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EXISTENCE AND UNIQUENESS OF THE SOLUTIONS OF THE NONLINEAR IMPULSE DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper the system of ordinary impulse differential equations with nonlocal conditions is investigated. First, the boundary value problem is reduced to the equivalent integral equation. Further, using the fixed point theorem, conditions for the existence and uniqueness of the solution of the boundary value problem are obtained. The continuous dependence of the solutions on the right-hand side of the boundary conditions is also established.

Mathematics Subject Classification (2020): 34A12.

Key words: Existence and uniqueness, nonlinear impulse differential equations, nonlocal boundary conditions.

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1. Introduction. The theory of impulse differential equations is presented as a natural description of some real processes that are subject to certain perturbations, the duration of which is insignificant in comparison with the duration of the process. Examples of such problems arise mainly in physics, technology, biology, economics and other areas of natural science. The mathematical models of such processes are described by the differential equations, the solutions of which are the functions with discontinuities of the first kind at fixed or non-fixed time moments. Such differential equations are well studied in monographs [10, 11, 14, 24, 26, 27]. In those works, mainly differential equations with local conditions were studied. However, in recent years, interest has increased in differential equations with impulses and nonlocal boundary conditions, which describe many practical processes.

The problems with integral boundary conditions have been used to describe many phenomena in applied sciences. We refer the interested reader to [2-9, 12, 13, 15-21, 23, 28-33] for the examples and references.

To date, there are a large number of works devoted to ordinary differential equations with impulses and nonlocal boundary conditions, in which existence theorems for the solutions under various types of the nonlocal conditions have been proved [2-9, 12, 13, 15-21, 23, 28-33].

Note that numerical methods for the multipoint and integral boundary value problems for ordinary differential equations of the first order were developed in [1, 22].

In this paper, we study a nonlocal boundary value problem for the systems of ordinary differential equations with impulses, the boundary conditions of which include pointwise and integral terms. Note that the investigated boundary value problem is rather general. In particular cases, it covers the Cauchy problem, non-separated two-point boundary value problem, and the "pure" integral condition. The questions of the existence and uniqueness of the solution of the considered boundary value problem, as well as the continuous dependence of the solution on the right-hand side of the boundary conditions are investigated.

2. Formulation of the problem. We consider the existence and uniqueness problems for the solution to the following system of differential equations

$$\dot{x}(t) = f(t, x(t)), \qquad t \in [0, T], t \neq t_i, \qquad i = 1, 2, ..., p \tag{2.1}$$

with nonlocal boundary condition

$$Ax(0) + \int_0^T n(t)\, x(t)\, dt + Bx(T) = C, \tag{2.2}$$

and impulses

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), \qquad i = 1, 2, ..., p, \tag{2.3}$$

where $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T; A, B \in R^{n\times n}$, $n(t) \in R^{n\times n}$ are given matrices; $C \in R^n$ is a given vector, moreover, det $N \neq 0$, wher e $N = A + \int_0^T n(t)\, dt + B$; $I_i : R^n \to R^n$ are given functions; $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, where

$x(t_i^+) = \lim\limits_{h\to 0^+} x(t_i+h)$, $x(t_i^-) = \lim\limits_{h\to 0^+} x(t_i-h) = x(t_i)$ are right and left limits of the function $x(t)$ at the point $t = t_i$, correspondingly.

3. Some auxiliary results and facts. Here we give some definitions and auxiliary facts that will be used below.

By $C\left([0,T] : R^n\right)$ we define the Banach space of the continuous vector functions $x(t)$, defined on the interval $[0,T]$, with values from R^n and with the norm $\|x\| = \max\limits_{[0,T]} |x(t)|$, where $|\cdot|$ stands for the norm in R^n.

By $PC\left([0,T], R^n\right)$ we denote the linear space

$PC\left([0,T], R^n\right) = \{x : [0,T] \to R^n; \quad x(t) \in C\left((t_i, t_{i+1}], R^n\right), \ i = 0, 1, ..., p;$ moreover $x(t_i^+)$ and $x(t_i^-)$, $i = 1, 2, ..., p$ exist and are bounded; $x(t_i^-) = x(t_i)\}$.

Obviously, the linear space $PC\left([0,T]; R^n\right)$ is a Banach space with the norm $\|x\|_{Pc} = \max\left\{\|x\|_{C((t_i,t_{i+1}])}, \ i = 0, 1, ..., p\right\}$.

We define the solution to the boundary value problem (2.1)–(2.3) as follows.

DEFINITION 3.1. The function $x \in PC\left([0,T] : R^n\right)$ is called to be a solution to boundary value problem (2.1)–(2.3), if for arbitrary $t \in [0,T]$, $t \neq t_i$, $i = 1, 2, ..., p$,

$$\dot{x}(t) = f(t, x(t))$$

and for $t = t_i$ $i = 1, 2, ..., p$ $0 < t_1 < t_2 < ... < t_p < T$
is valid

$$\Delta x(t_i^+) = x(t_i^+) - x(t_i) = I_i(x(t_i)).$$

Additionally the function $x(t)$ satisfies to boundary condition (2.2).

Introduce the functions

$$K(t,\tau) = \begin{cases} N^{-1}(A + \int_0^t n(\tau)\, d\tau), & 0 \le \tau \le t, \\ -N^{-1}\left(\int_t^T n(\tau)\, d\tau + B\right), & t < \tau \le T. \end{cases}$$

LEMMA 3.1. *Let* $y \in C\left([0,T]; R^n\right)$ $a_i \in R^n$ $i = 1, 2, ..., p$. *The differential equation*

$$\dot{x}(t) = y(t) \tag{3.1}$$

with impulses

$$x(t_i^+) - x(t_i) = a_i; i = 1, 2, ..., p, \ 0 < t_1 < t_2 < ... < t_p < T, \tag{3.2}$$

and nonlocal boundary conditions

$$Ax(0) + \int_0^T n(t)\, x(t)\, dt + Bx(T) = C \tag{3.3}$$

has the unique solution $x(t) \in PC\left([0,T], R^n\right)$ *that is expressed by the formula*

$$x(t) = N^{-1}C + \int_0^T K(t,\tau)\, y(\tau)\, d\tau + \sum_{0<t_i<T} K(t, t_i)\, a_i \tag{3.4}$$

for $t \in (t_i, t_{i+1}]$, $i = 0, 1, ..., p$.

Proof. Let the function $x(t) \in PC\left([0,T], R^n\right)$ be a solution to boundary value problem (3.1)–(3.3). Then integrating equation (3.1) on the interval $t \in (0,\, t_{i+1})$we get

$$\int_0^t y(s)ds = \int_0^t \dot{x}(s)ds = \left[x(t_1) - x(0^+)\right] + \left[x(t_2) - x({t_1}^+)\right] + ... + \left[x(t) - x({t_i}^+)\right] =$$

$$= -x(0) - \left[x({t_1}^+) - x(t_1)\right] - \left[x({t_{21}}^+) - x(t_2)\right] - ... - \left[x({t_i}^+) - x(t_i)\right] + x(t).$$

Considering condition (3.2) in the last equality we obtain

$$x(t) = x(0) + \int_0^t y(s)ds + \sum_{0<t_i<t} a_i. \tag{3.5}$$

Now we require that the function $x(t) \in PC\left([0,T], R^n\right)$defined by equality (3.5) satisfy the boundary condition (3.3)

$$(A + \int_0^T n\,(t)\,dt + B)\,x(0) = C - \int_0^T n\,(t) \int_0^t y\,(sdsdt) -$$

$$- \int_0^T n\,(t) \sum_{0<t_i<t} a_i dt - B \int_0^T y\,(t)\,dy - B \sum_{0<t_i<T} a_i. \tag{3.6}$$

Since det $N \neq 0$ from (3.6) we have

$$x(0) = N^{-1} \left[C - \int_0^T n\,(t) \int_0^t y\,(s)\,dsdt - \int_0^T n\,(t) \sum_{0<t_i<T} a_i dt - B \int_0^T y\,(t)\,dy \right.$$
$$\left. -B \sum_{0<t_i<T} a_i \right]. \tag{3.7}$$

Now we put the value $x(0)$ defined by equality (3.7) into (3.6). It gives

$$x(t) = N^{-1} \left[C - \int_0^T n\,(t) \int_0^t y\,(s)\,dsdt - \int_0^T n\,(t) \sum_{0<t_i<T} a_i dt - B \int_0^T y\,(t)\,dy \right.$$
$$\left. -B \sum_{0<t_i<T} a_i \right] + \int_0^t y\,(s)\,ds + \sum_{0<t_i<t} a_i. \tag{3.8}$$

Since

$$\int_0^T n(t)\int_0^t y(s)\,dsdt = \int_0^T\int_t^T n(s)\,dsy(t)\,dt\int_0^T n(t)\int_0^t y(s)\,dsdt$$
$$= \int_0^T n(t)\sum_{0<t_i<T} a_i dt = \sum_{0<t_i<T}\int_{t_i}^T n(t)\,dta_i,$$

is valid then from (3.8) we obtain

$$x(t) = N^{-1}\left[C - \int_0^T\int_t^T n(s)\,dsy(t)\,dt - \sum_{0<t_i<T}\int_{t_i}^T n(t)\,dta_i - B\int_0^T y(t)\,dy\right.$$
$$\left. -B\sum_{0<t_i<T} a_i\right] + \int_0^t y(s)\,ds + \sum_{0<t_i<t} a_i. \quad (3.9)$$

Making the simplifications below

$$x(t) = N^{-1}\left[C - \int_0^t\int_\tau^T n(s)\,dsy(\tau)\,d\tau - \sum_{0<t_i<t}\int_{t_i}^T n(t)\,dta_i - B\int_0^t y(s)\,ds - B\sum_{0<t_i<t} a_i\right] +$$
$$-N^{-1}\left(\int_t^T\int_\tau^T n(s)\,dsy(\tau)\,d\tau - \sum_{t<t_i<T}\int_{t_i}^T n(t)\,dta_i - B\int_t^T y(s)\,ds - B\sum_{t<t_i<T} a_i\right) +$$
$$+\int_0^t y(s)\,ds + \sum_{0<t_i<t} a_i$$

we get

$$x(t) = N^{-1}C + \int_0^t\left[E - N^{-1}B - N^{-1}\int_\tau^T n(s)\,ds\right]y(\tau)\,d\tau$$
$$+ N^{-1}\int_t^T\left[-B - \int_\tau^T n(s)\,ds\right]y(s)\,ds$$
$$+ \sum_{0<t_i<t}\left[E - N^{-1}B - N^{-1}\int_{t_i}^T n(t)\,dt\right]a_i$$
$$+ \sum_{t<t_i<T}\left[-N^{-1}B - N^{-1}\int_{t_i}^T n(t)\,dt\right]a_i.$$

This implies the validity of representation (3.4). □

Note. Formula (3.4) implies the following statements:

(1) The constant vector function $x(t) = N^{-1}C$ is a solution of the differential equation

$$\dot{x}(t) = 0$$

with nonlocal condition

$$Ax(0) + \int_0^T n(t)\, x(t)\, dt + Bx(T) = C.$$

(2) The function $x(t) = \int_0^T K(t,s)\, y(s)\, ds$ is a solution of the differential equation $\dot{x}(t) = y(t)$
with nonlocal condition

$$Ax(0) + \int_0^T n(t)\, x(t)\, dt + Bx(T) = 0.$$

Here the matrix function $K(t,s)$ is indeed Green's function for the considered problem.

(3) Piecewise constant function

$$x(t) = \sum_{0<t_i<t} K(t,t_i)\, a_i, \qquad i = 1,\, 2,\, ...,\, p,$$

is a solution of the differential equation

$$\dot{x}(t) = 0$$

with impulses

$$x({t_i}^+) - x(t_i) = a_i, \quad i = 1,\, 2,\, ...,\, p$$

and boundary condition

$$Ax(0) + \int_0^T n(t)\, x(t)\, dt + Bx(T) = 0.$$

Lemma 3.2. *Suppose that $f \in C([0,T] \times R^n,\, R^n)$ and $I_i(x) \in C(R^n)$. Then the function $x(t) \in PC([0,T],\, R^n)$ is a solution of boundary value problem (2.1)–(2.3) if and only if the function $x(t) \in PC([0,T],\, R^n)$ would be a solution to the following impulsive integral equation*

$$x(t) = N^{-1}B + \int_0^T K(t,s)\, f(s, x(s))\, ds + \sum_{i=1}^{P} K(t,t_i)\, I_i(x(t_i)), \tag{3.10}$$

for $t \in (t_i,\, t_{i+1}), \quad i = 0,\, 1,\, ...,\, p$.

Proof. Let $x(t) \in PC\left([0,T], R^n\right)$ be a solution to the boundary value problem (2.1)–(2.3). Then similarly to Lemma 1 one may show that the function $x(t) \in PC\left([0,T], R^n\right)$ satisfy integral equation (3.10).

The opposite statement is also true. By direct calculation, one can make sure that the solution to integral equation (3.10) also satisfies equation (2.1), boundary condition (2.3), and impulse conditions (2.2).

The lemma is proved. □

4. Main results. The first main result of this section is based on the Banach fixed point principle. On the basis of this principle, a theorem on the existence and uniqueness of the solution to boundary value problem (2.1)–(2.3) is proved.

THEOREM 4.1. *Suppose that the following conditions are satisfied:*

(H1) There exists a constant $M \geq 0$ such that

$$|f(t, x) - f(t, y)| \leq M\, |x - y|,$$

for any $t \in [0,T]$ and for any $x, y \in R^n$.

(H2) There exist constants $l_i \geq 0$, $i = 1, 2, ..., p$ such that

$$|I_i(x) - I_i(y)| \leq l_i\, |x - y|$$

for any $x, y \in R^n$.

Then if

$$L = S\left(MT + \sum_{k=1}^{p} l_k\right) < 1, \tag{4.1}$$

boundary value problem (2.1)–(2.3) *has a unique solution.*

Here the number S is defined by the relation

$$S = \max_{0 \leq t, s \leq T} \|K(t,s)\|.$$

Proof. For the proof, we use the Banach fixed point principle.

Define the operator $F : PC\ ([0,T]; R^n) \to PC\ ([0,T] \times R^n)$ by the relation

$$(Fx)(t) = N^{-1}B + \int_0^T K(t,s)\, f(s, x(s))\, ds + \sum_{k=1}^{P} K\left(t; t_k\right) I_k\left(x\left(t_k\right)\right) \tag{4.2}$$

for $t \in (t_i, t_{i+1}), \quad i = 0, 1, 2, ..., p$.

Obviously, the fixed points of the operator$F$ are solutions to boundary value problem (2.1)–(2.3). Using the principle of contracting operators, we will show that the operator Fdefined by equality (4.2) has a unique fixed point.

Set $M_f = \max\limits_{[0,T]} |f\,(t, 0)|$ $m_I = \max\limits_{k \in \{1,2,\ldots,p\}} |I_k\,(0)|$ and fix the number

$$r \geq \frac{\left\|N^{-1}B\right\| + S\left(M_f T + p m_I\right)}{1 - L}.$$

Now we show that $FB_r \subset B_r$, where

$$B_r = \{x \in PC([0,T], R^n) : \|x\|_{PC} \le r\}.$$

For $x \in B_r$ we have

$$\|(Fx)(t)\| \le \left\|N^{-1}B\right\| + \max_{[0,T]} \int_0^T |K(t,s)| \left[|f(s, x(s)) - f(s,0)| + |f(s,0)|\right] ds +$$

$$+ \max_{[0,T]} \sum_{k=1}^{P} |K(t; t_k)| \left[|I_k(x(t_k)) - I_k(0)| + |I_k(0)|\right] \le$$

$$\le \left\|N^{-1}B\right\| + S\left[(MTr + M_f T) + \left(\sum_{k=1}^{p} l_k\right) r + pm_I\right] \le r.$$

Let $x,\ y \in PC([0,T];\ R^n)$ are arbitrary fixed elements. Then for any $t \in (t_i,\ t_{i+1}]$ we have

$$|F(x)(t) - F(y)(t)| \le \int_0^T |K(t,s)| \cdot |f(s,\ x(s)) - f(s,\ y(s))|\ ds +$$

$$+ \sum_{k=1}^{P} |K(t_i,\ t_k)| \cdot |I_k(x(t_k)) - I_k(y(t_k))|.$$

Using conditions (H1), (H2) from the last inequality we get

$$|F(x)(t) - F(y)(t)| \le SNT\ \|x - y\| + S\sum_{k=1}^{P} l_k\, |x(t_k)) - y(t_k)|.$$

This inequality can be rewritten as follows

$$|F(x)(t) - F(y)(t)| \le \left[S\left(NT + \sum_{k=1}^{P} l_k\right)\right] \times \|x - y\|_{PC}.$$

Thus

$$\|F(x)(t) - F(y)(t)\| \le L\ \|x - y\|_{PC}.$$

Here, taking into account condition (4.1), we obtain that the operator$F$ is contracting. According to the fixed point principle, it can be concluded that the operator Fhas a unique fixed point. This is equivalent to the fact that nonlocal boundary value problem (2.1)–(2.3) has a unique solution. The theorem is proved. □

The second result of this section is devoted to establishing the existence of solutions to boundary value problem (2.1)–(2.3), which is based on the Schaufer fixed point theorem.

THEOREM 4.2. *Suppose that the following conditions are satisfied:*

(H3) The function $f : [0,T] \times R^n \to R$ *is continuous and there exists a constant* $N_1 > 0$ *such that*

$$|f(x,t)| \leq N_1$$

for all $t \in [0,T]$ *and* $x \in R^n$.

(H4) The functions $I_k : R^n \to R^n$ *are continuous and there exists constants* $N_2 > 0$*such that*

$$\max_{k\in\{1,2,\ldots,P\}} |I_k(x)| \leq N_2.$$

Then boundary value problem (2.1)–(2.3) *has at last one solution on* $[0,T]$.

Proof. Let us show that under the above conditions, the operator $F(x)(t)$defined by equality (4.2) has fixed points. This will be done in some steps:

Step 1. The operator F within the conditions of the theorem is continuous in $PC\left([0,T];R^n\right)$. Let $\{x_n\}$ be a functional sequence in the space $PC\left([0,T];R^n\right)$ and $x_n \to x$ $x \in PC\left([0,T];R^n\right)$. Then for any $t \in (t_i, t_{i+1}]$ and $i = 0,1,...,p$ is valid

$$|F(x_n)(t) - F(x)(t)| \leq \int_0^T |K(t,s)| \cdot |f(s,\, x_n(s)) - f(s,\, x(s))|\, ds$$
$$+ \sum_{k=1}^{P} |K(t,\, t_k)| \cdot |I_k(x_n(t_k)) - I_k(x(t_k))|\,.$$

Considering here conditions (H3), (H4) we get

$$|F(x_n)(t) - F(x)(t)| \leq ST \max_{S\in[0,T]} |f(s,\, x_n(s)) - f(s,\, x(s))|$$
$$+ S\sum_{k=1}^{P} |I_k(x_n(t_k)) - I_k(x(t_k))|\,.$$

Since the functions f and I_k, $k = 1,2,...,p$ are continuous this implies

$$\|F(x_n)(t) - F(x)(t)\|_{PC} \to 0$$

at $n \to \infty$.

Step 2. The mapping $F$is limited in the space $PC\left([0,T];R^n\right)$. This is equivalent to showing that for any $\eta > 0$, there exists such $l > 0$ that for any

$$x \in B_\eta = \{x \in PC\left([0,T];R^n\right) : \|x\| \leq \eta\}$$

takes place

$$\|F(x(\cdot))\| \leq l.$$

Applying the triangle inequality and using assumptions (H3) and (H4) fort $t \in (t_i, t_{i+1}]$ we get

$$|F(x)(t)| \leq \int_0^T |K(t,s)| \cdot |f(s,\, x(s))| \; ds + \sum_{i=1}^{P} |K(t,\, t_i)| \cdot |I_i(x(t_i))| \, .$$

Thus,

$$|F(x)(t)| \leq S \; [TN_1 + pN_2] = l.$$

Step 3. The operator F maps a bounded set into an equicontinuous subset of the space $PC\ ([0,T]\,; R^n)$. Let $\tau_1,\, \tau_2 \in (t_i,\, t_{i+1}]$ and $\tau_1 < \tau_2$. B_η be a bounded set in Step 2 and let $x \in B_\eta$. Then we have:

$$\begin{aligned} F(x)(\tau_2) - \; & F(x)(\tau_1) \\ = N^{-1} & \int_0^{\tau_2} \left(A + \int_0^s n\,(\tau)\, d\tau \right) f(s,\, x(s))\, ds - N^{-1} \int_{\tau_2}^T \int_s^T n\,(\tau)\, d\tau f(s,\, x(s))\, ds \\ - N^{-1} & \int_0^{\tau_1} \left(A + \int_0^s n\,(\tau)\, d\tau \right) f(s,\, x(s))\, ds + N^{-1} \int_{\tau_1}^T \int_s^T n\,(\tau)\, d\tau f(s,\, x(s))\, ds \\ = N^{-1} & \int_{\tau_1}^{\tau_2} \left(A + \int_0^s n\,(\tau)\, d\tau \right) f(s,\, x(s)\, ds + N^{-1} \int_{\tau_1}^{\tau_2} \int_s^T n\,(\tau)\, d\tau f(s,\, x(s)\, ds \\ = & \int_{\tau_1}^{\tau_2} f(s,\, x(s)\, ds. \end{aligned}$$

From this we obtain

$$|F(x)(\tau_1) - F(x)(\tau_2)| \leq \int_{\tau_1}^{\tau_2} |f(s,\, x(s))| \; ds.$$

At $\tau_1 \to \tau_2$ the right side of the previous inequality tends to zero. Taking into account that the mapping F is continuous and equicontinuous, we come to the conclusion that the mapping

$$F : PC\ ([0,T]\,; R^n) \to PC\ ([0,T]\,; R^n)$$

completely continuous.

Step 4. Let us show that the set

$$\Delta = \{x \in PC\,([0,T] : R^n) : x = \lambda\, F(x)\},$$

is bounded for some $0 < \lambda < 1$. Let for some $0 < \lambda < 1$ the equality $x = \lambda\,(Fx)$ is valid. Then for any $t \in (t_i,\, t_{i+1}]\,,\, i = 0, 1, ..., p$ we have

$$x(t) = \lambda \left[N^{-1}B + \int\limits_0^T K(t,s)\, f(s, x(s))\, ds + \sum_{k=1}^{P} K(t_i, t_k)\, I_n\ (x(t_k)) \right].$$

From this, considering conditions (H3) and (H4) (as in Step 2) for any $t \in [0, T]\,,$ we get

$$|F(x)(t)| \le [N_1 T + pN_2]\ S.$$

Consequently, for any $t \in [0, T]$ we obtain

$$\|x\|_{PC} \le [N_1 T + pN_3]\, S = R.$$

This shows that the set Δ is bounded. Hence, all conditions of the Schaufer fixed point theorem are satisfied. It follows that the operator F has fixed points, which are solutions to boundary value problem (2.1)–(2.3).

The theorem is proved. □

Now let us show the continuous dependence of the solutions of problem (2.1)–(2.3) on the right-hand side of (2.2).

THEOREM 4.3. *Let conditions (H1), (H2) be satisfied and $L < 1$. Then for any $B_1, B_2 \in R^n$ and for the corresponding solutions x_1, x_2 of the following boundary value problems*

$$\dot{x}_j(t) = f(t, x_j(t)), \qquad t \in [0,\, T]\,,\, t \ne t_i, \qquad i = 1,\, 2,\, ...,\, p, \tag{4.3}$$

$$Ax_j(0) + \int_0^T n\,(t)\, x_j\,(t)\, dt = B_j, \tag{4.4}$$

$$x_j(t_i^+) - x_j(t_i) = I_i(x_j(t_i)), \quad i = 1,\, 2,\, ...,\, p,\ j = 1, 2, \tag{4.5}$$

the estimate

$$\|x_1\,(t) - x_2\,(t)\| \le (1 - L)^{-1} \left\|N^{-1}\right\| \|B_1 - B_2\|$$

is true.

Proof. Let $B_1, B_2 \in R^n$ be arbitrary points and x_1, x_2 be corresponding solutions to problem (4.3)–(4.6). Then we can write

$$x_1(t) - x_2\,(t) = N^{-1}\,[B_1 - B_2] + \int_0^T K(t,s)\ [f(s, x_1(s)) - f(s, x_2(s))]\ ds +$$

$$+ \sum_{k=1}^{P} K(t, t_k)\ [I_k(x_1(t_k)) - I_k(x_2(t_k))]\,. \tag{4.6}$$

Now, using conditions (H1) and (H2), from (4.6) we obtain

$$|x_1(t) - x_2(t)| \le \left\| N^{-1} [B_1 - B_2] \right\| + SM \int_0^T |x_1(\tau) - x_2(\tau)| \, d\tau$$
$$+ S \sum_{i=1}^{p} l_i \, |x_1(t_k) - x_2(t_k)| \, .$$

From this

$$\|x_1(t) - x_2(t)\| \le \left\| N^{-1} \right\| \|B_1 - B_2\| + S \left(MT + \sum_{i=1}^{p} l_i \right) \|x_1(t) - x_2(t)\| \, .$$

Since $L < 1$ it follows from the last inequality that

$$\|x_1(t) - x_2(t)\| \le (1 - L)^{-1} \left\| N^{-1} \right\| \|B_1 - B_2\| \, .$$

The theorem is proved. □

Note that the scheme proposed in this work can be successfully applied in more complex boundary value problems with impulses. For example, for the boundary value problem, when (2.2) involves three-point or multi-point and integral terms.

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