)} y_{k-1} + \frac{1}{(a_k - \lambda)} y_k, \quad k \in \mathbb{N}.$$

Then,

$$(\Delta_{a,b} - \lambda I)^{-1} = (s_{nk}) = \begin{pmatrix} \frac{1}{(a_0 - \lambda)} & 0 & 0 & \cdots \\ \frac{-b_0}{(a_0 - \lambda)(a_1 - \lambda)} & \frac{1}{(a_1 - \lambda)} & 0 & \cdots \\ \frac{b_0 b_1}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} & \frac{-b_1}{(a_1 - \lambda)(a_2 - \lambda)} & \frac{1}{(a_2 - \lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $S_n=\sum_{k=0}^{\infty}|s_{nk}|$. Then, for each $n\in\mathbb{N}$, the series S_n is convergent since it is finite. Next, we prove that $\sup_n S_n$ is finite.

Since $\lim_{k\to\infty}\frac{|b_k|}{|a_k-\lambda|}=\frac{|b|}{|a-\lambda|}=q<1$. Then there exists $k_0\in\mathbb{N}$ and $q_0 < 1$ such that $\frac{|b_k|}{|a_k-\lambda|}<q_0<1$, for all $k\geq k_0+1$. Then, for each $n\geq k_0+1$, we can prove that

$$S_n\leq\frac{1}{|a_n-\lambda|}\left[1+q_0+q_0^2+\ldots+q_0^{n-k_0-1}m_{k_0}\right],$$

where $m_{k_0}=1+\frac{|b_{k_0}|}{|a_{k_0}-\lambda|}+\frac{|b_{k_0}||b_{k_0-1}|}{|a_{k_0}-\lambda||a_{k_0-1}-\lambda|}+\ldots+\frac{|b_{k_0}||b_{k_0-1}|\ldots|b_0|}{|a_{k_0}-\lambda||a_{k_0-1}-\lambda|\ldots|a_0-\lambda|}$. Then $m_{k_0}\geq 1$ and so

$$S_n\leq\frac{m_{k_0}}{|a_n-\lambda|}\left[1+q_0+q_0^2+\ldots+q_0^{n-k_0-1}\right].$$

Thus, $\sup_n S_n<\infty$, since $q_0 < 1$.

Now it is easy to see that $\lim_{n\to\infty}|s_{nk}|=0$, for all $k\in\mathbb{N}$, since $\lim_{n\to\infty}\left|\frac{s_{n+1,k}}{s_{n,k}}\right|=\lim_{n\to\infty}\left|\frac{b_n}{a_{n+1}-\lambda}\right|=\left|\frac{b}{a-\lambda}\right|<1$. So, the columns of $(\Delta_{a,b}-\lambda I)^{-1}$ are in *c*₀. Then, from Lemma 1.1, we have $(\Delta_{a,b}-\lambda I)^{-1}\in B(c_0)$ and so $\lambda\notin\sigma(\Delta_{a,b},c_0)$. Thus $\sigma(\Delta_{a,b},c_0)\subseteq D\cup E$.

Conversely, suppose that $\lambda\notin\sigma(\Delta_{a,b},c_0)$. Then $(\Delta_{a,b}-\lambda I)^{-1}\in B(c_0)$. Since $(\Delta_{a,b}-\lambda I)^{-1}$-transform of the unit sequence $e_1=(1,0,0,\ldots)$ is in c_0, we have $\lim_{k\to\infty}\left|\frac{b_k}{a_{k+1}-\lambda}\right|=\left|\frac{b}{a-\lambda}\right|\leq 1$ and $\lambda\neq a_k$, for all $k\in\mathbb{N}$. Then $\{\lambda\in\mathbb{C}:|a-\lambda|<|b|\}\subseteq\sigma(\Delta_{a,b},c_0)$ and $\{a_k:k\in\mathbb{N}\}\subseteq\sigma(\Delta_{a,b},c_0)$. But, $\sigma(\Delta_{a,b},c_0)$ is a compact set, and so it is closed. Then $D=\{\lambda\in\mathbb{C}:|a-\lambda|\leq|b|\}\subseteq\sigma(\Delta_{a,b},c_0)$ and $E=\{a_k:a_k\notin D\}\subseteq\sigma(\Delta_{a,b},c_0)$. This completes the proof. □

Theorem 2.3. $\sigma_p(\Delta_{a,b},c_0)=\begin{cases}E, & \text{if there exists } m\in\mathbb{N}: a_i\neq a_j\ \forall i,j\geq m;\\ \varnothing, & \text{otherwise}\end{cases}$

Proof. Consider the equation $\Delta_{a,b}x=\lambda x$ for any x in c_0. Then $(a_0-\lambda)x_0=0$ and $(a_k-\lambda)x_k+b_{k-1}x_{k-1}=0$, for all $k=1,2,3,\ldots$. Hence, for all $\lambda\notin\{a_k:k\in\mathbb{N}\}$, we have $x_k=0$, for all $k\in\mathbb{N}$. So, $\lambda\notin\sigma_p(\Delta_{a,b},c_0)$. This shows that $\sigma_p(\Delta_{a,b},c_0)\subseteq\{a_k:k\in\mathbb{N}\}$.

Now, if $\lambda = a_i$ and there exists $j \in \mathbb{N}$, such that $a_i = a_j$, then we can easily see that $x_k = 0$ for all $k < \max\{i, j\}$. Then we have the following cases:

***Case* (i)**: Let (a_k) be such that $a_i \neq a_j$ for all $i, j \in \mathbb{N}$ and let, $\lambda = a_0$. If $x_0 = 0$, then $x_k = 0,$ for all $k \in \mathbb{N}$ and so $\lambda \notin \sigma_p(\Delta_{a,b}, c_0)$. Also, if $x_0 \neq 0$ then we have $x_{k+1} = \dfrac{-b_k}{a_{k+1} - a_0} x_k \neq 0$, for all $k \in \mathbb{N}$, and hence $\lim\limits_{k\to\infty} \left|\dfrac{x_{k+1}}{x_k}\right| = \left|\dfrac{b}{a - a_o}\right|$. But $\left|\dfrac{b}{a - a_o}\right| \neq 1$, since $a_0 \neq a + b,\ a_0 \neq a - b$. Then, $x \in c_0$ if and only if $|a - a_0| > |b|$. Then $a_0 \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a - a_0| > |b|$.

Similarly, we can prove that $a_k \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a - a_k| > |b|$. Thus $\sigma_p(\Delta_{a,b}, c_0) = E$ in this case.

***Case* (ii)**: If (a_k) is such that there exists $m \in \mathbb{N}$ with $a_i \neq a_j$ for all $i, j \geq m$, then we can prove, as in Case (i), that $a_k \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a - a_k| > |b|$. Thus $\sigma_p(\Delta_{a,b}, c_0) = E$.

***Case* (iii)**: If (a_k) is not as in Case (i) or Case (ii), that is for all $m \in \mathbb{N}$ there exist $i < m$ and $j \geq m$ such that $a_i = a_j$, then we have $x = \theta$. Thus $\sigma_p(\Delta_{a,b}, c_0) = \varnothing$ in this case.

This completes the proof. $\square$

It is well known that if $T : c_0 \to c_0$ is a bounded matrix operator with the matrix A, then the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose A^t of the matrix A.

Theorem 2.4. (i) $\{\lambda \in \mathbb{C} : |a - \lambda| < |b|\} \subseteq \sigma_p(\Delta_{a,b}^*, c_0^*)$,

(ii) $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_{a,b}^*, c_0^*)$,

(iii) $\left\{\lambda \in \mathbb{C} : \sup\limits_n \left|\dfrac{a_n - \lambda}{b_n}\right| < 1\right\} \subseteq \sigma_p(\Delta_{a,b}^*, c_0^*)$,

(iv) $\sigma_p(\Delta_{a,b}^*, c_0^*) \subseteq \left\{\lambda \in \mathbb{C} : \inf\limits_n \left|\dfrac{a_n - \lambda}{b_n}\right| < 1\right\}$,

(v) $\sigma_p(\Delta_{a,b}^*, c_0^*) \subseteq (D \cup E) \backslash G$,

where the set G is defined as:

$\lambda \in G$ if and only if there exists $k_0 \in \mathbb{N}$ such that $|a_k - \lambda| = |b_k|$, for all $k \geq k_0$.

Proof. (i) Suppose that $\Delta_{a,b}^* f = \lambda f$ for $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$. Then, by solving the system of equations $a_0 f_0 + b_0 f_1 = \lambda f_0$ and $a_k f_k + b_k f_{k+1} = \lambda f_k$, $k \geq 1,$ we obtain

$f_{k+1} = \dfrac{(\lambda - a_k)}{b_k} f_k, \quad k \in \mathbb{N}$. Thus, $\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \subseteq \sigma_p(\Delta^*_{a,b}, c^*_0)$. This completes the proof of (*i*).

(***ii***) Clearly, for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, ..., f_k, 0, 0, ...)$ is an eigenvector of the operator $\Delta^*_{a,b}$ corresponding to the eigenvalue $\lambda = a_k$, where $f_n \neq 0$ for all $n = 0, 1, 2, ...k$ and $f_n = \dfrac{\lambda - a_{n-1}}{b_{n-1}} f_{n-1}$, for all $n = 1, 2, 3, ..., k$. Thus $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta^*_{a,b}, c^*_0)$. This completes the proof of (*ii*).

(***iii***) We have $f_k = \dfrac{(\lambda - a_0)(\lambda - a_1)...(\lambda - a_{k-1})}{b_0 b_1 ... b_{k-1}} f_0$, $k = 1, 2, 3, ...$. Then

$$\sum_k |f_k| = |f_0| + \sum_{k=1}^{\infty} \left| \frac{(\lambda - a_0)(\lambda - a_1)...(\lambda - a_{k-1})}{b_0 b_1 ... b_{k-1}} \right| |f_0| \leq |f_0| + |f_0| \sum_{k=1}^{\infty} \left[\sup_n \left| \frac{\lambda - a_n}{b_n} \right| \right]^k .$$

Thus, $\left\{ \lambda \in \mathbb{C} : \sup_n \left| \dfrac{a_n - \lambda}{b_n} \right| < 1 \right\} \subseteq \sigma_p(\Delta^*_{a,b}, c^*_0)$. This completes the proof of (*iii*).

(***iv***) Let $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*_0)$. Then there exists $f \neq \theta$ in c^*_0 such that $\Delta^*_{a,b} f = \lambda f$. Then the series $\sum_k |f_k|$ is convergent, and so,

$$|f_0| + |f_0| \sum_{k=1}^{\infty} \left[\inf_n \left| \frac{\lambda - a_n}{b_n} \right| \right]^k \leq |f_0| + \sum_{k=1}^{\infty} \left| \frac{(\lambda - a_0)(\lambda - a_1)...(\lambda - a_{k-1})}{b_0 b_1 ... b_{k-1}} \right| |f_0| = \sum_k |f_k| < \infty .$$

This implies that $\inf_n \left| \dfrac{\lambda - a_n}{b_n} \right| < 1$. This completes the proof of (*iv*).

(***v***) Let $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*_0)$. Then there exists $f \neq \theta$ in c^*_0 such that $\Delta^*_{a,b} f = \lambda f$. Then the series $\sum_k |f_k|$ is convergent. If $f_0 \neq 0$ and $f_k = 0$ for all $k \in \mathbb{N} \setminus \{0\}$, then $\lambda = a_0$. Similarly, we can have $\lambda = a_k$, for some $k \geq 1$. Then, λ may belongs to the set $\{a_k : k \in \mathbb{N}\}$. On the other hand, if $\lambda \notin \{a_k : k \in \mathbb{N}\}$, then $f_k \neq 0$ for all $k \in \mathbb{N}$ and so, by using d'Alembert criterion, we must have $\lim_{k \to \infty} \left| \dfrac{f_{k+1}}{f_k} \right| = \left| \dfrac{\lambda - a}{b} \right| \leq 1$. Hence, $\sigma_p(\Delta^*_{a,b}, c^*_0) \subseteq \{\lambda \in \mathbb{C} : |a - \lambda| \leq |b|\} \cup \{a_k : k \in \mathbb{N}\}$. Also, if there exists $k_0 \in \mathbb{N}$ such that $|a_k - \lambda| = |b_k|$, for all $k \geq k_0$, then we have the series $\sum_k |f_k|$ be not convergent, and so $\lambda \notin G$. This completes the proof of (*v*).

In general, $\sigma_p(\Delta^*_{a,b}, c^*_0) \neq \{\lambda \in \mathbb{C} : |a - \lambda| < |b|\}$. This can be shown in the following example.

Example 2.5 Let $a_k = \left(\dfrac{k+1}{k+3}\right)^2$, and $b_k = \left(\dfrac{k+1}{k+2}\right)^2$. Then, $\sup\limits_k |a_k| = \mathop{\mathrm{sub}}\limits_k |b_k| = a = b = 1$, and $a_k \neq a \pm b$, for all $k \in \mathbb{N}$. Clearly, $0 \notin \{\lambda \in \mathbb{C} : |a-\lambda| < |b|\}$. But, $0 \in \sigma_p(\Delta^*_{a,b}, c^*_0)$ since there exists $f = (f_0, f_1, f_2, \ldots)$ such that $f_0 \neq 0$ and $f_{k+1} = \dfrac{(0-a_k)}{b_k} f_k$ and we can easily see that $\sum\limits_k |f_k| = |f_0| + 4|f_0| \sum\limits_{k=1}^{\infty} \left(\dfrac{1}{k+2}\right)^2 < \infty$. This proves that $\sigma_p(\Delta^*_{a,b}, c^*_0) \neq \{\lambda \in \mathbb{C} : |a-\lambda| < |b|\}$. □

Theorem 2.6. If there exists $m \in \mathbb{N}$ such that $a_i \neq a_j$ for all $i, j \geq m$, then:

(i) $\{\lambda \in \mathbb{C} : |a-\lambda| < |b|\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(ii) $\{a_k : |a - a_k| \leq |b|\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(iii) $\left\{\lambda \in \mathbb{C} : \sup\limits_k \left|\dfrac{a_k - \lambda}{b_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(iv) $\sigma_r(\Delta_{a,b}, c_0) \subseteq \left\{\lambda \in \mathbb{C} : \inf\limits_k \left|\dfrac{a_k - \lambda}{b_k}\right| < 1\right\}$,

(v) $\sigma_r(\Delta_{a,b}, c_0) \subseteq (D \cup E) \backslash G$.

Proof. ***(i)*** Let $\lambda \in \mathbb{C}$ with $|a-\lambda| < |b|$. Then, the operator $(\Delta_{a,b} - \lambda I)$ is triangle except may be for $\lambda = a_k$, for some $k \in \mathbb{N}$, and consequently the operator $(\Delta_{a,b} - \lambda I)$ has an inverse. Further, by Theorem 2.3, we see that the operator $(\Delta_{a,b} - \lambda I)$ is one to one for $\lambda = a_k$, for some $k \in \mathbb{N}$, when $|a-\lambda| < |b|$. So, $(\Delta_{a,b} - \lambda I)^{-1}$ exists.

Also, if $\lambda \in \mathbb{C}$ with $|a-\lambda| < |b|$, then $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*_0)$ and so $(\Delta^*_{a,b} - \lambda I)$ is not one to one. Hence, by Lemma 1.2, the range of the operator $(\Delta_{a,b} - \lambda I)$ is not dense in c_0. Thus $\{\lambda \in \mathbb{C} : |a-\lambda| < |b|\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$.

(ii) It is clear that, for all a_k with $|a - a_k| \leq |b|$, we have $a_k \notin \sigma_p(\Delta_{a,b}, c_0)$. Then, the operator $(\Delta_{a,b} - a_k I)^{-1}$ exists. On the other hand $a_k \in \sigma_p(\Delta^*_{a,b}, c^*_0)$, and so $(\Delta^*_{a,b} - \lambda I)$ is not one to one. Then it is easy to see that $(\Delta_{a,b} - \lambda I)$ is not dense in c_0. Thus, $\{a_k : |a - a_k| \leq |b|\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$.

(iii) Let $\lambda \in \mathbb{C}$ with $\sup\limits_k \left|\dfrac{a_k - \lambda}{b_k}\right| < 1$. Then it is easy to see that $|a-\lambda| \leq |b|$. Similarly, as in (i) we can prove that $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$ which shows that

$\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{a_k - \lambda}{b_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$.

(*iv*) For all $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$, we have $(\Delta_{a,b} - \lambda I)^{-1}$ exists and defined on a set which is not dense in c_0. Then, $(\Delta^*_{a,b} - \lambda I)$ is not one to one, and therefore $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*_0)$. This implies that $\lambda \in \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{a_k - \lambda}{b_k}\right| < 1\right\}$. Thus, $\sigma_r(\Delta_{a,b}, c_0) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{a_k - \lambda}{b_k}\right| < 1\right\}$.

(*v*) The proof is similar to that of (*iv*). □

Theorem 2.7. If for every $m \in \mathbb{N}$ there exists $i < m$ and $j \geq m$ such that $a_i = a_j$, then:

(i) $\{\lambda \in \mathbb{C} : |a - \lambda| < |b|\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(ii) $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(iii) $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{a_k - \lambda}{b_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_{a,b}, c_0)$,

(iv) $\sigma_r(\Delta_{a,b}, c_0) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{a_k - \lambda}{b_k}\right| < 1\right\}$,

(v) $\sigma_r(\Delta_{a,b}, c_0) \subseteq (D \cup E) \setminus G$.

Proof. The proof is similar to that of Theorem 2.6. □

Also, is true the following theorem

Theorem 2.8. $\sigma_r(\Delta_{a,b}, c_0) = \sigma_p(\Delta^*_{a,b}, c^*_0) \setminus \sigma_p(\Delta_{a,b}, c_0)$.

Theorem 2.9. If there exists $m \in \mathbb{N}$ such that $a_i \neq a_j$ for all $i, j \geq m$, then:

(i) $\sigma_c(\Delta_{a,b}, c_0) \subseteq \{\lambda \in \mathbb{C} : |a - \lambda| = |b|\}$,

(ii) $\sigma_c(\Delta_{a,b}, c_0) \subseteq \{\lambda \in \mathbb{C} : |a - \lambda| \leq |b|\} \cap \left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - a_k}{b_k}\right| \geq 1\right\}$,

(iii) $G \subseteq \sigma_c(\Delta_{a,b}, c_0)$.

Proof. The proof immediately follows from Theorem 2.2, Theorem 2.3 and Theorem 2.6 because the parts $\sigma_p(\Delta_{a,b}, c_0)$, $\sigma_r(\Delta_{a,b}, c_0)$ and $\sigma_c(\Delta_{a,b}, c_0)$ of the spectrum $\sigma(\Delta_{a,b}, c_0)$ of $\Delta_{a,b} \in B(c_0)$ are disjoint and their union is $\sigma(\Delta_{a,b}, c_0)$.

Theorem 2.10. If for every $m \in \mathbb{N}$ there exists $i < m$ and $j \geq m$ such that $a_i = a_j$, then:

(i) $\sigma_c(\Delta_{a,b}, c_0) \subseteq \{\lambda \in \mathbb{C} : |a - \lambda| = |b|\} \cup E$,

(ii) $\sigma_c(\Delta_{a,b}, c_0) \subseteq (D \cup E) \cap \left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{a_k - \lambda}{b_k}\right| \geq 1\right\}$,

(iii) $G \subseteq \sigma_c(\Delta_{a,b}, c_0)$.

Proof. The proof is similar to that of Theorem 2.9.

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c_0 ARDICILLIQLAR FƏZASINDA $\Delta_{a,b}$ ÜMUMİLƏŞMİŞ FƏRQ OPERATORUNUN SPEKTRİ HAQQINDA

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XÜLASƏ

Məqalənin əsas məqsədi yeni təyin olunmuş $\Delta_{a,b}$ ümumiləşmiş fərq operatorunun c_0 ardıcıllıqlar fəzasında spektrini təyin etməkdən ibarətdir. Məlum Δ fərq operatoru və onun ümumiləşmələri olan $B(r,s)$ və Δ_v operatorları və həmçinin sağ sürüşmə və Zveyer operatorları daxil edilən ümumiləşmiş fərq operatorları sinfinə daxildirlər. İşdə $\Delta_{a,b}$ operatorunun məhdudluğu göstərilmiş və onun norması hesablanmışdır.

О СПЕКТРЕ ОБОБЩЕННОГО РАЗНОСТНОГО ОПЕРАТОРА $\Delta_{a,b}$ ПО ПРОСТРАНСТВУ ПОСЛЕДОВАТЕЛЬНОСТЕЙ c_0

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РЕЗЮМЕ

Основная цель настоящей работы является определение спектра одного нового обобщенного разностного оператора $\Delta_{a,b}$ по пространству последовательностей c_0. Класс введенных обобщенных разностных операторов включает в себя разностный оператор Δ и его обобщения $B(r,s)$ и Δ_v, а также операторы Звейера и правого сдвига. Доказана ограниченность оператора $\Delta_{a,b}$ и найдена его норма в пространстве последовательностей c_0.