Boundedness and Compactness of Commutators of the Riesz Potential in General Morrey-Type Spaces

Dauren Matin

Azerbaijan, Baku

09.12.2022

Цель работы: find compactness conditions for sets and some operators in general spaces of Morrey type.
Note that the boundedness of known classical operators (the Riesz potential, the Hardy-Littlewood maximal and fractionally maximal operators, etc.) in Morrey-type spaces has been fairly well studied. Vagif Guliyev, Hussein Guliyev, V.I. Burenkov, Y.Savano, I.Chen, A.Gogatishvilli, T.Mizuhara, E.Nakai, and. others.
Note that the question of compactness of sets and operators in Morrey-type spaces has been little studied.

Morrey space

In 1938 C.Morrey introduced the Morrey space in his paper *ON THE SOLUTIONS OF QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS.*

Let , $0 \leq \lambda \leq \frac{n}{p}$, 0<p$\leq \infty$. They say that the function $f \in M_p^{\lambda}(\mathbb{R}^n)$, if $f \in L_p^{loc}(\mathbb{R}^n)$ и

$$\|f\|_{M_p^{\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\ r>0} r^{-\lambda} \left(\int\limits_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty$$

When $\lambda = 0$

$$\|f\|_{M_p^0(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left(\int\limits_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} = \|f\|_{L_p(\mathbb{R}^n)}$$

When $\lambda = \frac{n}{p}$

$$\|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} \equiv \|f\|_{L_\infty(\mathbb{R}^n)} = ess \sup_{x\in\mathbb{R}^n} |f(x)| < \infty$$

When $\lambda < 0,\ \lambda > \frac{n}{p}$

$$\|f\|_{M_p^{\lambda}(\mathbb{R}^n)}$$

– trivially.

Generalized Morrey space

Let $1 \le p \le \infty$, w be a measurable non-negative function on $(0,\infty)$ that is not equivalent to zero. The generalized Morrey space $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_p^{w(\cdot)}} \equiv \sup_{x\in\mathbb{R}^n} \sup_{r>0} \Big(w(r) \, \|f\|_{L_p(B(x,r))} \Big) < \infty$$

(T.Mizuhara 1991, E.Nakai 1994, V.Guliyev 1994)
The space $M_p^{w(\cdot)}$ coincides with the classical Morrey space M_p^{λ} for $w(r) = r^{-\lambda}$, for $0 \le \lambda \le \frac{n}{p}$.

Morrey type local space

Let $0 < p, \theta \leq \infty$, and let w be a non-negative measurable function on $(0, \infty)$ denoted by $LM_{p\theta}^{w(\cdot)}$ Morrey-type local space. This is the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm:

$$\|f\|_{LM_{p\theta}^{w(\cdot)}} \equiv \|f\|_{LM_{p\theta}^{w(\cdot)}(R^n)} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)} < \infty$$

If $w(r) = r^{-\lambda - \frac{1}{\theta}}$, then $LM_{p\theta}^{w} = LM_{p\theta}^{\lambda}$

$$\|f\|_{LM_{p\theta}^{\lambda}} = \left(\int_0^\infty \left(r^{-\lambda} \|f\|_{L_p(B(0,r))} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty$$

(V.S. Guliyev 1994 *Integral operators in spaces of functions on homogeneous groups and on domains on* R^n doctoral dissertation)

Global Morrey-type spaces

Let $0 < p, \theta \leq \infty$, and let w be a non-negative measurable function on $(0, \infty)$ denoted by $GM_{p\theta}^{w(\cdot)}$ The global Morrey space $GM_{p\theta}^{w(\cdot)} \equiv GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ is defined as set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{GM_{p\theta}^{w(\cdot)}} \equiv \sup_{x\in\mathbb{R}^n} \left\| w(r) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)} < \infty$$

If $\theta = \infty, w(r) \equiv 1$, then $LM_{p\infty}^1 \equiv GM_{p\infty}^1 \equiv L_p$

If $\theta = \infty, w(r) \equiv r^{-\lambda}$, then $GM_{p\infty}^{r^{-\lambda}} \equiv M_p^\lambda$

(V.I. Burenkov, H.V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces, Studia Mathematica 163 (2004), no. 2, 157-176.)

Let $0 < p, \theta \leq \infty$. Denote by Ω_θ the set of all functions that are non-negative, measurable on $(0, \infty)$, not equivalent to 0, and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

The space $LM_{p\theta}^{w(\cdot)}$ is non-trivial if and only if $w \in \Omega_\theta$

Denote by $\Omega_{p\theta}$ the set of all functions that are non-negative, measurable on $(0, \infty)$, not equivalent to 0, and such that for some $t > 0$

$$\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty, \quad \|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

The space $GM_{p\theta}^{w(\cdot)}$ is non-trivial if and only if $w \in \Omega_{p\theta}$

Theorem *(Frechet-Kolmogorov)* Let $1 \leq p < \infty$. The set $S \subset L_p(\mathbb{R}^n)$ is precompact if and only if

1. $\sup_{f \in S} ||f||_{L_p(\mathbb{R}^n)} < \infty$
2. $\lim_{u \to 0} \sup_{f \in S} ||f(\cdot + u) - f(\cdot)||_{L_p(\mathbb{R}^n)} = 0$
3. $\lim_{R \to \infty} \sup_{f \in S} ||f||_{L_p({}^cB(0,R))} = 0$

Theorem *(Frechet-Kolmogorov)* Let $1 \leq p < \infty$. The set $S \subset L_p(\mathbb{R}^n)$ is precompact if and only if

1. $\sup\limits_{f \in S} ||f||_{L_p(\mathbb{R}^n)} < \infty$
2. $\lim\limits_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0$,
3. $\lim\limits_{R \to \infty} \sup\limits_{f \in S} ||f||_{L_p({}^c B(0,R))} = 0$

where for any $\delta > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$

$$(A_\delta f)(x) = \frac{1}{|B(x,\delta)|} \int\limits_{B(x,\delta)} f(y) dy$$

Теорема 1. Пусть $1 \le p \le \infty$ и $w \in \Omega_{p\infty}$. Предположим, что множество $S \subset M_p^{w(\cdot)}$ удовлетворяет следующим условиям:

$$\sup_{f \in S} \|f\|_{M_p^{w(\cdot)}} < \infty, \tag{1}$$

$$\lim_{u \to 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}} = 0, \tag{2}$$

$$\lim_{r \to \infty} \sup_{f \in S} \left\| f \chi_{{}^c B(0,r)} \right\|_{M_p^{w(\cdot)}} = 0. \tag{3}$$

Тогда S является предкомпактным множеством в $M_p^{w(\cdot)}$.
Где ${}^c B(0,r)$ дополнение шара $B(0,r)$ с центром в точке 0, и радиусом r.
$w(r) = r^{-\lambda}$
Y. Chen, Y. Ding, X. Wang, Potential Anal. (2009), no. 4, 301-313.
Y. Chen, Y. Ding, Canad. J. Math. (2012) no. 2, 257-281.
$w(r) = 1$ Фреше-Колмогоров

Замечание. Отметим, что условие (1) в теореме является необходимым, так как любое предкомпактные множество в нормированном пространстве является ограниченным.
Что касается условий (2) и (3), то они не являются необходимыми.
Существует пример множества S из M_p^λ которое является предкомпактным но для него условия (2) и (3) не выполняются.
Пример: Множество S, состоящее только из одной функции
$f_0(x) = |x|^{\lambda - \frac{1}{p}} \in M_p^\lambda$ предкомпактно, но условия (2) и (3) не выполняются.

Теорема 2. Пусть $1 < p < \infty$, $0 < \theta < \infty$, $w \in \Omega_\theta$ и выполнено условие $t^{-\frac{n}{p}} \left\| w(r) r^{\frac{n}{p}} \right\|_{L_\theta(0,t)} \le c_1 \|w(r)\|_{L_\theta(t,\infty)}$. Для того, чтобы множество $S \subset LM_{p\theta}^{w(\cdot)}$ было предкомпактным в $LM_{p\theta}^{w(\cdot)}$ необходимо и достаточно, чтобы

$$\sup_{f \in S} \|f\|_{LM_{p\theta}^{w(\cdot)}} < \infty, \tag{4}$$

$$\lim_{R_1 \to 0^+} \sup_{f \in S} \left\| f \chi_{B(0,R_1)} \right\|_{LM_{p\theta}^{w(\cdot)}} = 0, \tag{5}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(B(0,R_2) \setminus B(0,R_1))} = 0 \tag{6}$$

и

$$\lim_{R_2 \to \infty} \sup_{f \in S} \left\| f \chi_{{}^c B(0,R_2)} \right\|_{LM_{p\theta}^{w(\cdot)}} = 0. \tag{7}$$

Если $\theta = \infty$, тогда условия (7)- (10) будут достаточными условиями для предкомпактности множеств $S \subset LM_{p\theta}^{w(\cdot)}$.

Theorem 3.
Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $w \in \Omega_{p\theta}$, and $S \subset GM_{p\theta}^{w(\cdot)}$. If

$$\sup_{f \in S} \|f\|_{GM_{p\theta}^{w(\cdot)}} < \infty, \tag{8}$$

$$\lim_{R_1 \to 0^+} \sup_{f \in S} \left\| f \chi_{B(0,R_1)} \right\|_{GM_{p\theta}^{w(\cdot)}} = 0, \tag{9}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(B(0,R_2) \setminus B(0,R_1))} = 0 \tag{10}$$

and

$$\lim_{R_2 \to \infty} \sup_{f \in S} \left\| f \chi_{{}^c B(0,R_2)} \right\|_{GM_{p\theta}^{w(\cdot)}} = 0. \tag{11}$$

Then the set S is precompact in $GM_{p\theta}^{w(\cdot)}$

Theorem 4. Let $1 \leq p \leq \theta \leq \infty$ and $w \in \Omega_{p\theta}$. Assume that the set $S \subset GM_{p\theta}^{w(\cdot)}(R^n)$ satisfies the following conditions:

$$\sup_{f \in S} \|f\|_{GM_{p\theta}^{w(\cdot)}} < \infty,$$

$$\lim_{u \to 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} = 0$$

и

$$\lim_{r \to \infty} \sup_{f \in S} \left\| f \chi_{^{c}B(0,r)} \right\|_{GM_{p\theta}^{w(\cdot)}} = 0$$

Then S is a precompact set in $GM_{p\theta}^{w(\cdot)}(R^n)$.

Riesz potential I_α

Riesz potential I_α of order $\alpha (0 < \alpha < n)$ is defined as follows

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

Given a function $b \in L_{loc}(\mathbb{R}^n)$, denote by M_b the multiplication operator $M_b f = bf$, where f is a measurable function. Then the commutator for the Riesz potential I_α and the operator M_b is defined by the equality

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \frac{[b(x)-b(y)]\, f(y)}{|x-y|^{n-\alpha}} dy.$$

A function $b(x) \in L_\infty(\mathbb{R}^n)$ belongs to the space $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int \ limits_Q |b(x) - b_Q| \, dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty,$$

where Q is a cube from $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|} \int\limits_Q f(y) dy$.

Denote by $VMO(\mathbb{R}^n)$ the BMO-closure of the space $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support.

Theorem A *(V. S. Guliyev 2012)* Let
$1 < p < \infty$,$0 < \alpha < n(1 - \frac{1}{q})$, $1 < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$,
$b \in BMO(\mathbb{R}^n)$ и (w_1, w_2) satisfy the following condition

$$\int_r^\infty \frac{ess \inf_{t<s<\infty} w_1(s) dt}{t} \leq C w_2(r). \qquad (12)$$

where C does not depend on r.
Then $[b, I_\alpha]$ is bounded in $M_p^{w_1}(\mathbb{R}^n)$ to $M_q^{w_2}(\mathbb{R}^n)$.

Theorem 5. *Let* $0 < \alpha < n(1 - \frac{1}{q})$, $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in VMO(\mathbb{R}^n)$, *function* $w_1, w_2 \in \Omega_{p,\infty}$ *satisfy the condition*

$$\int_r^\infty \frac{ess \inf_{t<s<\infty} w_1(s) dt}{t} \leq C w_2(r),$$

then the commutator $[b, I_\alpha]$ *is a compact operator from* $M_p^{w_1}$ *в* $M_q^{w_2}$.

Theorem 6. Let $1 \le p_1 < p_2 < \infty$, for $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$, or $1 \le \; p_1 < \infty$, $1 \le p_2 < \infty$, for $n(\frac{1}{p_1} - \frac{1}{p_2}) < \alpha < \frac{n}{p_1}$,
$0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le 1$, $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, and let

$$\left\| w_2(r) \frac{\frac{r}{p_2}}{(t+r)^{\frac{n}{p_1}-\alpha}} \right\|_{L_{\theta_2(0,\infty)}} \le c \, \| w_1(r) \|_{L_{\theta_1(t,\infty)}}$$

for all $t > 0$, where $c > 0$ do not depend on t, and $b \in VMO(\mathbb{R}^n)$. Moreover, Let the commutator $[b, I_\alpha]$ be bounded from $GM^{w_1}_{p_1\theta_1}$ to $GM^{w_2}_{p_2\theta_2}$. Then the commutator $[b, I_\alpha]$ is compact from $GM^{w_1}_{p_1\theta_1}$ в $GM^{w_2}_{p_2\theta_2}$

Publications in international journals:

1. Bokayev N., Burenkov, V. Matin D.T., On pre-compactness of a set in general local and global Morrey-type spaces,Astana, EURASIAN MATHEMATICAL JOURNAL , 2017 Volume 8 no. 3 -C.109-115
2. Bokayev N., Burenkov, V. Matin D.T., On the pre-compactness of a set in the generalized Morrey spaces, AIP Conference Proceedings 1759, 020108 (2016); doi: 10.1063/1.4959722
3. Bokayev N., Burenkov, V. Matin D.T., Sufficient conditions for the pre-compactness of sets in global Morrey-type spaces, AIP Conference Proceedings 1880, 030001 (2017); doi: 10.1063/1.5000600
4. Bokayev N., Burenkov, V. Matin D.T., On pre-compactness of a set in general local and global Morrey-type spaces, International conference on "OPERATORS IN MORREY-TYPE SPACES AND APPLICATIONS"Dedicated to 60th Birthday of Professor Vagif S. Guliyev-Kirsehir (TURKEY), 2017.-C.76-77.

Publications in publications recommended by the Committee for Control in the Sphere of Education and Science of the Ministry of Education of the Republic of Kazakhstan:

5. Bokayev N., Burenkov, V. Matin D.T., О достаточном условии предкомпактности множеств в обобщенных пространствах Морри, Вестник. Серия математика. -Караганда: КарГУ, 2016. no 4.(84).-C.18-26
6. N. Bokayev , D.T. Matin, О достаточных условиях компактности коммутатора для потенциала Рисса в обобщенных пространствах Морри, Вестник. Серия естественно-технических наук. -Астана: ЕНУ, 2016. no 6. (115) I часть.-C.8-13
7. Matin D.T., Предкомпактность множеств в глобальных пространствах типа Морри, Вестник. Серия естественно-технических наук. -Астана: ЕНУ, 2017. no 4. (119) I часть.-C.14-23

PAXMET!