ANALYTICAL SOLUTIONS OF THE KLEIN-GORDON EQUATION WITH THE WOODS-SAXON POTENTIAL FOR ARBITRARY l-STATE

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In this work the analytical solution of the radial Klein-Gordon equation for the standard Woods-Saxon potential [1] is presented. In our calculations we have applied the Nikiforov-Uvarov method [2-4] by using the Pekeris approximation [5] to the centrifugal potential for arbitrary l states. The bound state energy eigenvalues and corresponding eigenfunctions are obtained for various values of the quantum numbers n_r and l.

The standard Woods-Saxon potential is defined by

$$V(r) = -\frac{V_0}{1+e^{\frac{r-R_0}{a}}} \quad (a \ll R_0) \,, \tag{1}$$

where V_0 is the potential depth, R_0 is the width of the potential or the nuclear radius and the parameter a is the thickness of the superficial layer inside which the potential falls from value $V=0$ outside of a nucleus up to value $V=-V_0$ inside a nucleus. At $a=0$ one gets the simple potential well with jump of potential on the surface of a nucleus.

The radial part of the Klein-Gordon equation [6] with Woods-Saxon potential is

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{(E-V)^2 - m_0^2c^4}{\hbar^2c^2} - \frac{l(l+1)}{r^2}\right]R(r) = 0\,,\ (0 \le r < \infty)\,, \tag{2}$$

where l is the angular momentum quantum number.

After introducing the new function $u(r) = rR(r)$, Eq.(2) takes the form

$$\frac{d^2u(r)}{dr^2} + \left[\frac{(E-V)^2 - m_0^2c^4}{\hbar^2c^2} - \frac{l(l+1)}{r^2}\right]u(r) = 0\,. \tag{3}$$

For the bound states $E^2 < m_0^2c^4$, we get

$$0 \le n_r < \frac{\sqrt{1 + \frac{192a^4l(l+1)}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2} \,, \tag{4}$$

$$0 < V_0 < \frac{4\hbar ca\sqrt{3l(l+1)}}{R_0^2}\,, \tag{5}$$

$$-\frac{V_0}{2} + \frac{4a\hbar^2c^2l(l+1)}{R_0^3V_0} - \frac{\hbar^2c^2}{2a^2V_0}\left[-n_r + \frac{\sqrt{1 + \frac{192\,a^4l(l+1)}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2}\right]^2 < E_{n_rl} < -\frac{V_0}{2} + \frac{4a\hbar^2c^2l(l+1)}{R_0^3V_0} + \frac{\hbar^2c^2}{2a^2V_0}\left[-n_r + \frac{\sqrt{1 + \frac{192\,a^4l(l+1)}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2}\right]^2 , \tag{6}$$

where n_r is the radial quantum number.

The exact energy eigenvalues of the Klein-Gordon equation with the Woods-Saxon potential are derived as

$$E_{n_r l} = -\frac{V_0}{2}\left(1-\frac{32a^3 l(l+1)}{R_0^3\left[\left(\sqrt{1+\frac{192a^4 l(l+1)}{R_0^4}-\frac{4V_0^2a^2}{\hbar^2c^2}}-2n_r-1\right)^2+\frac{4V_0^2a^2}{\hbar^2c^2}\right]}\right)\Bigg\{1\mp\frac{2c}{V_0}\times$$

$$\left(\sqrt{1+\frac{192a^4 l(l+1)}{R_0^4}-\frac{4V_0^2a^2}{\hbar^2c^2}}-2n_r-1\right)\left\{\left[\left(\sqrt{1+\frac{192a^4 l(l+1)}{R_0^4}-\frac{4V_0^2a^2}{\hbar^2c^2}}-2n_r-1\right)^2+\frac{4V_0^2a^2}{\hbar^2c^2}\right]\times \quad (7)\right.$$

$$\left(m_0^2c^2+\frac{\hbar^2 a l(l+1)}{R_0^3}\left(1-\frac{4a}{R_0}+\frac{12a^2}{R_0^2}\right)\right)\left[\left(\sqrt{1+\frac{192a^4 l(l+1)}{R_0^4}-\frac{4V_0^2a^2}{\hbar^2c^2}}-2n_r-1\right)^2+\frac{4V_0^2a^2}{\hbar^2c^2}-\right.$$

$$\left.\left.\frac{32a^3 l(l+1)}{R_0^3}\right]^{-2}-\frac{\hbar^2}{16a^2}\right\}^{\frac{1}{2}}\Bigg\}.$$

If all three conditions (4), (5) and (6) are satisfied simultaneously, the bound states exist. From Eq.(4) is seen that if $l=0$, then one gets $n_r<0$. Hence, the Klein-Gordon equation for the standard Woods-Saxon potential with zero angular momentum has not bound states.

According to Eq.(7) the energy eigenvalues depend on the depth of the potential V_0, the width of the potential R_0, and the surface thickness a. Any energy eigenvalue must be less than V_0. If constraints imposed on n_r, V_0 and $E_{n_r l}$ are satisfied, the bound states appear. From Eq.(5) is seen that the potential depth increases when the parameter a increases, but the parameter R_0 is decreasing and vice versa. Therefore, one can say that the bound states exist within this potential. Thus, the energy spectrum Eq.(7) are limited, i.e. we have only the finite number of energy eigenvalues.

The corresponding radial wave functions $u_{n_r l}(z)$ are given by expressions

$$u_{n_r l}(z) = C_{n_r l}\, z^{\varepsilon}(1-z)^{\sqrt{\varepsilon^2-\beta^2+\gamma^2}}\, P_{n_r}^{(2\varepsilon,\, 2\sqrt{\varepsilon^2-\beta^2+\gamma^2})}(1-2z),$$

where

$$\varepsilon^2 = -\frac{\left(E^2-m_0^2c^4\right)a^2}{\hbar^2c^2}+\frac{48a^4 l(l+1)}{R_0^4},\ \beta^2 = \frac{2EV_0a^2}{\hbar^2c^2}-\frac{8a^3 l(l+1)}{R_0^3}\left(1-\frac{6a}{R_0}\right),$$

$\gamma^2 = -\frac{V_0^2a^2}{\hbar^2c^2}+\frac{48a^4 l(l+1)}{R_0^4}$, $\quad z=\left(1+e^{\frac{r-R_0}{a}}\right)^{-1}$, $\quad P_{n_r}^{(2\varepsilon,2\sqrt{\varepsilon^2-\beta^2+\gamma^2})}(1-2z)$ is the Jacobi polynomials and $C_{n_r l}$ are the normalization constants determined using $\int_0^\infty \left[u_{n_r l}(r)\right]^2 dr = 1$ constraint.

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