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HOLDER WEIGHT ESTIMATES OF SINGULAR INTEGRALS GENERATED BY GENERALIZED SHIFT OPERATOR

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Systematic investigations of multidimentional singular integrals generated by generalized shift operator begin from the appropriate papers where for these integrals, Privalov type theorems were proved. In the given paper these integrals, that depend on the pole are studied in Holder weight spaces $H^{\gamma}_{\alpha\beta}$*. Sufficient conditions for α, β, γ providing their invariance, were found.*

Key words: generalized shift, singular integral, characterization of the core, Holder space, weighted estimates.

1. Singular integral generated by a generalized shift operator.

Let R_m be Euclidean space of dimension m (m≥2),

$R_m^+ = \{(x_1,.....,x_{m-1},x_m) \in R_m : x_m > 0\}, s_m^+ = \{x \in R_m^+ : |x| = 1\}$ T^Y

be a generalized shift operator (briefly GS0) ([5]), which acts according to the law

$$T^s u(x) = C_\nu \int_0^\pi u(x^{'} - s^{'}; \sqrt{x_m^2 - 2x_m s_m \cos\alpha + s_m^2}) \sin^{2\nu-1}\alpha d\alpha \qquad (1)$$

where

$$\nu > 0 \quad x = (x^{'}, x_m), s = (s^{'}, s_m), x^{'}, s^{'} \in R_{m-1}, c_\nu = \Gamma(\nu + \frac{1}{2}) / \Gamma(\frac{1}{2})\Gamma(\nu)$$

It is known that this shift is closely connected with the Bessel differential operator $B_x^m = \frac{\partial^2}{\partial x_m^2} + \frac{2\nu}{x_m}\frac{\partial}{\partial x_m}$

The singular integral

$$Au(x) = V.p.\int_{R_m^+} \frac{f(x,\theta)}{|s|^{m+2\nu}} [T^s u(x)] s_m^{2\nu} ds = \lim_{\varepsilon \to +0} A_\varepsilon u(x) \qquad (2)$$

where

$$A_\varepsilon u(x) = \int\limits_{\{s\in R_m^+ \cdot |s|>\varepsilon} \frac{f(x,\theta)}{|s|^{m+2\nu}}\left[T^s u(x)\right] s_m^{2\nu} ds,\ \ \theta = s/|s|,\ \ \varepsilon>0,\ \ x\in R_m^+$$

is called a singular integral depends on a pole (briefly SI) generated by GSO T^S ([1]).

Let a and b be arbitrary numbers such that 0 <a<b ≤ +∞. Then for any point $x\in R_m^+$ the following equality holds

$$\begin{aligned} &\int\limits_{\{s\in R_m^+ : a<|s|<b\}} f(x, s/|s|)|s|^{-m-2\nu}\left[T^S u(x)\right] s_m^{2\nu} ds = \\ &= \frac{1}{2} c_\nu \int\limits_{\{y\in R_{m+1} : a< r_{\tilde{x}y}^{-m-2\nu}<b\}} f(\tilde{x},\tilde{\theta}) r_{\tilde{x}y}^{-m-2\nu} u(y'; \sqrt{y_m^2+y_{m+1}^2})|y_{m+1}|^{2\nu-1} dy \end{aligned} \tag{3}$$

where

$$\tilde{x} = (x', x_m, 0), y = (y', y_m, y_{m+1}), dy = dy_1, ..., dy_{m+1})$$

$$\tilde{\theta} = (\frac{x'-y'}{r_{\tilde{x}y}}; \frac{\sqrt{(x_m-y_m)^2+y_{m+1}^2}}{r_{\tilde{x}y}}), r_{\tilde{x}y} = |\tilde{x}-y|$$

If we assume in (3) u (x) ≡ 1 (then $T^s u(x) \equiv 1$ ≡ 1) and pass to the polar coordinates, we obtain

$$\begin{aligned} &\int\limits_{S_m^+} f(x,\theta)\theta_m^{2\nu} dS(\theta) = \frac{1}{2} C_\nu \int\limits_{S_{m+1}} f(x,\tilde{\theta})|\theta_{m+1}|^{2\nu-1} dS(\tilde{\theta}), \\ &S_{m+1} = \{y\in R_{m+1} : |y|=1\} \end{aligned} \tag{4}$$

Later on "C" is a constant; its exact value is not essential for us; $a(x) \prec b(x)$ means that $a(x) \le cb(x)$, where c doesn't depend on x.

In the case when $a(x) \prec b(x)$ and $b(x) \prec a(x)$ we will write $a(x) \sim b(x)$

2. Holder space with weight $H_{\alpha,\beta}^{\gamma}(R_m^+)$

Let γ> 0, α> 0, β be a real number, $\rho(x) = x_m^\alpha(1+|x|)^{\beta-\alpha}, x\in R_m^+$ By definition ([3]) $u\in H_{\alpha,\beta}^{\gamma}(R_m^+)$ if $\lim\limits_{x\to\infty} u(x)\rho(x) = \lim\limits_{x\to 0} u(x)\rho(x) = 0$ and the norm

$$\|u\|_{H_{\alpha\beta}^{\lambda\gamma}} = \sup_{x,y\in R_m^+} (|u(x)\rho(x) - u(y)\rho(y)| d^{-\gamma}(x,y))$$

is finite, where

$$d(x,y) = |x-y|(1+|x|)(1+|y|)^{-1}$$

If the contrary is not stipulated, then later on we will assume that

$$0<\gamma<1,\ 0<\alpha-\gamma<1, 0<\beta+\gamma<m. \tag{5}$$

Let $x\in R_m^+$ denote

$$\omega_x = \{s\in R_m^+ : |s-x| < \frac{x_m}{2}\},\ \tilde{\omega}_x = \{y\in R_{m+1} : |\tilde{x}-y| < \frac{x_m}{2}\},$$

$$\psi_\gamma(x) = x_m^{\gamma-\alpha}(1+|x|)^{-1} \equiv \rho^{-1}(x)(x_m(1+|x|)^{-2})^\gamma$$

The spaces $H_{\alpha\beta}^{\gamma}$ can be determined in terms of inequalities. The following lemma is valid.

Lemma 2. ([4]). Let 0 <γ<α, β + γ> 0 $u \in H_{\alpha\beta}^{\gamma}$ if

$a)\exists C_1(u), \forall x \in R_m^+, |u(x)| \leq C_1(u)\Psi_{\gamma(x)}$,

$b)\exists C_2(u), \forall x \in R_m^+, \forall y \in \omega_x^{'}$,

$|u(x)-u(y)| \leq C_2(u)\rho^{-1}(x)d^{\gamma}(x,y)$.

Moreover,

$$(\min C_1(u) + \min C_2(u)) \sim \|u\|.$$

We cite the important corollary to this lemma.

Corollary 1. If $u \in H_{\alpha\beta}^{\gamma}$ then

$a)\ |u(y', \sqrt{y_m^2 + y_{m+1}^2})| \prec \|u\|(|y_m| + |y_{m+1}|)^{\gamma-\alpha}(1+|y|)^{-l} \sim \|u\|(|y_m| + |y_{m+1}|)^{\gamma-\alpha}(1+|y'|+|y_m|+|y_{m+1}|)^{-l}$;

$b)\ \forall x \in R_m^+, \forall y \in \omega_x^{'}\ |u(y', \sqrt{y_m^2 + y_{m+1}^2}) - u(x)| \prec c\|u\|\rho^{-1}(x)d^{\gamma}(\tilde{x}, y) \sim \|u\|\rho^{-1}(x)(|\tilde{x} - y|/(1+|x|^2)^{\gamma}$

Suppose $\mu = (\gamma+\beta) + (1+\gamma-\alpha)$. By virtue of (5) $\mu > 0$. Taking into account that $m + 2\nu + l = (-1) + \mu + 2\nu$, we obtain from the latter:

$$i_2(x; A_x) \leq \int_0^{\infty} y_{m+1}^{2\nu-1} dy_{m+1} \int_0^{\infty} (y_m + y_{m+1})^{\gamma-\alpha} dy_m \int_{R_{m-1}} (|z| + x_m + y_m + y_{m+1})^{m-1+\mu+2\nu} dz < \psi_{\gamma}(x), (|x| \geq 1$$

Let $|x| < 1$ and $y \in B_x$. Then for $|y| \geq 1$ $|y| \sim |y| + 1 \sim |y| + 1 + |x|$ and also for $|y| < 1$ $(1+|y|) \sim 1$ and $|x - y| \sim |y| + x_m \sim |y^{'}| + |y_m| + |y_{m+1}| + x_m$.

Therefore

$$i_2(x; B_x) \leq c \int_{\{y \in R_{m+1} : |y| < 1\}} \frac{(|y_m| + |y_{m+1}|)^{\gamma-\alpha} |y_{m+1}|^{2\nu-1}}{(|y^{'}| + |y_m| + |y_{m+1}| + |x_m|)^{m+2\nu+l}} dy +$$
$$+ \int_{\{y \in R_{m+1} : |y| \geq 1\}} \frac{(|y_m| + |y_{m+1}|)^{\gamma-\alpha} |y_{m+1}|^{2\nu-1}}{(|y| + 1 + |x|)^{m+2\nu+l}} dy \prec (x_m^{\gamma-\alpha} + \frac{1}{(1+|x|)^{\beta+\gamma}}) < \psi_{\gamma}(x) \qquad (6)$$

The validity of estimate (10) for $i_2(x; C_x)$ is proved by analogous reasonings.

Thus, we proved that $i_2(x; R_m \setminus \omega_x') < \psi_{\gamma}(x)$ and $|i_2(x)| < \|u\| \|f\| \Psi_{\gamma}(x)$ (7)

So, the absolute convergence of integrals $i_1(x), i_2(x)$ is proved.

Theorem 1. Let $u \in H_{\alpha\beta}^{\gamma}$ u and (5) be full filled. If $f(x,\theta),\ x \in R_m^+, \theta \in S_m^+$ is bounded and

$$\int_{S_m^+} f(x,\theta)\theta_m^{2\nu} ds(\theta) = 0 \qquad (*)$$

then at each point $x \in R_m^+$ there exists SI $Au(x)$ generated by GSI T^y, and the following equality holds

$$Au(x) = v.p.\int_{R_m^+} f(x,\theta)\,|S|^{-m-2\nu}\,[T^Y u(x)]S_m^{2\nu}ds =$$

$$= \frac{1}{2}C_\upsilon \int_{\omega_x'} f(x,\widetilde{\theta})\,|\widetilde{x}-y|^{-m-2\nu}\,(u(y';\sqrt{y_{m+1}^2+y_m^2})-u(x))\,|y_{m+1}|^{2\nu-1}\,dy+$$

$$+\frac{1}{2}C_\upsilon \int_{R_{m+1}\setminus\omega_x'} f(x,\widetilde{\theta})\,|\widetilde{x}-y|^{-m-2\nu}\,u(y';\sqrt{y_{m+1}^2+y_m^2})\,|y_{m+1}|^{2\nu-1}\,dy.$$

Proof. From (4) by virtue of (*) we obtain

$$\int_{S_{m+1}} f(x,\widetilde{\theta})\,|\theta_{m+1}|^{2\upsilon-1}\,dS(\theta)=0 \qquad (**)$$

Let $u\in H^{\gamma}_{\alpha,\beta}$, $x=(x',x_m)\in R_m^+, 0<\varepsilon<\frac{x_m}{2}$

Then from (3) we obtain

$$Au(x)=\frac{1}{2}C_\nu\Big(\int_{S_{m+1}} f(x,\widetilde{\theta})\,|\theta_{m+1}|^{2\nu-1}\,dS(\theta)\Big)[u(x)]\ln\frac{x_m}{2}+\frac{1}{2}C_\nu i_1(x;\omega_x'(\varepsilon))+i_2(x),$$

where

$$\omega_x'=\{y\in R_{m+1}:\varepsilon<|\widetilde{x}-y|<\frac{x_m}{2}\}$$

Now taking into account (**) and the absolute convergence of integrals $i_1(x;\omega_x')$ and $i_2(x)$, passing to the limit as ε → +0, we prove the theorem

3. Boundedness in $H^{\gamma}_{\alpha\beta}$.

$$C_\delta : f(x,\theta)\in C_\delta;\sup|f(x,\theta)-f(y,\theta)|<K_\delta(f)\cdot\varphi_\delta(x,y),\quad 0\le\delta\le 1$$

where

$$\varphi_\delta(x,y)=|\big(x-y|/\max(|x-y|,1+\min\{|x|;|y|\}\big)^{\delta},\ \forall\widetilde{\delta}\in(0;\delta]$$

$a)\ x\in A_x\cup B_x\cup\omega_x,\ \varphi_\delta(x,y)\le|x-y|^{\widetilde{\delta}}\,(1+|y|)^{-\widetilde{\delta}}$

$b)\ x\in C_x,\ \varphi_\delta(x,y)\le|x-y|^{\widetilde{\delta}}\,(1+|x|)^{-\widetilde{\delta}}$.

Theorem 2. Let f satisfy condition (*) and

$$|f(x,\theta_1)-f(x,\theta_2)|\le c_f\,|\theta_1-\theta_2|^{\mu},\ f(x,\theta)\in C_\delta(R_m^+)$$

If $0<\gamma<\delta\le 1, 0<\alpha-\gamma<1, \beta+\gamma<m,\ 0<\gamma\le\mu,\ 0<\mu\le 1$.

where C_f is a constant, $x\in R_n^+,\ \theta_1,\theta_2\in S_m^+$

then SI operator generated by GSO T^Y :

$$A:u\to Au(x)\equiv v.p.\int_{R_m^+} f(x,\theta)\,|s|^{-m-2\nu}\,[T^S u(x)]S_m^{2\nu}ds$$

is bounded in $H^{\gamma}_{\alpha\beta}$

Proof. By virtue of theorem 1 from (7) we obtain

$$Au(x)\le\frac{1}{2}C_\nu C(|i_1(x;\omega_x')|+|i_2(x)|)<\|f\|\|u\|\Psi_\gamma(x), \qquad (8)$$

By virtue of lemma 2, to prove the theorem it suffices to show that $\forall x \in R_m^+$ and $|h| \leq x_m / 8$

$$|Au(x) - Au(x+h)| \leq c\|u\|\rho^{-1}(x)(1+|x|)^{-2\nu}|h|^{\nu} \tag{9}$$

where c is independent of x and h.

Suppose

$$\omega_1(x) = \omega(\tilde{x}, 2|h|), \quad \omega_2(x) = \omega(\tilde{x}+h, 3|h|), \quad \omega_3(x) = \omega(\tilde{x}+h, \frac{x_m}{2} - |h|),$$

obviously $\omega_1(x) \subset \omega_2(x) \subset \omega_3(x) \subset \omega_x^{'}$

Subject to (*) and (4) one can prove that

$$Au(x) - Au(x+h) = \sum_{i=1}^{5} J_i(x;h) \tag{10}$$

where

$$J_1(x;h) = \left(\int\limits_{\omega_2} + \int\limits_{\omega_x^{'} \setminus \omega_3} \right) K(y,x)(u_1(y) - u(x)) |y_{m+1}|^{2\nu-1}\, dy$$

$$J_2(x;h) = -\int\limits_{\omega_2} K(y,x+h)(u_1(y) - u(x)) |y_{m+1}|^{2\nu-1}\, dy$$

$$J_3(x;h) = -\int\limits_{\omega_x^{'} \setminus \omega_3} K(y,x+h) u_1(y) |y_{m+1}|^{2\nu-1}\, dy$$

$$J_4(x;h) = \int\limits_{\omega_x^{'} \setminus \omega_3} (K(y,x) - K(y,x+h)(u_1(y) - u(x)) |y_{m+1}|^{2\nu-1}\, dy$$

$$J_5(x;h) = \int\limits_{R_{m+1} \setminus \omega_x^{'}} (K(y,x) - K(y,x+h) u_1(y) |y_{m+1}|^{2\nu-1}\, dy$$

where

$$K(y,\tilde{x}) = f(x,\tilde{\theta}) / r_{y\tilde{x}}^{m+2\upsilon}, \quad r_{y\tilde{x}} = |\tilde{x} - y|, \; u_1(y) = u(y^{'}, \sqrt{y_m^2 + y_{m+1}^2})$$

Using easy calculations, one can prove that for $x \in R_m^+$, $y \in R_{m+1} \setminus \omega_2$ and $|h| \leq x_m/8$

$$r_{y\tilde{x}} \sim r_{y\tilde{x}+h} \tag{11}$$

$$|K(y,x) - K(y,x+h)| \prec C_f(\varphi_\delta(\tilde{x}, \tilde{x}+h) + \frac{|h|^\mu}{|\tilde{x} - y|^\mu}) r_{y\tilde{x}}^{-(m+2\nu)}, \tag{12}$$

Not let us majorize $J_i(x;h|, \quad i = \overline{1,5}$

Taking into account b) of corollary 1 and also (11), we obtain

$$J_1(x;h) = c\left(\int\limits_{\omega_2} + \int\limits_{\omega_x^{'} \setminus \omega_3} \right) \frac{|f(x,\tilde{\theta})|}{|y - \tilde{x}|^{m+2\upsilon}} \rho^{-1}(x) \left(\frac{|y - \tilde{x}|}{1+|x|^2} \right)^{\gamma} |y_{m+1}|^{2\nu-1}\, dy \prec$$

$$\prec c_f \rho^{-1}(x)(1+|x|)^{-2\gamma} \left(\int\limits_{\omega_2} + \int\limits_{\omega_x^{'} \setminus \omega_3} \right) \frac{|y_{m+1}|^{2\nu-1}}{|(\tilde{x}+h) - y|^{m+2\nu-\gamma}} \prec c_f \rho^{-1}(x)(1+|x|)^{-2\gamma}|h|^{\gamma}$$

The following expression is proved analogously

$$J_2(x;h) < c_f \rho^{-1}(x)(1+|x|)^{-2\gamma}\,|h|^{\gamma}$$

$$J_3(x;h) = cc_f\|u\| \int\limits_{\omega_x^{'}\setminus\omega_3} \frac{(|y_m|+|y_{m+!})^{\gamma-\alpha}|}{r^{m+2\upsilon}_{y\,\widetilde{x+h}}(1+|y|)^{l}}\,|y_{m+1}|^{2\upsilon-1}\,dy \prec \|f\|\|u\|\Psi_{\gamma}(x) \int\limits_{\omega_x^{'}\setminus\omega_3} \frac{|y_{m+1}|^{2\nu-1}}{r^{m+2\nu}_{\widetilde{x+h}}}\,dy$$

Taking into account that $A=\{y\in R_{m+1} :|\tilde{x}+h-y|<\frac{x_m}{2}+\frac{|h|}{2}\}\supset\omega_x^{'}$ and passing to the polar coordinates, we obtain

$$\int\limits_{\omega_x^{'}\setminus\omega_3} \frac{|y_{m+!}^{2\nu-1}|}{r^{m+2\nu}_{y\,\widetilde{x+h}}}dy \le \int\limits_{A\setminus\omega_3} \frac{|y_{m+1}|^{2\nu-1}}{r^{m+2\nu}_{\widetilde{x+h}}}\,dy \le c\frac{h}{x_m}$$

Hence

$$|J_3(x;h)| < \|f\|\|u\|\Psi_{\gamma}(x)\,|h|\,x_m^{-1} < \|f\|\|u\|\rho^{-1}(x)(1+|x|)^{-2\gamma}\,|h|^{\gamma}$$

Let us majorize $J_4(x;h)$. Taking into account (9) and (11), we obtain

$$|J_4(x;h)| \prec c_f\|u\|\rho^{-1}(x)(1+|x|)^{-2\gamma} \int\limits_{\omega_3\setminus\omega_2} \left(|h|^{\delta}\,(1+|x|)^{-\delta}+|h|^{\mu}|\tilde{x}-y|^{-\mu}\right)|\tilde{x}-y|^{\gamma-m-2\nu}|y_{m+1}|^{2\nu-1}\,dy \prec$$

$$\prec c_f\|u\|\rho^{-1}(x)(1+|x|)^{-2\gamma}\,|h|^{\gamma}$$

Let us majorize $|J_5(x;h)|$. Subject to (12) and (11) we obtain

$$|J_5(x;h)| \prec c_f\|u\|(|h|^{\delta}\,(1+|x|)^{-\delta}+|h|^{\mu}\,x_m^{-\mu})i_2(x;R_{m+1}\setminus\omega_x^{'}) \prec$$
$$\prec (c_f+\|f\|)(|h|/x_m)^{\gamma}\psi_{\gamma}(x) \prec \left(c_f+\|f\|\right)\|u\|\rho^{-1}(x)(1+|x|^{-2\gamma})\,|h|^{\gamma}$$

Thus, theorem 2 is completely proved.

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ÜMUMİLƏŞMİŞ SÜRÜŞMƏ OPERATORUNUN DOĞURDUĞU SİNQULYAR İNTEQRALLAR ÜÇÜN ÇƏKİLİ HÖLDER QİYMƏTLƏNDİRMƏLƏRİ

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XÜLASƏ

İşdə $H_{\alpha\beta}^{\gamma}$ çəkili Hölder fəzalarında ümumiləşmiş sürüşmə ilə polyusdan asılı xarakteristikalı sinqulyar inteqralın məhdudluğu üçün α, β, γ parametrləri və xarakteristikanın dəyişənlərə görə hamarlıq tərtibləri üzərinə kafi qədər geniş şərtlər tapılmışdır.

Açar sözlər: ümumiləşmiş sürüşmə, sinqulyar inteqral, nüvənin xarakteristikası, Hölder fəzası, çəkili qiymətləndirmələr.

ВЕСОВЫЕ ГЕЛЬДЕРОВЫЕ ОЦЕНКИ ДЛЯ СИНГУЛЯРНЫХ ИНТЕГРАЛОВ, ПОРОЖДЕННЫХ ОПЕРАТОРОМ ОБОБЩЕННОГО СДВИГА

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РЕЗЮМЕ

В работе найдены достаточные условия на α, β, γ и на степени гладкости по переменным характеристики, обеспечивающие ограниченность в пространствах Гельдера $H_{\alpha\beta}^{\gamma}$ с весом сингулярных интегралов с обобщенным сдвигом характеристикой, зависящей от полюса.

Ключевые слова: обобщенный сдвиг, сингулярный интеграл, характеристика ядра, пространство Гельдера, весовые оценки.

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