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***BRIEF PAPER***

# CONVEXITY OF WEIGHTED GINI MEAN

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Abstract. In this note, we establish the convexity of weighted Gini mean for data $X = \{x_1, x_2, ..., x_n\}$ such that $x_i > 0$ for all $i$.

Keywords: convexity, Gini mean.

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## 1. Introduction

The Gini means of the data $X = \{x_1, x_2, \cdots, x_n\}$ with $x_i > 0$ for $i = 1, \cdots, n$ are defined by [3]

$$G(p,q;X) = \begin{cases} \left(\dfrac{\sum\limits_{i=1}^{n} x_i^p}{\sum\limits_{i=1}^{n} x_i^q}\right)^{1/(p-q)}, & p \neq q, \\ \exp\left(\dfrac{\sum\limits_{i=1}^{n} x_i^p \ln x_i}{\sum\limits_{i=1}^{n} x_i^p}\right), & p = q, \end{cases} \tag{1}$$

for any real numbers $p, q \in \mathbb{R}$. It is easy to see that $G(0,-1;X)$ is the harmonic mean, $G(0,0;X)$ is the geometric mean, and $G(1,0;X)$ is the arithmetic mean of $X$. There has been a number of literature such as [1, 2, 5, 7, 8, 9] and the related references therein about inequalities and properties of Gini means.

In this note, we will study the convexity of weighted Gini means defined as

$$G(p,q;W,X) = \begin{cases} \left(\dfrac{\sum\limits_{i=1}^{n} (w_i x_i)^p}{\sum\limits_{i=1}^{n} (w_i x_i)^q}\right)^{1/(p-q)}, & p \neq q, \\ \exp\left(\dfrac{\sum\limits_{i=1}^{n} (w_i x_i)^p \ln(w_i x_i)}{\sum\limits_{i=1}^{n} (w_i x_i)^p}\right), & p = q, \end{cases} \tag{2}$$

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where $W = (w_1, \cdots, w_n) \in \mathbb{R}^n$ is a positive weight vector such that $\sum\limits_{i=1}^{n} w_i = 1$. Clearly, $nG(1, 0; W, X)$ is the weighted arithmetic mean of $X$. The weight vector determines the relative importance of each quantity on the average. In weighted mean, instead of each of the data points contributing equally to the final average, some data points contribute more than others. The notion of weighted mean plays an important role in descriptive statistics [4] and also occurs in many other areas of mathematics.

The main result in this note is established as follows.

**Theorem 1.1.** *The following three statements hold:*
*(i)* $G(p, q; W, X)$ *is increasing with respect to* $p \in (-\infty, +\infty)$ *and* $q \in (-\infty, +\infty)$*;*
*(ii)* $\ln G(p, q; W, X)$ *is convex with respect to* $p$ *and* $q$ *when* $(p, q) \in (-\infty, 0) \times (-\infty, 0)$*;*
*(iii)* $\ln G(p, q; W, X)$ *is concave with respect to* $p$ *and* $q$ *when* $(p, q) \in (0, +\infty) \times (0, +\infty)$.

## 2. Proof of Theorem 1.1

To begin with, we note that the logarithm of weighted Gini mean (2) can be expressed by

$$\ln G(p, q; W, X) = \begin{cases} \dfrac{1}{p-q} \displaystyle\int_q^p \frac{\sum\limits_{i=1}^{n} (w_i x_i)^s \ln(w_i x_i)}{\sum\limits_{i=1}^{n} (w_i x_i)^s} \mathrm{d}s, & p \neq q, \\ \dfrac{\sum\limits_{i=1}^{n} (w_i x_i)^p \ln(w_i x_i)}{\sum\limits_{i=1}^{n} (w_i x_i)^p}, & p = q. \end{cases} \tag{3}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}s} \left( \frac{\sum\limits_{i=1}^{n} (w_i x_i)^s \ln(w_i x_i)}{\sum\limits_{i=1}^{n} (w_i x_i)^s} \right) = \frac{\sum\limits_{i<j} (w_i x_i)^s (w_j x_j)^s \left(\ln(w_i x_i) - \ln(w_j x_j)\right)^2}{\left(\sum\limits_{i=1}^{n} (w_i x_i)^s\right)^2} > 0, \tag{4}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \left( \frac{\sum\limits_{i=1}^{n} (w_i x_i)^s \ln(w_i x_i)}{\sum\limits_{i=1}^{n} (w_i x_i)^s} \right) =$$

$$= -\frac{\sum\limits_{i<j} (w_i x_i)^s (w_j x_j)^s ((w_i x_i)^s - (w_j x_j)^s)(\ln(w_i x_i) - \ln(w_j x_j))^3}{(\sum\limits_{i=1}^{n} (w_i x_i)^s)^3} =$$

$$= \begin{cases} \geq 0, & s \leq 0, \\ \leq 0, & s \geq 0. \end{cases} \tag{5}$$

Hence, the integrand in (3) is increasing for $s \in (-\infty, +\infty)$, convex for $s \in (-\infty, 0)$, and concave for $s \in (0, +\infty)$.

Suppose that $f(t)$ is a function and its integral arithmetic mean is defined as

$$\phi(p, q) = \begin{cases} \frac{1}{p-q} \int_q^p f(t) \mathrm{d}t, & p \neq q, \\ f(p), & p = q. \end{cases} \tag{6}$$

It is straightforward to check that if $f(t)$ is differentiable and increasing on some interval $I$, then $\phi(p,q)$ is also increasing with respect to $p$ and $q$ on $I$. Furthermore, if $f(t)$ is twice differentiable and convex on $I$, then $\phi(p,q)$ is also convex with respect to $p$ and $q$ on $I$ (see e.g. [6] Lemma 2.1).

Therefore, by using (3), (4), (5) and the fact that logarithmic function is increasing, we conclude that Gini mean $G(p,q;W,X)$ is increasing with respect to $p \in (-\infty,+\infty)$ and $q \in (-\infty,+\infty)$, logarithmically convex with respect to $p$ and $q$ for $(p,q) \in (-\infty,0) \times (-\infty,0)$, logarithmically concave with respect to $p$ and $q$ for $(p,q) \in (0,+\infty) \times (0,+\infty)$. $\square$

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