AMS: 517.95

REDUCTION OF THE INVERSE PROBLEM WITH RESPECT TO DOMAIN TO VARIATIONAL STATEMENT AND ITS INVESTIGATION

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In the present paper, we consider the inverse problem relatively domain and suggest a new approach for reducing the inverse problem for a domain to an equivalent problem in a variational setting and give an effective solution algorithm for solving such problems.

Keywords: inverse problem, support function, elliptic equation, variational problem, numerical methods, basis functions.

1.Introduction

A wide class of practical problems are reduced to the inverse problem with respect to domain. As an example we can show problems of elasticity theory, diffusion problems, the problems arising in hydrodynamics. Papers concerning inverse problems usually deal with inverse problems for an unknown function (coefficients and functions occurring in the boundary and initial conditions). But in our case, a domain is sought and the investigation of considered problems is related with some strong difficulties. In order to avoid these difficulties and for investigation of such problems for the first time the considered inverse problem is reduced to variational statement. As the obtained variational problem is a domain dependent variantal problem, the investigation of such problems encountered some difficulties. Here, we give an effective solution algorithm for solving such problems.

2. Statement of the problem

Let D be r-dimensional domain, i.e. $D \subset R^r$ and $x = (x_1, x_2 ..., x_r) \in D$. Denote by S_D the boundary of the domain D, $S_D = \partial D$. Assume that the boundary S_D is in the space C^2.

Let's consider the following inverse problem

$$-\Delta u + a(x)u = f(x), \; x \in D, \tag{1}$$

$$u(x) = 0 \quad , x \in S_D. \tag{2}$$

$$\frac{\partial u(x)}{\partial n} = 0 \; , \; x \in S_D \tag{3}$$

where the functions a and f are continuously differentiable functions in R^r and $a(x) > 0$ for all($x \in D$). Denote by K the set of convex domains set with a boundary from C^2.

Our goal is to find a pair $(D,u) \in K \times C^2(D)$ such that the function $u = u(x)$ satisfies equation (1) and boundary conditions (2), (3) in the domain D. As it is seen, condition (2) is a Dirichlet condition and condition (3) is a Neumann condition. At first we consider the following unkown domain variational problem for solving inverse problem (1)-(3),

$$J(D,u) = \int_D F(x,u(x),u_x(x))dx \to \min, \; D \in K, \; u \in C^2(D), \tag{4}$$

$$u(x) = 0 \; , \; x \in S_D. \tag{5}$$

We assume that $F(x,u,p)$ is a continuously differentiable function on own variables in $D \times R \times R^r$.

If boundary condition is satisfied for $D \in K, \quad u \in C^2(D)$,the pair (D, u) is said to be a possible pair. Denote by M all possible pairs set. The pair $(D^*, u^*) \in M$ is called an optimal pair if it gives a minimum to functional (4) in the set M.

Give the following theorem which is obtained in [7].

Theorem 1. Let the pair $(D^*, u^*) \in M$ be an optimal pair for variational problem (4), (5). Then the function $u^* = u^*(x)$ is a solution of the following Euler equation in the domain D^*

$$F_u(x,u(x),u_x(x)) - \sum_{i=1}^{r} \frac{d}{dx_i} F_{u_{x_i}}(x,u(x),u_x(x)) = 0, \quad x \in D^* \tag{6}$$

and moreover, in the boundary S_{D^*}, the condition

$$F(x,u^*(x),u_x^*(x))-\sum_{i=1}^{r}u_{x_i}^*(x)F_{u_{x_i}}(x,u^*(x),u_x^*(x))=0, \quad x\in S_{D^*} \tag{7}$$

is satisfied.

As it is seen, this theorem is proved for convex domains. But one can obtain a similar result for doubly connected domain D with internal and external boundaries S_1 and S_2. For that, we must use the expansion

$$\int_D F dx = \int_{D_2} F dx - \int_{D_1} F dx,$$

where D_1 and D_2 are convex domains bounded by the boundaries S_1 and S_2.

Now, take the function $F(x,u,p)$ in the following form

$$F(x,u,u_x)=|u_x|^2+a(x)u^2-2f(x)u, \tag{8}$$

where

$$|u_x|^2=|u_{x_1}|^2+|u_{x_2}|^2+....+|u_{x_r}|^2.$$

As the functions a and f are continuously differentiable functions in R^r, the function $F(x,u,p)$ is also a continuously differentiable function on $D\times R\times R^r$. Apply the theorem mentioned above to this function. It is clear that

$$F_u=2a(x)u-2f(x),\ F_{u_{x_i}}=2u_{x_i}.$$

Hence

$$\sum_{i=1}^{r}\frac{d}{dx_i}F_{u_{x_i}}=2\Delta u.$$

So, from condition (6), we see that the function $u^*=u^*(x)$ satisfies equation (1) in the domain D^*. From condition (7) we get the following boundary condition:

$$|u_x^*|^2+au^{*2}-2f(x)u^*-2|u_x^*|^2=0,\ x\in S_D.$$

If we take into account the condition $u^*(x)=0$, $x\in S_{D^*}$, we get

$$|u_x^*|=0,\ x\in S_{D^*},$$

or

$$u_x^*(x)=0,\ x\in S_{D^*}.$$

From the equation of the derivative with respect to the normal $\frac{\partial u(x)}{\partial n}$, we get that function (3) satisfies the boundary condition as well. Thus, we proved the following theorem.

Theorem 2. Let the pair $(D^*, u^*) \in M$ be an optimal pair for problem (4), (5). Then it is a solution of problem (1)-(3) as well.

This theorem shows that if instead of the inverse problem (1)-(3) we take the function $F(x,u,p)$ in the form (8), we can investigate the variational problem (4), (5). Notice that the inverse of this fact is not true in general. Though the functional $J(D,u)$ is convex with respect to the functional u, this functional is not convex with respect to D in general.

3. Algorithm for numerical solution

Using obtained results, for numerical solution of the problem (4)-(5) the following methods is proposed.

Let the system of the functions $\{\varphi_k(x)\}, k = 1,2,...$ form a basis in the space $C^2(D)$. Then function $u = u(x)$ may be expanded by this basis

$$u = \sum_{i=1}^{\infty} \alpha_k \varphi_k(x).$$

We take this into account in problem (4), (5) and get:

$$I(D,\alpha) = \int_D \Phi(x,\alpha)dx \to \min, \; D \in K, \; \alpha = (\alpha_{1,}\alpha_2, \alpha_3, ...), \tag{9}$$

$$\sum_{i=1}^{\infty} \alpha_k \varphi_k(x) = 0, \; x \in S_D, \tag{10}$$

where

$$\Phi(x,\alpha) = F(x, \sum_{i=1}^{\infty} \alpha_k \varphi_k(x), \sum_{i=1}^{\infty} \alpha_k \frac{\partial \varphi_k(x)}{\partial x}).$$

In our case, as $F(x,u,p)$ is in the form (8)

$$\Phi(x,\alpha) = \left| \sum_{k=1}^{\infty} \alpha_k \frac{\partial \varphi_k}{\partial x} \right|^2 + a(x) \left| \sum_{i=1}^{\infty} \alpha_k \varphi_k(x) \right|^2 - 2f(x) \sum_{i=1}^{\infty} \alpha_k \varphi_k(x). \tag{11}$$

For solving problem (9)-(10), calculate the first variation of functional (9). It is clear that

$$\frac{\partial I}{\partial \alpha} = \int_D \frac{\partial \Phi(x,\alpha)}{\partial \alpha} \delta\alpha dx.$$

Calculate the first variation of the functional $I(D,\alpha)$ with respect to the domain D. In [5,7] the functional of the form

$$G(D)=\int_D g(x)dx$$

is considered and for its first variation the formula

$$\delta G(D)=\int_{S_D} g(x)\delta P_D(n(x))ds\,,$$

is obtained. Here, the function $g(x)$ is a continuously differentiable function in R^r, $n(x)$ is external normal to the surface S_D at the point x and $P_D(x)$ is support function of the domain D and is determined as follows:

$$P_D(x)=\sup_{l\in D}(l,x),\quad x\in R^r\,.$$

We take into account this formula and get

$$\delta I(D,\alpha)=\int_{S_D}\Phi(x,\alpha)\delta P_D(n(x))ds+\int_D\frac{\partial\Phi(x,\alpha)}{\partial\alpha}\delta\alpha dx\,. \quad (12)$$

It is seen from this formula that the numbers $\alpha_1,\alpha_2,\alpha_3,\dots$ are found from the system of equations

$$\int_D\frac{\partial\Phi(x,\alpha)}{\partial\alpha}dx=0 \quad (13)$$

In our case, as $\Phi(x,\alpha)$ is in the form (11), the system of equations (13) will be a system of linear equations.

The set K may satisfy some additional restrictions as well. For example, the volume K may be the mentioned domains set with the given area of surface. In another case, the domains set K may be given as $D_0\subset D\subset D_1$ as well, where D_0,D_1 and R^r are the given domains. For practical problems, the set K may be given in the form of integral restrictions

$$\int_D g(x)dx=c$$

or

$$\int_D g(x)dx\le c\,.$$

In general, assume that there is a domain $G\subset R^r$ such that for arbitrary $D\in K$ contained in the set K $D\subset G$. We can state this condition in a simpler from as follows: we know that the optimal domain is contained in a certain domain G.

The obtained relations (12), (13) enable to solve problem (9), (10) approximately. For that we give the following algorithm.

Step 1. Take arbitrary domain $D^{(0)} \in K$ and the basis functions

$$\{\varphi_k^{(0)}(x)\}, k = 1,2,\dots .$$

Step 2. Solving the system of equations

$$\int_{D^{(0)}} \frac{\partial \Phi(x,\alpha)}{\partial \alpha} dx = 0 \ ,$$

we find the convergence $\alpha^{(0)} = (\alpha_1^0, \alpha_2^0, \alpha_3^0, \dots)$.

Step 3. Minimizing the linear functional

$$\int_{S_D{}^{(0)}} \Phi(x,\alpha^{(0)}) P_D(x) ds \to \min, \ \ D \in K \tag{14}$$

we find the convex function $P(x)$.

Step 4. The intermediate domain $\overline{D}^{(0)}$ is found as a subdifferential of the function $P(x)$ at the point $x = 0$ [8]. In other words,

$$\overline{D}^{(0)} = \partial P(0) = \{l \in R^r ; P(x) \geq (l,x), \ \forall x \in R^r\} \ .$$

Step 5. A new domain $D^{(1)}$ is found as follows:

$$D^{(1)} = (1-\mu)\overline{D}^{(0)} + \mu D^{(0)}, \quad 0 < \mu < 1.$$

Here, the domain μ may be choosen in different ways [7, 9].

If a new found domain $D^{(1)}$ satisfies definite exactness conditions, the iteration process is completed. On the contrary, for a new domain $D^{(1)}$ the iteration begins from the first step. The exactness condition may be given in different ways. For example,

$$\left| I(D^{(k+1)}, \alpha^{(k+1)}) - I(D^{(k)}, \alpha^{(k)}) \right| < \varepsilon \ .$$

Here, $\varepsilon > 0$ is said to be accuracy order of the method. Now, give some rules for choosing the quantity μ_k.

1) In general, the numbers μ_k may be chosen from the following condition

$$f_k(\mu) = I((1-\mu_k)D^{(k)} + \mu_k \overline{D}^{(k)}, \alpha_k) \to \min, \quad \mu \in [0,1].$$

2) The quantity μ_k may be given as a sequence satisfying the following conditions:

$$0 \leq \mu_k \leq 1, \quad \lim_{k \to \infty} \mu_k = 0, \quad \sum_{k=0}^{\infty} \mu_k = \infty \ .$$

For example,

$$\mu_k = \frac{1}{k+1}, \ k = 1,2,3\dots$$

3) The another method is to take $\mu_k = 1$ and verity that the value of the functional decreases. If the value of the functional doesn't decrease, then the value of μ_k decreases twice.

As we have noted above, we can get the similar result also for a doubly-connected domain with integral and external boundaries S_1, S_2. In this case we'll consider the following problem:

$$-\Delta u + au = f(x), \ x \in D, \tag{15}$$

$$u(x) = \varphi(x), \ x \in S_1, \tag{16}$$

$$\frac{\partial u(x)}{\partial n} = 0, \quad x \in S_2, \tag{17}$$

$$u(x) = 0, \quad x \in S_2. \tag{18}$$

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OBLASTA GÖRƏ TƏRS MƏSƏLƏNİN VARİASİONAL QOYULUŞA GƏTİRİLMƏSİ VƏ ONUN TƏDQİQİ

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XÜLASƏ

İşdə oblasta görə tərs məsələyə baxılır. Yeni yanaşma tətbiq edilərək, baxılan məsələ ona ekvivalent olan variyasiya məsələsinə gətirilir. Daha sonra, alınan variyasiya məsələsini həll etmək üçün həll alqoritmi təklif olunur.

Açar sözlər: tərs məsələ, dayaq funksiyası, elliptik tənlik, variasiya məsələsi, ədədi üsul, bazis funksiyaları.

ПРИВЕДЕНИЕ ОБРАТНОЙ ЗАДАЧИ К ВАРИАЦИОННОЙ ПОСТАНОВКЕ И ЕГО ИССЛЕДОВАНИЕ

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РЕЗЮМЕ

В работе рассматривается обратная задача относительно области. Применяя новый подход, эта задача приводится к вариационной постановке. Далее, предлагается алгоритм для решения вариационной задачи.

Ключевые слова: обратная задача, опорная функция, эллиптическое уравнение, вариационная задача, численный метод, базисные функции.

Redaksiyaya daxil oldu: 10.03.2011-ci il.
Çapa imzalandı: 19.12.2011-ci il.