Cones of monotone functions generated by a generalized fractional-maximal function

N.A. Bokayev,A.Gogatishvili,A.N. Abek.

Scientific seminar of the Institute of Applied Mathematics at Baku State University ,
December 9, 2022, Baku

09.12.2022

Abstract
In this paper, we define the function spaces generated by a generalized fractional maximal function. The questions of the embedding such spaces in rearrangement invariant spaces are considered. Estimates are obtained for a non-increasing rearrangement of the generalized fractional maximal function. Various cones of the non-increasing rearrangement of a generalized fractionally maximal function are defined and conditions for their mutual covering are found.

Цель работы:

1. Define a generalized fractional-maximal function $(M_{\Phi}f)(x)$;
2.Obtain the estimates for a non-increasing rearrangement of generalized fractional-maximal functions;
3.Construction of various cones of monotone functions generated by a non-increasing rearrangement of a generalized fractional-maximal function;
4. Consider the question of mutual coverings of such cones;
5. On embedding of generalized fractional-maximal function spaces in rearrangement invariant space

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Functional norm

A mapping $\rho : L_0^+ \to [0,\infty]$ is called a *functional norm* (short: FN), if the next conditions are met for all $f, g, f_n \in L_0^+, \quad n \in N$:

$(P1) \rho(f) = 0 \Rightarrow f = 0$, $\mu-$ almost everywhere (briefly: $\mu-$ a.e.);
$\rho(\alpha f) = \alpha \rho(f), \alpha \geq 0; \rho(f+g) \leq \rho(f) + \rho(g)$ (properties of the norm);

$(P2) f \leq g$, $(\mu-$ a.e.$) \Rightarrow \rho(f) \leq \rho(g)$ (monotony of the norm);

$(P3) f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f) (n \to \infty)$ (the Fatou property);

$(P4) 0 < \mu(\sigma) < \infty \Rightarrow \int\limits_\sigma f d\mu \leq c_\sigma \rho(f), f \in L_0^+.$;

$(P5) 0 < \mu(\sigma) < \infty \Rightarrow \rho(\chi_\sigma) < \infty$ (finiteness of the FN for characteristic functions (χ_σ) of sets of finite measure).

Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}, \lim\limits_{n\to\infty} f_n = f$ $(\mu-$ a.e.)

C.Bennett, R.Sharpley [1], as well as the concepts of an ideal space (briefly: IS) considered in the book by S.G.Crane, Yu.I.Petunin and E.M.Semenov [2].

Banach function space

Let ρ there be a functional norm. The set of functions $X = X(\rho)$ from L_0, for which $\rho(|f|) < \infty$ is called a *Banach function space* (briefly: BFS), generated by the FN ρ. For $f \in X$ we assume

$$\|f\|_X = \rho(|f|).$$

Example 1.1. Let $S = R^n$, $\mu \equiv \mu_n$ - be the Lebesgue measure in R^n, $1 \leq p \leq \infty$; $u \in L_0(R^n)$, $0 < u < \infty$, (μ-a.e.); $u \in L_p^{loc}(R^n)$, $\frac{1}{u} \in L_{p'}^{loc}(R^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$. The space $X = L_{p,u}(R^n)$ with a norm $f_X = fu_{L_p}$ i.e.:

$$\|f\|_X = \left(\int\limits_{R^n} |fu|^p d\mu \right)^{\frac{1}{p}}, \ 1 \leq p < \infty; \ \|f\|_X = \|fu\|_{L_\infty}, \ p = \infty$$

is a BFS.

Non-increasing rearrangement of function

Let $L_0 = L_0(R^n)$ - be the set of all Lebesgue measurable functions $f : R^n \to C$; $\dot{L}_0 = \dot{L}_0(R^n)$ - the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* defined by the equality:

$$f^*(t) = inf\{y \in [0;\infty) : \ \lambda_f(y) \le t\}, \ \ t \in R_+ = (0;\infty) \quad (1.3)$$

where

$$\lambda_f(y) = \mu_n \{x \in R^n : \ |f(x)| > y\}, \quad y \in [0,\infty) \quad (1.4)$$

-the Lebesgue distribution function.

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t} \int\limits_0^t f^*(\tau) d\tau; \ \ t \in R_+ \quad (1.7)$$

Rearrangement invariant space

A FN ρ is *rearrangement-invariant* if

$$f^* \leq g^* \Rightarrow \rho(f) \leq \rho(g). \qquad (1.11)$$

Banach function space (BFS) $X = X(\rho)$, generated by a rearrangement invariant functional norm (FN) ρ, will be called a *rearrangement invariant space* (in short: RIS).
Examples of RIS are Lebesgue spaces $L_p(R^n)$, Lorentz spaces, and Orlich spaces.

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Symmetrical rearrangement of function

Let $f^{\#}:\mathbb{R}^n\to\mathbb{R}^n$ denote a symmetrical rearrangement of f, i.e. a radially symmetric nonnegative decreasing right continuous function (as a function of $\rho=|x|$, $x\in\mathbb{R}^n$) that is equimeasurable with f. That is

$$f^{\#}(\rho)=f^*(v_n\rho^n);\ f^*(t)=f^{\#}\left(\left(\frac{t}{v_n}\right)^{\frac{1}{n}}\right),\ \rho,t\in\mathbb{R}_+. \qquad (1.5)$$

Here v_n is the volume of the n-dimensional unit ball.

Definition 1.6. For functional norm ρ, we introduce the *associated norm* $\rho^{'}$:

$$\rho^{'}(g) = sup\left\{ \int\limits_{S} fgd\mu : f \in L_0^+, \rho(f) \leq 1 \right\}, \ \ g \in L_0^+. \qquad (1.13)$$

Theorem 1.1. Let ρ be FN, $\rho^{'}$ - the associated norm,

$$X^{'} = X(\rho^{'}) = \{g \in L_0 : \rho^{'}(|g|) < \infty\} \qquad (1.14)$$

then $\rho^{'}$ is the FN; $X^{'}$ is the BFS with the norm

$$\|g\|_{X'} = \rho^{'}(|g|), \qquad (1.15)$$

moreover, the spaces X and $X^{'}$ are Banach spaces and the duality principle is valid for them

$$X^{''} = (X^{'})' = X, \qquad (1.16)$$

Everywhere in this work, we denote rearrangement invariant space (in short: RIS) by $E = E(R^n)$, and by $E^{'} = E^{'}(R^n)$ the associated rearrangement-invariant space and $\tilde{E} = \tilde{E}(R_+)$, $\tilde{E}^{'} = \tilde{E}^{'}(R_+)$ their Luxembourg representation, i.e. such RIS that

$$\|f\|_E = \|f^*\|_{\tilde{E}}, \quad \|g\|_{E'} = \|g^*\|_{\tilde{E}'}, \tag{1.17}$$

2. The generalized fractional-maximal function and estimate of its non-increasing rearrangement.

Let E be RIS and $\Phi : R_+ \to R+)$, $\Phi \in \ A_n(R)$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in E(\mathbb{R}^n) \cap L_1^{loc}(\mathbb{R}^n)$ by the equality

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} f(y)dy, \qquad (2.8)$$

where $B(x,r)$ is a ball with the center at the point x and radius r. That is, consider the operator M_Φ: $L_1^{loc}(R^n) \to \dot{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$ we obtain the classical fractional-maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{B(x,r)} |f(y)|dy. \qquad (2.9)$$

$$(M_\rho f)(x) = \sup_{r>0} \frac{\rho(r)}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \qquad (2.)$$

where $\rho : R+ \to R+$

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Hakim D.I., Nakai E., Sawano Y. Generalized fractional maximal operators and vector-valued inequalities on generalized Orlicz-Morrey spaces // Rev Mat Complut, 2015

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We denote by $M_E^\Phi = M_E^\Phi(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = inf\{\|f\|_E : f \in E(R^n), \;\; M_\Phi f = u\} \qquad (2.10)$$

Note that in the works of M.I. Goldman, E.G. Bakhtigareeva [3, 4, 6], the generalized Riesz potential was considered using the convolution operator:

$$A : E_1(R^n) \to \dot{L}_0(R^n), \qquad (2.11)$$

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int\limits_{R^n} G(x-y)f(y)dy, \qquad (2.12)$$

where the kernel $G(x)$ satisfies the conditions:

$$G(x) \cong \Phi(|x|), \;\; x \in R^n \qquad (2.13)$$

$$\Phi \in \Im_n(\infty); \;\; \exists c \in R_+.$$

The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha - n}, \;\; \alpha \in (0; n).$$

DEFINITION 1. Let $R \in (0;\infty]$. Let $\Phi : (0;R) \to R_+$. We say that the function Φ belongs to the class $A_n(R)$ if:
(1) Φ decreases and is continuous on $(0;R)$;
(2) $\Phi(r)r^n \uparrow, \quad r \in (0,R)$.
The function Φ belongs to the class $B_n(R)$ if the following conditions hold:
(1) Φ decreases and is continuous on $(0;R)$;
(2) There exists a constant $C \in R_+$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \leq C\Phi(r)r^n.$$

The function Φ belongs to the class $E_n(R)$ if

$$\int_0^{r^n} \frac{ds}{\Phi(s^{1/n})s} \leq \frac{C}{\Phi(r)}, r \in (0;R).(2.7)$$

For example, $\Phi(t) = t^{-\alpha} \in A_n(\infty), \quad 0 < \alpha < n$

Note that relation (2.7) is equivalent to the inequality (can be obtained by a change of variables):

$$\int_0^r \frac{dt}{\Phi(t)t} \leq \frac{C}{\Phi(r)}. \qquad (2.7')$$

For example the function $\Phi(t) = t^{\alpha-n} \in A_n(\infty)$ $(0 < \alpha < n)$. The function $\Phi(t) = t^{-n} ln(1+t)^{\alpha} \in D_n(\infty)(\alpha > 0)$ and $\Phi(t) = t^{\alpha-n} \in D_n(\infty)$ $(0 < \alpha < n)$.thbbR$_+$.

Lemma 2.1. $E_n(R) \subset B_n(R) \subset A_n(R)$.

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Lemma 2.3. Let $\Phi \in \ B_n(0, \infty)$ and the kernel G defined by (2.14). Then the following inequality holds:

$$(M_\Phi f)(x) \leq (G * f)(x), \quad x \in R^n. \tag{2.14}$$

2. The estimates of non-increasing rearrangement of generalized fractional-maximal function

Theorem 2.1. Let $\Phi \in A_n(\infty)$. Then there exist a positive constant C, depending from n such that

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty), \qquad (2.15)$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Theorem 2.2. Let $\Phi \in A_n(\infty)$. Inequality (2.15) is sharp in the sense that for every $\varphi \in L_0^+(0,\infty;\downarrow)$ there exists a function $f \in L^+(\mathbb{R}^n)$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\Phi f)^*(t) \ge C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty) \qquad (2.23)$$

where, C is a positive constant which depends only on n.

In the case $\Phi(r) = r^{\alpha-n}$, $0 < \alpha < n$ for M_α Theorems 2.1 and 2.2 were proved by A. Cianchi, R. Kerman, B. Opic, L. Pick *A sharp rearrangement inequality for the fractional maximal operator.* Studia Mathematica. 138 (2000), no. 3., 277–284

Theorem 2.3. Let $\Phi \in \ B_n(\infty)$. Then there exist a positive constant C, depending from n such that

$$(M_\Phi f)^{**}(t) \leq C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0, \infty), \qquad (2.28)$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Theorem 2.4.
Let $\Phi \in D_n(\infty)$, then there is an inequality

$$(M_\Phi f)^*(t) \leq C \left(t\Phi(t^{1/n}) f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) f^*(\tau) \right), \quad t \in (0, \infty \qquad (1)$$

N.A. Bokayev, A. Gogatishvili,A.N. Abek,On estimates of non-increasing rearrangements of generalized fractional-maximal functions. Eurasian Math. Journal. 2022. Vol. 14, No. 2, pp. 8Ц18. (in print)

3. Cones generated by a generalized fractional maximal function.

Definition 3.1. Define $\Im_T = \{K(T)\}$ for $T \in (0, \infty]$ as a set of cones considering from measurable non-negative functions on $(0, T)$, equipped with positive homogeneous functionals $\rho_{KM(T)} : K(T) \to [0, \infty)$ with properties:
(1) $h \in K(T)$, $\alpha \geq 0 \Rightarrow \alpha h \in K(T)$, $\rho_{K(T)}(\alpha h) = \alpha \rho_{K(T)}(h)$;
(2) $\rho_{K(T)}(h) = 0 \Rightarrow h = 0$ almost everywhere on $(0, T)$.

Definition 3.2 [4]. Let $K(T), M(T) \in \Im_T$. The cone $K(T)$ *covers the cone* $M(T)$ (notation: $M(T) \prec K(T)$) if there exist $C_0 = C_0(T) \in R_+$, and $C_1 = C_1(T) \in [0, \infty)$ with $C_1(\infty) = 0$ such that for each $h_1 \in M(T)$ there is $h_2 \in K(T)$ satisfying

$$\rho_{K(T)}(h_2) \leq C_0 \rho_{M(T)}(h_1), \; h_1(t) \leq h_2(t) + C_1 \rho_{M(T)}(h_1), \; t \in (0, T). \tag{3.1}$$

The equivalence of the cones means mutual covering:

$$M(T) \approx K(T) \Leftrightarrow M(T) \prec K(T) \prec M(T). \tag{3.2}$$

Let E is rearrangement-invariant space (briefly: RIS). We consider the following four cones of decreasing rearrangements of generalized fractional maximal functions equipped with homogeneous functionals, respectively:

$$K_1 \equiv KM_E^{\Phi} = \{h \in L^+(\mathbb{R}_+) : h(t) = u^*(t),\ t \in \mathbb{R}_+,\ u \in M_E^{\Phi}\}, \tag{3.3}$$

$$\rho_{K_1}(h) = \inf\{\|u\|_{M_E^{\Phi}} : u \in M_E^{\Phi};\ u^*(t) = h(t),\ t \in \mathbb{R}_+\}; \tag{3.4}$$

$$K_2 \equiv K\widetilde{M}_E^{\Phi} = \{h : h(t) = u^{**}(t),\ t \in \mathbb{R}_+,\ u \in M_E^{\Phi}\}, \tag{3.5}$$

$$\rho_{K_2}(h) = \inf\{\|u\|_{M_E^{\Phi}} : u \in M_E^{\Phi};\ u^{**}(t) = h(t),\ t \in \mathbb{R}_+\}. \tag{3.6}$$

This means that the cones K_1 and K_2 consist of non-increasing rearrangements of generalized fractional maximal functions.

$$K_3 = \widetilde{K}_E^{\Phi} = \Big\{ h \in L^+(\mathbb{R}_+) : h(t) = \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) u^{**}(\tau), \ t \in \mathbb{R}_+, u \in \tag{3.7}$$

$$\rho_{K_3}(h) = inf \Big\{ \|u\|_{E(\mathbb{R}^n)}, u \in E(\mathbb{R}^n) : h(t) = \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) u^{**}(\tau), t \in \tag{3.8}$$

$$K_4 = \widetilde{\widetilde{K}}_E^{\Phi} = \Big\{ h \in L^+(\mathbb{R}_+) : h(t) = t\Phi(t^{1/n}) u^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) u^{*}(\tau), t \in \mathbb{R}_+, \ u \in E(\mathbb{R}^n) \Big\}, (3.9)$$

$$\rho_{K_4}(h) = inf \Big\{ \|u\|_{E(\mathbb{R}^n)}, u \in E(\mathbb{R}^n) : h(t) = t\Phi(t^{1/n}) u^{**}(t) +$$

$$+ \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) u^{*}(\tau), \ t \in \mathbb{R}_+ \Big\}. (3.10)$$

Note that in the works of Goldman M.L. [4], cones generated by generalized potentials are considered (also see [3]-[6], [12], [13]). They study the space of potentials $H_E^G \equiv H_E^G(\mathbb{R}^n)$ in n-dimensional Euclidean space:

$$H_E^G(\mathbb{R}^n) = \{u = G * f : f \in E(\mathbb{R}^n)\}, \qquad (3.11)$$

where $E(\mathbb{R}^n)$ is an rearrangement invariant space (RIS).

$$\|u\|_{H_E^G} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n); G * f = u\}, \qquad (3.12)$$

$$M(T) \equiv M_E^G(T) = \{h(t) = u^*(t), t \in (0;T), u \in H_E^G\},$$

$$\rho_{M(T)}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^*(t) = h(t), t \in (0;T)\};$$

$$\widetilde{M}(T) \equiv \widetilde{M}_E^G(T) = \{h(t) = u^{**}(t), t \in (0;T) : u \in H_E^G\},$$

$$\rho_{\widetilde{M}}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^{**}(t) = h(t)), t \in (0;T)\}.$$

In [4] was also considered the cone

$$K(T) = K_E^\Phi(T) = \left\{h(g;t) = h(t) = \int_0^T f_\Phi(t;\tau)g(\tau)d\tau, t \in \mathbb{R}_+ : g \in$$

equipped with a functional

Theorem 3.1. The cone generated by the generalized potential covers the cone generated by the generalized maximal function, i.e. $KM_E^{\Phi} \prec KM_E^{G}$.

Theorem 3.2. Let $\Phi \in B_n(\infty)$. Then

$$K_4 \prec K_3 \tag{3.19}$$

Theorem 3.3. Let $\Phi \in B_n(\infty) \cap A_n$. Then

$$K_3 \prec K_4. \tag{3.23}$$

Theorem 3.4. Let $\Phi \in B_n(\infty)$. Then

$$KM_E^{\Phi} \approx K\widetilde{M}_E^{\Phi} \approx \widetilde{K}_E^{\Phi}. \tag{3.28}$$

Theorem 3.5. Let $\Phi \in E_n(\infty)$. Then

$$K_1 \approx K_2 \approx K_3 \approx K_4. \tag{3.42}$$

Definition 4.1 [5]. The embedding $M(T) \hookrightarrow \widetilde{X}(0,T)$ means that $M(T) \subset \widetilde{X}(0,T)$ and there exists $C_{M(T)}(T) \in R_+$ such that

$$\|h\|_{\widetilde{X}(0,T)} \leq C_{M(T)}\rho_{M(T)}(h), \quad h \in M(T) \tag{4.1}$$

Lemma 4.1 (see[5,13]). Let $K(T), M(T) \in \Im_T$. For every BFS $\widetilde{X}(0,T)$ if $M(T) \prec K(T)$ then
$K(T) \hookrightarrow \widetilde{X}(0,T) \Rightarrow M(T) \hookrightarrow \widetilde{X}(0,T)$, and (see (3.1))

$$C_{M(T)} \leq C_0 C_{K(T)} + C_1 \|1\|_{\widetilde{X}(0,T)} \tag{4.2}$$

Let us formulate criteria for embedding of generalized fractional maximal functional spaces in rearrangement invariant spaces $X(\mathbb{R}^n)$:

$$M_E^{\Phi}(\mathbb{R}^n) \subset X(\mathbb{R}^n). \tag{4.3}$$

Theorem 4.1. Let $\Phi \in B_n(\infty)$. The embedding

$$M_E^{\Phi}(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n) \tag{4.7}$$

is equivalence to the next embedding

$$K_1 M_E^{\Phi}(\mathbb{R}_+) \mapsto \widetilde{X}(\mathbb{R}_+) \tag{4.8}$$

$$M_E^{\Phi}(\mathbb{R}^n) \subset X(\mathbb{R}^n). \tag{4.3}$$

$$\Leftrightarrow K_3 \hookrightarrow \widetilde{X}(\mathbb{R}_+). \tag{4.4}$$

Definition 4.2.[17]. Let E - RIS. By an optimal RIS we mean an RIS $X_0 = X_0(R^n)$ such that $M_E^{\Phi}(\mathbb{R}^n) \subset X(\mathbb{R}^n)$ holds for $X = X_0$ and if (4.3) is valid for another RIS X, then $X_0 = X$. Such an optimal RIS is called a rearrangement-invariant envelope of the space of potentials.

Theorem 4.3. Let $\Phi \in B_n(\infty)$. The embedding

$$M_E^{\Phi}(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n) \tag{4.7}$$

is equivalence to the next embedding

$$K_1 M_E^{\Phi}(\mathbb{R}_+) \mapsto \widetilde{X}(\mathbb{R}_+) \tag{4.8}$$

Theorem 4.5. Let $\Phi \in B_n(\infty)$. The optimal RIS $X_0 = X_0(\mathbb{R}^n)$ for embedding

$$M_E^{\Phi}(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$$

is defined by following norm:

$$\|f\|_{\widetilde{X}_0(0,\infty)} =$$

$$\sup_{g^*} \left\{ \int_0^\infty f^*(\tau) g^*(\tau) d\tau : g \in L_0(0,\infty), \sup_{\int_0^t h(s)ds \le \int_0^t g^*(s)ds} \left\| \int_t^\infty \Phi(s^{1/n}) s h(s) ds \right\|_{E'} \le 1 \right\}. (4.16)$$

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Спасибо за внимание!