DIAGONALIZATION OF KLEIN-GORDON EQUATION IN AN AXIAL COLOR MAGNETIC FIELD

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In Refs. [1,2] we demonstrated that due to presence of the Gell-Mann matrices in the original form of Dirac equation, it is impossible to write eigenvalue equation for energy operator of colored particle in an external color field. But this problem was solved for colored spinors in a chromomagnetic field. The same problem exists in motion of colored scalar particle in a color background. Klein-Gordon equation contains color matrices, which mix different color states and enable us to write eigenvalue equation for this operator. This problem can be solved by using diagonalization method described in Refs [1,2]. We aim to find unitary transformation in a color space, which reduces Klein-Gordon operator to its diagonal form. We consider motion of colored scalar particle in an axial chromomagnetic field.

We use constant vector potential introduced in [3] in order to generate chromomagnetic field. For the field directed along third axis of ordinary and color spaces we choose vector potential A_μ^a in the following form:

$$A_1^a = \sqrt{\tau}\delta_{1a}, \quad A_2^a = \sqrt{\tau}\delta_{2a}, \quad A_3^a = 0, \quad A_0^a = 0, \tag{1}$$

where τ is a constant and $\delta_{\mu a}$ is the Kroneker symbol. Then Klein-Gordon equation takes the form:

$$\left(P^2 - M^2\right)\psi = 0,$$

where $P_\mu = p_\mu + gA_\mu = p_\mu + gA_\mu^a \lambda^a / 2$; λ^a are the Gell-Mann matrices, and color index a runs $a = 1,\overline{8}$. Here g is the color interaction constant. The Klein-Gordon operator $H = P^2 - M^2$ in color space has the explicit matrix form:

$$H = \begin{pmatrix} \mathrm{P}^2 & -Gp_- & 0 \\ -Gp_+ & \mathrm{P}^2 & 0 \\ 0 & 0 & \pi^2 \end{pmatrix}. \tag{2}$$

Here $\mathrm{P}^2 = p^2 - M^2 - G^2/2$, $\pi^2 = p^2 - M^2$, $G = g\tau^{1/2}$, $p_\pm = p_1 \pm ip_2$. Hamiltonian (2) will get the diagonal form H' under some U transformation of the basic vectors of the color spin space

$$HU = UH'. \tag{3}$$

U transformation is unitary and determinant and trace of any matrix under this transformation is preserved

$$\det H' = \det H, \quad TrH = TrH'.$$

Determinant of H can be written down as a product of three factors:

$$\det H = \left(\mathrm{P}^2 + Gp_\perp\right)\left(\mathrm{P}^2 - Gp_\perp\right)\pi^2 = f_1 f_2 f_3, \tag{4}$$

where f_i denote $f_{1,2} = \mathrm{P}^2 \pm Gp_\perp$, $f_3 = \pi^2$ and $p_\perp^2 = p_1^2 + p_2^2$. Determinant of H' is product of diagonal elements

$$\det H' = \det\begin{pmatrix} h'_{11} & 0 & 0 \\ 0 & h'_{22} & 0 \\ 0 & 0 & h'_{33} \end{pmatrix} = h'_{11}\, h'_{22}\, h'_{33}.$$

We have two products: the product of three factors f_i in Eq.(4) and product of three diagonal elements h'_{ii} in $\det H'$; which should be equal:

$$f_1 f_2 f_3 = h'_{11}\, h'_{22}\, h'_{33}\,.$$

It can be found that, the sum of factors f_i in Eq.(4) is equal to the sum of the diagonal elements of H and, consequently, should be equal to the sum of h'_{jj}

$$\sum_i f_i = 3\mathrm{P}^2 + G^2 = \sum_j h_{jj} = \sum_j h'_{jj}\,.$$

Equalities of the product and sum allow us to assume that factors f_i in (4) are the same with the diagonal elements h'_{jj}, since we know invariance of trH and $\det H$ under U transformation. But, we encounter the question of places of f_i along the diagonal of H', i.e. which factor f_i corresponds to the which diagonal element h'_{jj}, since we have 3 different variants for this correspondence. Each variant can be offered as a candidate for H', while we have only one H'. Due to uniqueness of the diagonal form of the hermitian matrix, only one of the constructed variants will give us the correct H'. We may choose some variant for $f_i = h'_{jj}$ identification and then verify this choice by means of Eq. (3). According to the uniqueness of H' and U the only correct variant of identification will satisfy Eq. (3), i.e. it will be solved for the u_{ij} without mathematical nonsense. We make the identification below

$$h'_{11} \equiv f_2, \quad h'_{22} \equiv f_1, \quad h'_{33} \equiv f_3\,.$$

For this H' equation (3) has got the form:

$$\begin{pmatrix} \mathrm{P}^2 & -Gp_- & 0 \\ -Gp_+ & \mathrm{P}^2 & 0 \\ 0 & 0 & \pi^2 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}\begin{pmatrix} \mathrm{P}^2 - Gp_\perp & 0 & 0 \\ 0 & \mathrm{P}^2 + Gp_\perp & 0 \\ 0 & 0 & \pi^2 \end{pmatrix} \quad (5)$$

From the equality (5) we get the following system of linear equations for the u_{ij}:

$$\begin{cases} p_- u_{21} = -p_\perp u_{11} \\ p_- u_{22} = p_\perp u_{12} \\ u_{13} = u_{23} = u_{31} = u_{32} = 0 \end{cases}. \quad (6)$$

Let us express, that u_{ij} are complex and number of unknowns in (6) are more than number of equations. We can include relations between u_{ij}, which gives us unitarity of U matrix. Having solved these equations jointly we finally find explicit form of U matrix:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\alpha} & p_- e^{-i\alpha} / p_\perp & 0 \\ -p_+ e^{i\alpha} / p_\perp & e^{-i\alpha} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \quad (7)$$

Let us note, that U transformation contains one free parameter α, which cannot be fixed.

REFERENCES

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