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SOME INTEGRAL FORMULAS FOR CLOSED MINIMALLY IMMERSED HYPERSURFACE IN THE UNIT SPHERE S^{n+1}

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Abstract. In this paper, we obtain some integral formulas for minimal hypersurfaces in the unit sphere S^{n+1}.

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1. Introduction

The Clifford torus is the only minimal surface in S^3 with constant contact angle. The study of minimal surfaces has played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds [6].

Many works have been done related to integral formulas by many mathematicians (see [1], [3], [4] and [7]). For example, Cao [1] obtained integral formulas for minimal space-like hypersurfaces in (n+1)-dimensional indefinite space form. In addition, Ximin [7] gave similar integral formulas for minimal space-like hypersurfaces in (n+p)-dimensional indefinite space form.

Later, significant works in this direction have been obtained by Külahcı, Ergüt and Bektaş [3,4].

In this paper, we conduct a study about minimal hypersurfaces in the unit sphere S^{n+1}. However, to the best of our knowledge, these integral formulas have not been presented for closed minimally immersed hypersurface in the unit sphere S^{n+1}. Thus, the study is proposed to serve such a need.

2. Preliminaries

Let M be an n-dimensional hypersurface in a unit sphere S^{n+1}. We choose a local orthonormal frame field in $\{e_1, ..., e_{n+1}\}$ in S^{n+1}, so that, restricted to M, $e_1, ..., e_n$ are tangent to M. Let $w_1, ..., w_{n+1}$ denote the dual co-frame field in S^{n+1}. Then, in M

$$w_{n+1} = 0.$$

It follows from Cartan's Lemma that

$$w_{n+1,i} = \sum_j h_{ij} w_j, \qquad h_{ij} = h_{ji}. \tag{1}$$

The second fundamental form h and the mean curvature H of M are defined by

$$h = \sum_{i,j} h_{ij} w_i w_j e_{n+1} \qquad \text{and} \qquad H = \sum_i h_{ii}. \tag{2}$$

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We recall that M is by definition a minimal hypersurface if the mean curvature of M is identically zero. The connection form w_{ij} is characterized by the structure equations

$$\begin{cases} dw_i + \sum\limits_j w_{ij} \wedge w_j = 0, \ w_{ij} + w_{ji} = 0, \\ dw_{ij} + \sum\limits_k w_{ik} \wedge w_{kj} = \Omega_{ij} \\ \Omega_{ij} = \frac{1}{2} \sum\limits_{kl} R_{ijkl} \ w_k \wedge w_l, \end{cases} \tag{3}$$

where Ω_{ij}(resp.R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M. The Gauss equation is given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{4}$$

The covariant derivative $\bigtriangledown h$ of the second fundamental form h of M with components h_{ijk} is given by

$$\sum_k h_{ijk} w_k = dh_{ij} + \sum_k h_{jk} w_{ik} + \sum_k h_{ik} w_{jk}.$$

Then the exterior derivative of (1) together with the structure equations yield the following Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{5}$$

Similarly, we have the covariant derivative $\bigtriangledown^2 h$ of $\bigtriangledown h$ with components h_{ijkl} as follows

$$\sum_l h_{ijkl} w_l = dh_{ijk} + \sum_l h_{ljk} w_{il} + \sum_l h_{ilk} w_{jl} + \sum_l h_{ijl} w_{kl}$$

and it is easy to get the following Ricci formula

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl}. \tag{6}$$

From now on, we assume that M is minimal. Denote by $S = \sum\limits_{i,j} h_{ij}^2$ the square of length of h. The components of the Ricci curvature and the scalar curvature are given respectively by

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik} h_{jk}, \tag{7}$$

$$R = n(n-1) - S. \tag{8}$$

It follows from (8) that S is constant if and only if R is constant. For any fixed point p in M, we can choose a local orthonormal frame field $e_1, ..., e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{9}$$

Let $S = \sum\limits_{i,j} h_{ij}^2$. The following formulas can be obtained by a direct computation

$$\Delta h_{ij} = (n - S) h_{ij}, \tag{10}$$

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S-n). \tag{11}$$

The Gauss-Kronecker curvature K of M is defined by

$$K = \det(h_{ij}). \tag{12}$$

Let M be an n-dimensional closed minimally immersed hypersurface in the unit sphere S^{n+1}. Assume in addition that M has constant scalar curvature or constant Gauss-Kronecker curvature. In this paper we announce that if M has $(n-1)$ principal curvatures with the same sign everywhere, then M is isometric to a Riemannian product $S^1\left(\sqrt{\frac{1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$. This Riemannian product also correspond to Clifford Torus [5].

From now on, we assume that M is a Riemannian product.

Theorem 2.1. Let M be an n-dimensional compact Riemannian product $S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$, then

$$\int_M \left\{-\frac{1}{2}\sum R^2_{mijk}+\sum R^2_{mj}+nR\right\}dV\leq 0, \tag{13}$$

where $\sum R^2_{mijk}$ is the square length of the sectional curvature, $\sum R^2_{mj}$ is the square length of the Ricci curvature tensor, R is the scalar curvature, dV is the volume element of M.

Theorem 2.2. Let M be an n-dimensional compact Riemannian product $S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$, then

$$\int_M \left\{-\frac{1}{2}\sum R^2_{mijk}+\sum R^2_{mj}\right\}dV\leq \left\{n^2(n-1)-nS\right\}Vol(M), \tag{14}$$

where $\sum R^2_{mijk}$ is the square length of the sectional curvature, $\sum R^2_{mj}$ is the square length of the Ricci curvature tensor, S is the square of length of second fundamental form, dV is the volume element of M.

Theorem 2.3. Let M be an n-dimensional compact Riemannian product $S^1\left(\sqrt{\frac{1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$, then

$$\int_M \left\{-\frac{1}{2}\sum R^2_{mijk}-(3n-2)S+\frac{1}{n}S^2\right\}dV\leq n(-2n^2+3n-1)Vol(M), \tag{15}$$

where $\sum R^2_{mijk}$ is the square length of the sectional curvature, S is the square of length of second fundamental form, dV is the volume element of M.

3. Proof of theorems

Proof of Theorem 2.1. From the definition of Laplacian we have

$$\Delta h_{ij}=\sum h_{im}R_{mkjk}+\sum h_{km}R_{mijk}. \tag{16}$$

From (4),(16) and taking into consideration M is minimal , we get

$$\sum h_{ij}\Delta h_{ij}=\sum h_{ij}h_{mk}R_{mijk}+\sum h_{ij}h_{im}R_{mkjk},$$

$$\sum h_{ij}\Delta h_{ij}=\frac{1}{2}\sum(h_{ij}h_{mk}-h_{mj}h_{ik})R_{mijk}+\sum(h_{ij}h_{im}-h_{ii}h_{jm})(-R_{mj}). \tag{17}$$

If the equality (4) is used in the first term at the right side of the equality (17), we find

$$\frac{1}{2}\sum(h_{mk}h_{ij}-h_{mj}h_{ik})=\frac{1}{2}\left[-(\delta_{mk}\delta_{ij}-\delta_{mj}\delta_{ik})-R_{mijk}\right]. \tag{18}$$

If the equality (4) is used in the second term at the right side of the equality (17), we have

$$\sum(h_{im}h_{ji}-h_{ii}h_{jm})=(\delta_{im}\delta_{ji}-\delta_{ii}\delta_{jm})+R_{ijim}. \tag{19}$$

If (18) and (19) are written in (17), we obtain

$$\begin{aligned}\sum h_{ij}\Delta h_{ij} &= \frac{1}{2}\sum\left[-(\delta_{mk}\delta_{ij}-\delta_{mj}\delta_{ik})-R_{mijk}\right]R_{mijk}+\\ &\quad+\sum\left[(\delta_{im}\delta_{ji}-\delta_{ii}\delta_{jm})+R_{ijim}\right](-R_{mj})\end{aligned}$$

or

$$\begin{aligned}\sum h_{ij}\Delta h_{ij}=-\frac{1}{2}\sum R^2_{mijk}+\sum R^2_{mj}+\frac{1}{2}\sum\left[-(\delta_{mk}\delta_{ij}-\delta_{mj}\delta_{ik})\right]R_{mijk}+\\ +\sum\left[(\delta_{im}\delta_{ji}-\delta_{ii}\delta_{jm})\right](-R_{mj}).\end{aligned} \tag{20}$$

After some calculation last two terms at the right side of (20) are obtained as the following:

$$\frac{1}{2}\sum\left[-(\delta_{mk}\delta_{ij}-\delta_{mj}\delta_{ik})\right]R_{mijk}+\sum\left[(\delta_{im}\delta_{ji}-\delta_{ii}\delta_{jm})\right](-R_{mj})=nR. \tag{21}$$

If (21) is written in (20), we find

$$\sum h_{ij}\Delta h_{ij}=-\frac{1}{2}\sum R_{mijk}^2+\sum R_{mj}^2+nR. \tag{22}$$

Since $\int\limits_M\left\{\sum h_{ij}\Delta h_{ij}\right\}dv\leq 0$ [2], we have the following:

$$\int\limits_M\left\{-\frac{1}{2}\sum R_{mijk}^2+\sum R_{mj}^2+nR\right\}dV\leq 0.$$

This completes proof of theorem 2.1.

Proof of Theorem 2.2. If (8) is considered in (13), the proof of the theorem 2.2 is trivial.

In order to prove theorem 2.3, we need the following lemma.

Lemma. Let $a_1,...,a_n$ be real numbers, then

$$\sum(a_i)^2\geq\frac{1}{n}\left(\sum a_i\right)^2, \tag{23}$$

where the equality sign holds when and only when $a_1=...=a_n$.

Proof of Theorem 2.3. If (9) is considered in (7), we have

$$R_{mj}=(n-1)\delta_{mj}-\sum\lambda_j^2\delta_{mj}. \tag{24}$$

If (23) is used in (24), we find

$$\begin{aligned}\sum R_{mj}^2 &= n(n-1)^2-2(n-1)S+\sum\lambda_j^4\geq\\ &\geq n(n-1)^2-2(n-1)S+\frac{1}{n}\left(\sum\lambda_j^2\right)^2\end{aligned}$$

$$\sum R_{mj}^2\geq n(n-1)^2-2(n-1)S+\frac{1}{n}S^2. \tag{25}$$

If (25) and (8) are used in (13), we obtain

$$\int\limits_M\left\{-\frac{1}{2}\sum R_{mijk}^2+n(2n^2-3n+1)-S(3n-2)+\frac{1}{n}S^2\right\}dV\leq 0$$

or

$$\int\limits_M\left\{-\frac{1}{2}\sum R_{mijk}^2-(3n-2)S+\frac{1}{n}S^2\right\}dV\leq n(-2n^2+3n-1)Vol(M).$$

This completes the proof.

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