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COMPLETENESS OF ELEMENTARY SOLUTIONS FOR A CLASS OF FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS

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In the paper, the completeness of the system of elementary solutions of a class of fourth order homogeneous operator-differential equations in the space of all the solutions from $W_2^4(R_+;H)$ is proved. Notice that the found completeness conditions are expressed by the operator coefficients of the equation under consideration.

Key words: *operator-differential equation, elementary solution, self-adjoint operator, Hilbert space.*

In the Hilbert space H consider the polynomial operator pencil

$$P(\lambda)=(\lambda E-A)^3(\lambda E+A)+\sum_{j=1}^{3}\lambda^{4-j}A_j\,, \qquad (1)$$

where E is a unit operator, A, A_j, $j=1,2,3$ are linear operators, moreover A is a self-adjoint positive-definite operator with domain of definition $D(A)$, the operators A_jA^{-j}, $j=1,2,3$ are bounded in H.

Denote by H_θ the scale of Hilbert spaces generated by the operator A, i.e. $H_\theta=D(A^\theta)$, $\theta\ge 0$, $(x,y)_\theta=(A^\theta x, A^\theta y)$, $x,y\in D(A^\theta)$. For $\theta=0$ we assume that $H_0=H$, $(x,y)_0=(x,y)$, $x,y\in H$.

Following the book [1, chp.1], define the space of the vector-functions

$$W_2^4(R_+;H)=\left\{u(t):u(t)\in L_2(R_+;H_4),\frac{d^4u(t)}{dt^4}\in L_2(R_+;H)\right\}$$

with the norm

$$\|u\|_{W_2^4(R_+;H)} = \left(\|u\|^2_{L_2(R_+;H_4)} + \left\| \frac{d^4u}{dt^4} \right\|^2_{L_2(R_+;H)} \right)^{1/2},$$

where $L_2(R_+;H)$ denotes the Hilbert space of all the vector functions $f(t)$ determined in $R_+ = [0,+\infty)$ with values from H that have the finite norm

$$\| f \|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \| f(t) \|^2 dt \right)^{1/2}.$$

Here and in sequel, the derivatives are understood in the sense of distributions theory [1].

Under $\sigma_\infty(H)$ we'll understand the set of completely continuous operators acting in H. It is known that if $B \in \sigma_\infty(H)$, then $(B^*B)^{1/2}$ is a completely continuous self-adjoint operator in H. The eigen values of the operator $(B^*B)^{1/2}$ will be called the s-*numbers* of the operator B:

$$s_k(B) = \mu_k((B^*B)^{1/2}),\ k = 1,2,...\ .$$

We'll numerate the non-zero s-numbers of the operator B in descending order with regard to their multiplicity. Denote

$$\sigma_p = \left\{ B : B \in \sigma_\infty(H);\ \sum_{k=1}^{\infty} s_k^p(B) < \infty \right\}, \quad 0 < p < \infty .$$

It is well known that theories appear while considering concrete model problems. Many problems of mechanics urge on the matter of the completeness of elementary solutions of boundary value problems for operator-differential equations. Note that we can meet the equation considered below while solving the stability problems of plates made of plastic material [2].

In the given paper we prove a theorem on the completeness of descending elementary solutions of the equation

$$P(d/dt)u(t) \equiv \left(\frac{d}{dt} - A \right)^3 \left(\frac{d}{dt} + A \right) u(t) + \sum_{j=1}^{3} A_j \frac{d^{4-j}u(t)}{dt^{4-j}} = 0,\ t \in R_+, \qquad (2)$$

with the boundary condition

$$u(0) = \varphi, \quad \varphi \in H_{7/2}, \qquad (3)$$

in the corresponding space of solutions of problem (2), (3) assuming that $A^{-1} \in \sigma_p$. In this case, pencil (1) has a discrete spectrum.

Let $\lambda_n (\operatorname{Re} \lambda_n < 0)$, $n = 1,2,...$ be the eigen values of pencil (1), $\{\psi_{0,n}, \psi_{1,n}, ..., \psi_{m,n}\}$ be the system of eigen and adjoint vectors responding to the eigen value λ_n:

$$P(\lambda_n)\psi_{0,n}=0,\ P(\lambda_n)\psi_{1,n}+P'(\lambda_n)\psi_{0,n}=0,$$

$$P(\lambda_n)\psi_{2,n}+P'(\lambda_n)\psi_{1,n}+\frac{P''(\lambda_n)}{2!}\psi_{0,n}=0,$$

$$P(\lambda_n)\psi_{3,n}+P'(\lambda_n)\psi_{2,n}+\frac{P''(\lambda_n)}{2!}\psi_{1,n}+\frac{P'''(\lambda_n)}{3!}\psi_{0.n}=0,$$

$$P(\lambda_n)\psi_{k,n}+P'(\lambda_n)\psi_{k-1,n}+\frac{P''(\lambda_n)}{2!}\psi_{k-2,n}+\frac{P'''(\lambda_n)}{3!}\psi_{k-3,n}+\psi_{k-4,n}=0,$$

$$k=4,5,...,m.$$

Then the vector-functions

$$u_{k,n}(t)=e^{\lambda_n t}\left(\psi_{k,n}+\frac{t}{1!}\psi_{k-1,n}+...+\frac{t^k}{k!}\psi_{0,n}\right),\ k=0,1,...,m,$$

belong to $W_2^4(R_+;H)$, satisfy equation (2) and are called *elementary solutions* of homogeneous equation (2). Obviously, the elementary solutions $u_{k,n}(t)$ satisfy the following boundary conditions:

$$u_{k,n}(0)=\psi_{k,n},\ k=0,1,...,m.$$

In the paper [3] it is determined that problem (2), (3) has a unique solution from the space $W_2^4(R_+;H)$ for any $\varphi\in H_{7/2}$. The set of all such solutions denote by $W(P)$. It is clear from the theorems on intermediate derivatives and on traces [1, chp.1] that the set $W(P)$ is a closed subspace of the space $W_2^4(R_+;H)$.

The question is: when the elementary solutions of equation (2) are complete in $W(P)$?

The following theorem is valid.

Theorem. *Let A be a self-adjoint positive-definite operator, A_jA^{-j}, $j=1,2,3$ are bounded operators in H and*

$$\frac{3\sqrt{3}}{16}\left\|A_1A^{-1}\right\|_{H\to H}+\beta_0^{-1/2}\left\|A_2A^{-2}\right\|_{H\to H}+\alpha^{-1/2}\left\|A_3A^{-3}\right\|_{H\to H}<1,$$

where β_0 is the solution of the equation $\beta^3+2\beta^2-39\beta-140=0$ from the interval $(0,16)$,

$$\alpha=\frac{4}{3^{4/3}\left(9+\sqrt{57}\right)}\left[2\cdot 3^{7/3}+\left(9+\sqrt{57}\right)^{5/3}+4\cdot 3^{2/3}\cdot\left(9+\sqrt{57}\right)^{1/3}\right],$$

and one of the following conditions is fulfilled:

a) $A^{-1} \in \sigma_p \quad (0 < p \le 1)$*;*

b) $A^{-1} \in \sigma_p \quad (0 < p < \infty), \quad A_j A^{-j} \in \sigma_\infty(H), \quad j = 1,2,3$.

Then the system of elementary solutions of problem (2), (3) is complete in $W(P)$.

Proof. From the uniqueness of the solution of problem (2), (3) (see [3]) and the theorem on traces [1, chp.1] it follows that

$$c_1 \|\varphi\|_{H_{7/2}} \le \|u\|_{W_2^4(R_+;H)} \le c_2 \|\varphi\|_{H_{7/2}} . \tag{4}$$

According to the theorem conditions, the system $\{\psi_{k,n}\}$ is complete in the space $H_{7/2}$ (see [3]). Then for any $\varepsilon > 0$ there exists the number N and the numbers $c_{k,n}^N$ such that

$$\left\| \varphi - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N \psi_{k,n} \right\|_{H_{7/2}} < \varepsilon . \tag{5}$$

By virtue of $\psi_{k,n} = u_{k,n}(t)\big|_{t=0}$ and $\varphi = u(t)\big|_{t=0}$ for the solution

$$u(t) - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N u_{k,n}(t)$$

from (4) we get

$$\tilde{c}_1 \left\| \varphi - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N \psi_{k,n} \right\|_{H_{7/2}} \le \left\| u(t) - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N u_{k,n}(t) \right\|_{W_2^4(R_+;H)} \le$$

$$\le \tilde{c}_2 \left\| \varphi - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N \psi_{k,n} \right\|_{H_{7/2}} . \tag{6}$$

Now, taking into account inequality (5), from (6) we have:

$$\left\| u(t) - \sum_{n=1}^{N} \sum_{k} c_{k,n}^N u_{k,n}(t) \right\|_{W_2^4(R_+;H)} < \tilde{c}_2 \varepsilon = \varepsilon_1 ,$$

that means the system of the elementary solutions of problem (2), (3) is complete in the space $W(P)$. The theorem is proved.

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BİR SİNİF DÖRDTƏRTİBLİ OPERATOR-DİFERENSİAL TƏNLİKLƏRİN ELEMENTAR HƏLLƏRİNİN TAMLIĞI HAQQINDA

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XÜLASƏ

Məqalədə $W_2^4\left(R_+;H\right)$-dan olan bütün həllər fəzasında bir sinif bircins dördtərtibli operator-diferensial tənliklərin elementar həllərinin tamlığı isbat edilmişdir. Qeyd edək ki, tamlıq üçün tapılan şərtlər tədqiq edilən tənliyin operator əmsalları ilə ifadə olunubdur.

Açar sözlər: operator-diferensial tənlik, elementar həll, öz-özünə qoşma operator, Hilbert fəzası.

О ПОЛНОТЕ ЭЛЕМЕНТАРНЫХ РЕШЕНИЙ ОДНОГО КЛАССА ОПЕРАТОР-НО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЧЕТВЕРТОГО ПОРЯДКА

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РЕЗЮМЕ

В статье доказана полнота системы элементарных решений одного класса однородных операторно-дифференциальных уравнений четвертого порядка в пространстве всех решений из $W_2^4\left(R_+;H\right)$. Отметим, что найденные условия полноты выражены операторными коэффициентами исследуемого уравнения.

Ключевые слова: операторно-дифференциальное уравнение, элементарное решение, самосопряженный оператор, гильбертово пространство.

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