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ON NONCLASSICAL LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

SHAKIR FORMANOV [1], ASKAR AKHMEDOV [2], LOLA SHARIPOVA [3], §

Abstract. Nonclassical limit theorems, unlike classical ones, do not require satisfaction of the uniform limit smallness condition. In the nonclassical situation of summation of independent random variables, the class of limit distributions is extended maximally, and it coincides with the set of all distributions. In the paper, it is carried out the comparative analysis of the classical Kolmogorov's model in the theory of summation of independent random variables with problems of the central limit problem in the nonclassical formulation. Also nonclassical versions of the central limit theorem are proved. In this connection, modified version of the known Stein's method is used. This version is based on a characterized property of the normal distribution.

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1. Introduction

Monograph by Gnedenko and Kolmogorov [2], being now the bibliographical rarity among theoretical-probabilistic literature, has been used for a long time not only as a source of necessary materials for specialists but also as an excellent text-book by limit theorems for sums of independent random variables (r.v.'s). This book was published in 1949, and it was devoted to account of basic results and methods of classical theory of limit theorems. This theory takes its origin from Bernoulli theorems concerning the law of large numbers and the original version of the Mouivre-Laplace central limit theorem (CLT). Classical theory of summation of independent r.v.'s has stimulated still now appearance and development of new fields of modern probability theory in spite of the fact that the great part of achievements of this theory relates to the last century.

2. Central limit problem

The foundation of the classical theory of summation of independent r.v.'s, as it was set forth in [2], was the following model of summation suggested in 1932-33 by Kolmogorov.

Let

$$X_{n1}, X_{n2}, \dots, X_{nn}, \dots \tag{1}$$

be an array of independent r.v.'s given on the same probabilistic space and satisfying to the following two conditions:

[1] Institute of Mathematics and Information Technologies, Academy of Sciences of Uzbekistan, e-mail: shakirformanov@yandex.ru

[2] South-Kazakhstan Technical University, Kazakhstan,

[3] Tashkent Institute of Railway Engineering, Uzbekistan, e-mail: slola@mail.ru

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(I) Inside any array, r.v.'s are independent, i.e. the characteristic function (ch.f.) $f_n(t)$, $t = (t_1, \dots, t_n)$ of collections $(X_{n1}, X_{n2}, \dots, X_{nn})$ and ch.f. $f_{nj}(t_j)$ of the r.v. X_{nj} are connected with the equality

$$f_n(t) = f_{n1}(t_1) \cdot \ldots \cdot f_{nn}(t_n), \quad t \in \mathbb{R}^n, \quad n \geq 1.$$

(US) For any $\varepsilon > 0$

$$\sup_j P\left(|X_{nj}| \geq \varepsilon\right) \to 0, \quad n \to \infty. \tag{2}$$

This condition (US) is called *the uniform limit smallness condition for r.v.'s* X_{nj}, and it is equivalent to

$$\sup_j |1 - f_{nj}(t)| \to 0, \quad n \to \infty \tag{3}$$

for any $t \in \mathbb{R}^1$.

Then sums

$$S_n = X_{n1} + \cdots + X_{nn} + \ldots,$$

formed on the base of the array (1), are considered.

The sum S_n can contain both the finite and infinite number of summands (for the last case corresponding series S_n are considered as convergent ones). It is raised the question on construction of asymptotical approximations for the distribution function (d.f.) F_n of the r.v. S_n when d.f.'s F_{nj} of summands are given. Approach of the sequence F_n with its approximation — d.f. G (if such approximation exists) — is understood in the sense of topology of the weak convergence.

Statements like the law of large numbers and the central limit theorem known at the beginning of thirties of the XX century, i.e. at the time of appearance of Kolmogorov's model, were relative mainly to the special case of this model — to the scheme of growing sums of independent r.v.'s in the form

$$X_{nj} = \frac{X_j}{b_n}, \quad b_n \to \infty, \quad \text{as} \quad n \to \infty.$$

It should be noted in this connection, the idea of arrays, may be not in such explicit form, entered in the probability theory much earlier then Kolmogorov's model. So, it was present in the known Poisson's theorem on approach of the binomial distribution by the Poisson one. Absolute estimates of the rate of convergence were present in the proof of the known Lyapunov's limit theorem [2] formally operated with the scheme of growing sums, what extended this theorem on the arrays automatically. A family of r.v.'s formed arrays of special form has been used explicitly at creation of the theory of stochastic differential equations by S. Bernshtein. However, just Kolmogorov suggested to consider the model containing the conditions (I) and (US) and raised the following two questions:

a) to describe the class $[G]$ of possible limit distributions for S_n;

b) to find necessary and sufficient conditions guaranteeing the weak convergence of distributions $F_n(x) = P(S_n < x)$ to the concrete distribution F from the class $[G]$.

Then these two questions in total have been said to be *the central limit problem of the theory of summation of independent r.v.'s* ([3], chapters 22–24). Kolmogorov has expressed the hypothesis that the class $[G]$ coincided with the set of infinitely divisible distributions introduced by Bruno de Finneti not long before. Kolmogorov's hypothesis, which should be dated just as the model itself by 1932-33, was confirmed in the work by his student Bawly for the case when X_{nj} have finite second moments; and the central limit problem was solved on the whole by A.Ya. Khinchin (see [2]).

The condition (I) in Kolmogorov's model has old traditions. Concept of independence is fundamental for the probability theory on the whole, and it plays the important role in the theory of summation of r.v.'s. Specialists even have made sure of necessary presence of (I) among conditions of limit theorems generalizing Bernoulli and Mouivre-Laplace theorems. This condition has been considered as much natural in initial investigations by approximation of distributions so that it has been not stipulated especially. But fundamental concepts like Markov chains, martingales, weakly dependent r.v.'s have appeared in next development of probability theory, and they permitted to generalize essentially the original Kolmogorov's model.

The condition (US) (in terms of ch.f.'s, it is equivalent to condition (3)), requiring uniform limit smallness of summands, has been interpreted for a long time by specialists as essentially important part of the Kolmogorov model and also in the theory of summation of r.v.'s on the whole since rejection of it has reduced to maximal extension of the class of possible limit distributions for sums S_n to the dimension of the set $\mathcal{F}$ of all distributions on $\mathbb{R}^1$. It has arisen natural doubt in possibility to construct any theory of limit theorems rich in content and comparable with the Kolmogorov model in such general situation.

In the second one-half of sixtieth of the last century, this problem was studied by V.M. Zolotarev and his students. As a final result, the theory of limit theorems based only on the condition (I) was created successfully (corresponding conditions composed the base of [7]).

At present, following to V.M. Zolotarev, theorems on limit distributions for S_n proved without the condition (US) are said to be *nonclassical*. Success of creation of the new theory of nonclassical theorems is connected mainly with the use of special characteristics of r.v.'s similar by properties to mathematical expectations and variances (in contrast to the classical case, these characteristics exist for any distribution).

3. Nonclassical limit theorems in the case of convergence to the normal distribution

We give below some nonclassical limit theorems in the form of the central limit theorem (CLT).

Suppose r.v.'s X_{nj} of the sequence (1) satisfy to the following conditions:
for any j

$$\mathbf{E}X_{nj} = 0, \quad \mathbf{E}X_{nj}^2 = \sigma_{nj}^2, \quad \sum_j \sigma_{nj}^2 = 1. \tag{4}$$

Introduce the following notations:

$$F_{nj}(x) = P(X_{nj} < x), \quad f_{nj}(t) = \mathbf{E}e^{itX_{nj}}, \quad j = 1, 2, \ldots;$$

$$L_n(\varepsilon) = \sum_j \int\limits_{|x|>\varepsilon} x^2 dF_{nj}, \quad \varepsilon > 0,$$

$$R_n(\varepsilon) = \sum_j \int\limits_{|x|>\varepsilon} |x|\,|F_{nj}(x) - \Phi_{nj}(x)|\,dx,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad \Phi_{nj}(x) = \Phi\left(\frac{x}{\sigma_{nj}}\right), \quad j = 1, 2, \ldots \ .$$

As it is well known, the Levy distance $L(\cdot,\cdot)$ between distributions F and G is determined by the equality

$$L(F,G) = \inf\{\varepsilon > 0 : F(x-\varepsilon) - \varepsilon \le G(x) \le F(x+\varepsilon) + \varepsilon, \ -\infty < x < \infty\}$$

and this distance metrizes the weak convergence in the space of distributions. Recall, validity of CLT for the sequence (1) is equivalent to fulfillment of the relation

$$L(F_n, \Phi) \to 0, \quad \text{as} \quad n \to \infty,$$

where $F_n(x) = P(S_n < x)$.

Further, one can verify validity of the following statement.

Lemma 1. *Let for the ch.f. $f(t)$ the following conditions hold:*

$$f'(0) = 0, \quad f''(0) = -\sigma^2 > -\infty.$$

Then for any t

$$|1 - f(t)| \leq \frac{\sigma^2 t^2}{2}. \tag{5}$$

Proof. One can easily check that for any real α

$$\left|e^{i\alpha} - 1 - i\alpha\right| \leq \frac{\alpha^2}{2}.$$

By virtue of $f'(0) = 0$, the last inequality implies

$$|1 - f(t)| = \left|\int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right) dF(x)\right| \leq \int_{-\infty}^{\infty} \left|e^{itx} - 1 - itx\right| dF(x) \leq \frac{\sigma^2 t^2}{2}.$$

Remark 1. *Formulation of the given lemma in the form of problem can be found in the problem-book by Prokhorov A.V., Ushakov V.G., and Ushakov N.G., Problems in Probability Theory, Moscow, 1986 (problem 4.95, p.91).*

Now, by virtue of (4) and (5), for any real t, we have

$$\sup_j |1 - f_{nj}(t)| \leq \frac{t^2}{2} \sup_j \sigma_{nj}^2. \tag{6}$$

In turn, relations (3) and (6) imply that if

$$\sup_j \sigma_{nj}^2 \to 0, \quad \text{as} \quad n \to \infty, \tag{7}$$

then the uniform limit smallness condition (US) holds. Hence, if (4) is valid, (7) can be accepted as the condition (US). In [2], the following classical version of CLT is given.

Theorem. (*Lindeberg-Feller*) *Let relations* (4) *and* (7) *hold. Then convergence $L(F_n, \Phi) \to 0$ takes place if and only if the following Lindeberg condition holds: for any $\varepsilon > 0$*

$$L_n(\varepsilon) \to 0, \quad \text{as} \quad n \to \infty. \tag{8}$$

In [7], the following theorem, being a nonclassical version of CLT, is proved.

Theorem A. (*Zolotarev*) *C*onvergence

$L(F_n, \Phi) \to 0$, as $n \to \infty$ is valid if and only if for unlimited increase of n the following two conditions hold: *1)*

$$\alpha_n = \sup_j L\left(F_{nj}, \Phi_{nj}\right) \to 0; \tag{9}$$

2) for any $\varepsilon > 0$

$$\Delta_n(\varepsilon) = \sum_{j \in A_n} \int_{|x| \geq \varepsilon} x^2 dF_{nj}(x) \to 0, \tag{10}$$

where the set A_n contains those values of the index j for which

$$\sigma_{nj}^2 \leq \sqrt{\alpha_n}. \tag{11}$$

Probabilistic sense of conditions (9)–(11) is consisted in the following: at first one choose from the sequence (1) the summands for which the condition (US) (or (7)) holds, and then fulfillment of the Lindeberg condition (8) is required for them.

To prove theorem A, at first necessity of the condition 1) is proved, and then the relation (10) is used to prove necessity of (11). Proof of Theorem A in the part of sufficiency of given conditions was made by direct probabilistic methods (without use of the method of ch.f.'s).

Theorem A stated below generalizes the Lindeberg-Feller theorem since fulfillment of conditions of the last one implies that relations (9) and (10) take place.

In monograph [4], V.I. Rotar proved the following theorem in which two conditions of Theorem A are reduced into one condition.

Theorem B. (*Rotar*) *Convergence*

$$L(F_n, \Phi) \to 0, \quad \text{as} \quad n \to \infty$$

is valid if and only if the condition

$$R_n(\varepsilon) \to 0, \quad \text{as} \quad n \to \infty \tag{12}$$

holds for any $\varepsilon > 0$.

At the beginning of seventieth of the last century, Ch. Stein [5] suggested sufficiently universal method for the proof of CLT based on a characteristic property of a normal distribution.

Lemma 2. *(Stein) Let for the r.v.* w, $\mathbf{E}w = 0$, $Dw = 1$. *The r.v.* w *has a normal distribution if and only if the following equality*

$$\mathbf{E}g'(w) = \mathbf{E}wg(w) \tag{13}$$

holds for any piecewise continuously differentiable function $g(\cdot)$, $\mathbb{R} \xrightarrow{g} \mathbb{R}$ *such that*

$$\int_{-\infty}^{\infty} \left| g'(x) \right| e^{-x^2/2} dx < \infty.$$

Equality (13) implies that the value

$$\left| \mathbf{E}g'(w) - \mathbf{E}wg(w) \right|$$

is represented the "amount of normality for the distribution of a r.v. w". This statement contains very convenient version for the proof of CLT: if a sequence of r.v.'s $\{w_n, \ n \geq 1\}$ satisfies to the equality

$$\lim_{n\to\infty} \mathbf{E}g'(w_n) = \lim_{n\to\infty} \mathbf{E}w_n g(w_n),$$

then the distribution of w_n is asymptotically normal.

In [5], Ch. Stein expressed opinion that his method does not have any relation to the known analytical method of ch.f.'s. However, Tikhomirov disproved this opinion. Combining Stein's ideas with the method of ch.f.'s, he has established unimprovable estimates for the rate of convergence in CLT for stationary processes satisfying to the condition of uniform strong mixing by Rosenblatt [6]. Later on, methods based on works [5], [6], have been said to be in collecting sense the Stein-Tikhomirov (S-T) method. In [1], Formanov suggested a modified version of the S-T method in terms of ch.f.'s.

Introduce the following class of ch.f.'s

$$\mathcal{F}(f) = \left\{ f(t) : f'(0) = 0, \ \ f''(0) = -\sigma^2 > -\infty \right\}.$$

Lemma 3. *For a function* $f \in \mathcal{F}(f)$ *to be the ch.f. of the normal distribution with the parameter* $(0, \sigma^2)$ *it is necessary and sufficient that it satisfies to the differential equation*

$$f'(t) + \sigma^2 t f(t) = 0 \tag{14}$$

with the initial condition $f(0) = 1$.

Proof of this lemma is evident, and we omit it. The mentioned lemma is used as the starting point for introduction of the new version of the S-T method.

Determine in the class of ch.f.'s $\mathcal{F}(f)$ the S-T operator with the help of the equality

$$\Delta(f) = f'(t) + \sigma^2 t f(t). \tag{15}$$

By virtue of lemma 3, the following statement on validity of the equivalent implication holds:

$$\{\Delta(f)=0\} \Longleftrightarrow \left\{f(t)=e^{-\frac{\sigma^2 t^2}{2}}\right\}. \tag{16}$$

If we consider equality (15) as the linear differential equation with the initial condition $f(0)=1$, then we make sure of validity of the equality

$$f(f)-e^{-\frac{\sigma^2 t^2}{2}}=e^{-\frac{\sigma^2 t^2}{2}}\int\limits_0^t \Delta(f(u))e^{\frac{\sigma^2 u^2}{2}}du. \tag{17}$$

This equality (17) also verifies validity of (16). It should be also noted that the sign of integration variable in (17) coincides with the sign of t.

(17) implies validity of the equality

$$\sup_{|t|\le T}\left|f(f)-e^{-\frac{\sigma^2 t^2}{2}}\right|\le T\sup_{|t|\le T}|\Delta(f(t))| \tag{18}$$

for any $T>0$. With regard to relations (15)– (18) we can conclude: to prove validity of CLT for a sequence of r.v.'s (1), it is sufficient to show that for $n\to\infty$

$$\lim_{n\to\infty}\sup_{|t|\le T}\left|\Delta\left(f_n\left(t\right)\right)\right|\to 0. \tag{19}$$

Here T is any positive number,

$$f_n(t)=\mathbf{E}e^{itS_n}=\mathbf{E}e^{it(X_{n1}+\ldots+X_{nn}+\ldots)}\ .$$

The following lemma contains one of important properties of the operator $\Delta(\cdot)$ (S-T-operator).

Lemma 4. *Let ch.f.'s $f,g\in\mathcal{F}(\cdot)$. Then*

$$\Delta(f\cdot g)=f\Delta(g)+g\Delta(f). \tag{20}$$

Proof. Condition of lemma implies $f'(0)=g'(0)=0$. Let

$$\sigma_1^2=-f''(0),\quad \sigma_2^2=-g''(0).$$

By definition of the operator $\Delta(\cdot)$,

$$\Delta(f\cdot g)=(f\cdot g)'+t\left[-\left(f\cdot g\right)''_{t=0}\right]f\cdot g=f'g+g'f+t\left(\sigma_1^2+\sigma_2^2\right)f\cdot g=$$

$$=\left(f'+t\sigma_1^2 f\right)\cdot g+\left(g'+\sigma_2^2 g\right)\cdot f=g\Delta(f)+f\Delta(g).$$

Since there is not an (US)-type condition in theorems A, B given below, they are nonclassical versions of the classical limit Lindeberg-Feller theorem containing necessary and sufficient conditions for validity of CLT . Characteristics $L_n(\varepsilon)$ and $R_n(\varepsilon)$ used in them are expressed by d.f.'s $F_{nj}(x)$. Below we give theorems on convergence of the distribution $F_n(x)$ to the normal law $\Phi(x)$, and conditions imposed upon in them are expressed in terms of ch.f.'s $f_{nj}(t)$. Such conditions have the right on existence because ch.f.'s also define uniquely distributions of r.v.'s. In the theory of limit theorems conditions on ch.f.'s appeared first in works by Cramer on asymptotical decompositions in CLT.

Theorem 5. *Under condition (4)*

$$\sup_x|F_n(x)-\Phi(x)|\to 0,\quad as\quad n\to\infty \tag{21}$$

if and only if for any $T>0$

$$\lim_{n\to\infty}\sup_{|t|\le T}\sum_j\left|\Delta\left(f_{nj}\left(t\right)\right)\right|=0. \tag{22}$$

Remark 2. *In theorem given below uniform convergence (21) is used instead of L-convergence ($L(f_n,\Phi)\to 0$) since in the case of CLT, these types of convergence are equivalent.*

Remark 3. *Since ch.f. and its derivatives are continuous functions on any finite segment, (22) could be formulated in the more simple form:*

$$\sum_j |\Delta(f_{nj}(t))| \to 0, \quad as \quad n \to \infty$$

for any $t \in \mathbb{R}$.

Before proof of theorem 1, we prove the following lemma which will be used further.

Lemma 6. *For any distribution function $F(x)$ such that*

$$\int_{-\infty}^{\infty} x dF(x) = 0,$$

the equality

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 dF(x) = 2 \int_{-\infty}^{0} x(1 - F(x) + F(-x))\, dx \tag{23}$$

holds.

Proof. In fact,

$$\int_{-\infty}^{\infty} x^2 dF(x) = \int_{-\infty}^{0} x^2 dF(x) + \int_{0}^{\infty} x^2 dF(x). \tag{24}$$

Integrate by parts the first integral in the right side of (24), supposing

$$u = x^2, \quad dv = dF.$$

Then

$$\int_{-\infty}^{0} x^2 dF(x) = \left(x^2 F(x)\right)\Big|_{-\infty}^{0} - 2 \int_{-\infty}^{0} xF(x)dx.$$

Keeping in mind

$$\lim_{x\to\infty} x^2 F(-x) = 0,$$

we obtain

$$\int_{-\infty}^{0} x^2 dF(x) = 2 \int_{-\infty}^{0} (-x)F(x)dx = 2 \int_{0}^{\infty} uF(-u)du. \tag{25}$$

The following equality

$$\int_{0}^{\infty} x^2 dF(x) = 2 \int_{0}^{\infty} u(1 - F(u))\, du \tag{26}$$

can be obtained analogously.

Now (23) follows from (24)–(26).

Proof of Theorem 1. In the part of sufficiency we can be restricted to establish the limit relation (19). Taking into account remark 2, for the last it is sufficient to show for any t

$$\lim_{n\to\infty} \Delta(f_n(t)) = 0. \tag{27}$$

Lemma implies $\Delta(\cdot)$ is the differentiation operation with regard to product of ch.f.'s. Application of this lemma allows to write the equality

$$\Delta(f_n(t)) = \sum_j \prod_{k=1}^{j-1} f_{nk}(t)\, \Delta(f_{nj}(t)) \prod_{k=j+1}^{\infty} f_{nk}(t)\,.$$

Hence,

$$|\Delta(f_n(t))| \le \sum_j |\Delta(f_{nj}(t))|. \tag{28}$$

Thus, sufficiency of the condition of theorem 1 follows from (27), (28) (in this connection, (19) should be taken into account).

Let's now prove necessity of the condition (22). Let CLT takes place, i.e. (21) holds. Then according to theorem B, for any $\varepsilon > 0$

$$\sum_j \int\limits_{|x|>\varepsilon} |x|\,|F_{nj}(x) - \Phi_{nj}(x)|\,dx = R_n(\varepsilon) \to 0. \tag{29}$$

Using the equality

$$\Delta(\varphi_{nj}(t)) = \Delta\left(e^{-\sigma_{nj}^2 t^2/2}\right) = 0,$$

correct for any j, we have

$$\left|\sum_j \Delta(f_{nj}(t))\right| = \left|\sum_j [\Delta(f_{nj}(t)) - \Delta(\varphi_{nj}(t))]\right| =$$

$$= \left|\sum_j \left(f'_{nj}(t) - \varphi'_{nj}(t)\right) + t\sum_j \sigma_{nj}^2 (f_{nj}(t) - \varphi_{nj}(t))\right| \le$$

$$\le \sum_j \left|f'_{nj}(t) - \varphi'_{nj}(t)\right| + |t|\sum_j \sigma_{nj}^2 |f_{nj}(t) - \varphi_{nj}(t)| = \Sigma_{n1}(t) + \Sigma_{n2}(t). \tag{30}$$

Estimate at first $\Sigma_{n2}(t)$. Taking into account $\mathbf{E}X_{nj} = 0$, $f''_{nj}(0) = \varphi''_{nj}(0) = -\sigma_{nj}^2$, we can write

$$|f_{nj}(t) - \varphi_{nj}(t)| = \left|\int\limits_{-\infty}^{\infty} e^{itx} d(F_{nj} - \Phi_{nj})\right| =$$

$$= \left|\int\limits_{-\infty}^{\infty} \left[e^{itx} - 1 - itx - \frac{(itx)^2}{2}\right] d(F_{nj} - \Phi_{nj})\right| \tag{31}$$

Integrate by parts in (31) supposing

$$u = e^{itx} - 1 - itx - \frac{(itx)^2}{2}, \quad dv = d(F_{nj} - \Phi_{nj}).$$

Then we obtain

$$|f_{nj}(t) - \varphi_{nj}(t)| \le \left[\left|e^{itx} - 1 - itx - \frac{(itx)^2}{2}\right| |F_{nj} - \Phi_{nj}|\right]_{-\infty}^{\infty} +$$

$$+ \left|\int\limits_{-\infty}^{\infty} it\left(e^{itx} - 1 - itx\right)(F_{nj} - \Phi_{nj})\,dx\right|. \tag{32}$$

One can easily be sure that the first summand in the right side of (32) is zero. Really, for any t

$$\left[\left|e^{itx} - 1 - itx - \frac{(itx)^2}{2}\right| |F_{nj} - \Phi_{nj}|\right]_{-\infty}^{\infty} \le \left[t^2x^2\,|F_{nj} - \Phi_{nj}|\right]_{-\infty}^{\infty} = 0.$$

Here we take into account the following relations

$$x^2\,|F(x) - \Phi(x)| \le x^2(1 - F(x)) + x^2(1 - \Phi(x)), \qquad \int\limits_{-\infty}^{\infty} x^2 dF = \int\limits_{-\infty}^{\infty} x^2 d\Phi,$$

$$F(-x)=o(x^2),\ \ 1-F(x)=o(x^2),\ \ \Phi(-x)=1-\Phi(x)\sim\frac{e^{-x^2/2}}{\sqrt{2\pi x}},\ x\to\infty.$$

Consider now the second summand in (32). We have

$$\left|\int\limits_{-\infty}^{\infty} it\left(e^{itx}-1-itx\right)\left(F(x)-\Phi(x)\right)dx\right|\le$$

$$\le|t|\int\limits_{|x|\le\varepsilon}\left|e^{itx}-1-itx\right|\left|F-\Phi\right|dx+$$

$$+|t|\int\limits_{|x|>\varepsilon}\left|e^{itx}-1-itx\right|\left|F-\Phi\right|dx=I_1+I_2. \tag{33}$$

Using the inequality

$$\left|e^{itx}-1-itx\right|\le\frac{t^2x^2}{2},$$

we obtain

$$I_1\le\frac{|t|^3}{2}\cdot\int\limits_{|x|\le\varepsilon}x^2\left|F-\Phi\right|dx\le\frac{|t|^3}{2}\cdot\varepsilon\cdot\int\limits_{|x|\le\varepsilon}|x|\left|F-\Phi\right|dx\le$$

$$\le\frac{|t|^3}{2}\cdot\varepsilon\left[\int\limits_{-\varepsilon}^{0}(-x)\left|F-\Phi\right|dx+\int\limits_{0}^{\varepsilon}x\left|F-\Phi\right|dx\right]. \tag{34}$$

Replacing $-x$ by x, we have

$$\int\limits_{-\varepsilon}^{0}(-x)\left|F-\Phi\right|dx=\int\limits_{0}^{\varepsilon}u\left|F(-u)-\Phi(-u)\right|du\le$$

$$\le\int\limits_{0}^{\varepsilon}uF(-u)du+\int\limits_{0}^{\varepsilon}u\Phi(-u)du. \tag{35}$$

Further, it is evident,

$$\int\limits_{0}^{\varepsilon}x\left|F-\Phi\right|dx\le\int\limits_{0}^{\varepsilon}x\left(1-F(x)\right)dx+\int\limits_{0}^{\varepsilon}x\left(1-\Phi(x)\right)dx. \tag{36}$$

Now taking into account (34)–(36), we obtain

$$I_1\le\frac{|t|^3}{2}\cdot\varepsilon\left[\int\limits_{0}^{\varepsilon}x(1-F(x)+F(-x))dx+\int\limits_{0}^{\varepsilon}x(1-\Phi(x)+\Phi(-x))dx\right]. \tag{37}$$

We have from the last estimate (37), with regard to lemma 5,

$$I_1\le\frac{|t|^3}{4}\cdot\varepsilon\left(\sigma_{nj}^2+\sigma_{nj}^2\right)=\frac{|t|^3}{2}\cdot\varepsilon\cdot\sigma_{nj}^2. \tag{38}$$

Now estimate I_2. Here, using the inequality

$$\left|e^{i\alpha}-1\right|\le|\alpha|,\quad \alpha\in\mathbb{R},$$

we have

$$I_2\le 2t^2\int\limits_{|x|>\varepsilon}|x|\left|F_{nj}(x)-\Phi_{nj}(x)\right|dx. \tag{39}$$

Sum by j in estimates (38), (39), taking into account (4) and (33), we obtain

$$\sum_j |f_{nj}(t)-\varphi_{nj}(t)| \le \frac{|t|^3}{2}\cdot\varepsilon + 2t^3\sum_j \int\limits_{|x|>\varepsilon} |x|\,|F_{nj}(x)-\Phi_{nj}(x)|\,dx \le$$

$$\le \frac{|t|^3}{2}\cdot\varepsilon + 2t^2 R_n(\varepsilon).$$

By arbitrariness of ε, we conclude from the last relation with regard to (29) that

$$\sup_{|t|\le T} \Sigma_{n2}(t) = o(1), \quad \text{as} \quad n\to\infty, \tag{40}$$

for any $T>0$.

Now take up estimate for $\Sigma_{n1}(t)$. Beforehand note that

$$\left|f'_{nj}(t)-\varphi'_{nj}(t)\right| = \left|\int\limits_{-\infty}^{\infty} x\left(e^{itx}-1-itx\right)d\left(F_{nj}-\Phi_{nj}\right)\right|.$$

If we integrate by parts under the module sign, supposing

$$u = x\left(e^{itx}-1-itx\right), \quad dv = d\left(F_{nj}-\Phi_{nj}\right),$$

then we obtain

$$\left|f'_{nj}(t)-\varphi'_{nj}(t)\right| \le \left|\int\limits_{-\infty}^{\infty} \left(e^{itx}-1-itx\right)\left(F_{nj}-\Phi_{nj}\right)dx\right| +$$

$$+\,|t|\left|\int\limits_{-\infty}^{\infty} \left(e^{itx}-1\right)\left(F_{nj}-\Phi_{nj}\right)dx\right| = I_3 + I_4. \tag{41}$$

Further we have

$$I_3 \le \frac{t^2}{2}\int\limits_{|x|\le\varepsilon} x^2|F_{nj}-\Phi_{nj}|dx + \int\limits_{|x|>\varepsilon} \left|e^{itx}-1\right|\,|F_{nj}-\Phi_{nj}|\,dx +$$

$$+\,|t|\int\limits_{|x|>\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx \le \frac{t^2}{2}\cdot\varepsilon\int\limits_{|x|\le\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx +$$

$$+2\,|t|\int\limits_{|x|>\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx. \tag{42}$$

We established in (33)–(38) that

$$\int\limits_{|x|\le\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx \le 2\sigma_{nj}^2. \tag{43}$$

It follows from (42) and (43) that

$$I_3 \le t^2\cdot\varepsilon\cdot\sigma_{nj}^2 + 2\,|t|\int\limits_{|x|>\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx. \tag{44}$$

Now, keeping in mind the trivial estimate $|e^{itx}-1|\le 2$, we obtain

$$I_3 \le t^2\cdot\varepsilon\int\limits_{|x|\le\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx + 2\,|t|\int\limits_{|x|>\varepsilon} |x|\,|F_{nj}-\Phi_{nj}|\,dx.$$

In turn, taking into account (43), we have

$$I_4 \leq 2t^2 \cdot \varepsilon \cdot \sigma_{nj}^2 + 2\,|t| \int\limits_{|x|>\varepsilon} |x|\,|F_{nj} - \Phi_{nj}|\,dx. \tag{45}$$

With regard to (41), (44), (45) we obtain finally

$$\sum_j \left| f'_{nj}(t) - \varphi'_{nj}(t) \right| \leq \frac{5}{2} t^2 \cdot \varepsilon + 4\,|t|\,R_n(\varepsilon). \tag{46}$$

By virtue of arbitrariness of ε we obtain from (46)

$$\sup_{|t|\leq T} \Sigma_{n1}(t) = o(1), \quad \text{as} \quad n \to \infty. \tag{47}$$

Now proof of necessity for the condition (22) follows from relations (30), (40), (47). Theorem 1 is proved on the whole.

Remark 4. *Theorem 1 can be considered as an analog of theorem B. Process of proving necessity of (22) shows that if the condition $R_n(\varepsilon) \to 0$ is valid, then (22) holds. Theorem 1 generalizes theorem B in the mentioned sense.*

We give now an analog of theorem A in terms of ch.f.'s. At first, note the following. Let for any t

$$\sup_j |f_{nj}(t) - \varphi_{nj}(t)| \to 0, \quad \text{as} \quad n \to \infty. \tag{48}$$

This condition (48) implies

$$\alpha_n = \sup_j L\left(F_{nj}, \Phi_{nj}\right) \to 0.$$

Arguments using in the proof of necessity for condition (22) show that if conditions (4) hold, the relation

$$\sup_j |\Delta\left(f_{nj}\left(t\right)\right)| \to 0 \tag{49}$$

for any t, is equivalent to (48).

Theorem 7. *Let conditions (4) hold. Then convergence*

$$\sup_x |F_n(x) - \Phi(x)| \to 0, \qquad n \to \infty$$

takes place if and only if the following conditions are valid:

1) $$\sup_j |\Delta\left(f_{nj}\left(t\right)\right)| \to 0, \qquad n \to \infty;$$

2) $$\sum_{j \in A_n} |\Delta\left(f_{nj}\left(t\right)\right)| \to 0, \qquad n \to \infty$$

for any t. Here $A_n = \left\{ j : \ \sigma_{nj}^2 < \sqrt[4]{\alpha_n} \right\}$.

We give only the scheme of the proof of theorem 2. We present the initial sum of r.v.'s S_n as

$$S_n = \sum_{j \in A_n} X_{nj} + \sum_{j \notin A_n} X_{nj} = \Sigma_{n1} + \Sigma_{n2}. \tag{50}$$

Conditions of theorem 2 imply

$$\lim_{n\to\infty} P\left(|\Sigma_{n2}| > \varepsilon\right) = 0 \tag{51}$$

for any $\varepsilon > 0$. In addition, one should to keep in mind, the number of summands in Σ_{n2}

$$\sum_{j \notin A_n} 1 \leq \frac{1}{\sqrt{\alpha_n}}.$$

Repeating reasonings carrying out in the part of sufficiency of conditions of theorem 1, one can establish,

$$\lim_{n\to\infty} P\left(\sum_{j\in A_n} X_n < x\right) = \Phi(x). \tag{52}$$

Sufficiency of conditions of theorem 2 follows from relations (50)–(52).

Necessity for conditions 1) and 2) is proved as follows: at first, necessity of 1) for validity of CLT in the nonclassical formulation is proved, and then this condition is used to prove necessity of 2).

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Formanov Shakir Kasimovich - was born on February, 20, 1941, at the Fergana region of Uzbekistan. He graduated from Tashkent State University in 1963. He got Ph.D degree in 1968 and Doctor of Sciences degree in 1973. Since 1994, he is active member of Uzbek Academy of Sciences. His current research interests include limit theorems of probability theory and mathematical statistics. Since 1976, he is a head of the department of probability theory and mathematical statistics (Institute of Mathematics of Uzbek Academy of Sciences) and professor of National University of Uzbekistan.

Akhmedov Askar Bakirovich - was born in 1954 at the Chimkent region of Kazakhstan. He graduated from Kazakh State University in 1976. He was a post-doctoral student in Moscow State University by supervision of academician Gnedenko B.V. He defended doctor thesis (Ph.D) in 1995. Presently, he is an associated-professor in South-Kazakhstan Technical University.

Sharipova Lola Djavlanovna - was born in 1968 in Tashkent, Uzbekistan. She graduated from Tashkent State University in 1990. She defended doctor thesis (Ph.D) in 2006. Presently, she works as associated professor in the Tashkent Institute of Railway Engineering.