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ON THE BEST POLYNOMIAL APPROXIMATIONS OF FUNCTIONS OF MANY REAL VARIABLES FROM THE SOBOLEV SPACE WITH DERIVATIVES OF BOUNDED VARIATION

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Abstract. The paper is devoted to estimates of the best joint polynomial approximations in uniform and integral metrics on a given bounded convex domain of the functions of many real variables (FMRV) either with convex derivatives of a certain order or from the seminormalized Sobolev space with derivatives of a certain order of bounded variation.

Keywords: exact estimates, best joint polynomial approximation, function in many real variables, seminormalized Sobolev space, derivatives of bounded variation.

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1. Introduction

In the paper [3] the exact (in the sense of the order of smallness) estimates of the best joint polynomial approximations of FMRV with convex (up or down) on the nonempty intersection of each straight line with given bounded convex domain derivatives of a certain order of the Lipschitz class in uniform and integral metrics are proved. These results are multidimensional analogs or multidimensional generalizations of some well-known results obtained for the polynomial approximations of functions of one variable.

The paper is devoted to some generalizations of the results of paper [3], related with joint polynomial approximations in uniform and integral metrics on a given bounded convex domain of FMRV either with convex derivatives of a certain order or from the seminormalized Sobolev space with derivatives of a certain order of bounded variation on the nonempty intersection of each straight line with the given bounded convex domain.

2. Some definitions and notations

In order to expound the results of this paper we need some definitions and notations. Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space equipped with the usual norm $||\cdot||(n \geq 1)$, G - bounded closed convex domain in $\mathbb{R}^n$, $\mathbf{e}$ - n-dimensional unit vector, $L_p(G)$ - quasi-normalized space of all measurable with respect to n-dimensional Lebesque measure real-valued functions on G whose pth power is integrable; this space is equipped by the quasi-norm

$$||f||_{p,G} = \left(\int\limits_G |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} \quad (0 < p < \infty),$$

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$$||f||_{\infty,G} := ess \sup\{|f(\mathbf{x})| : \mathbf{x} \in G\}.$$

For $0 < p \leq \infty$ and integer $l \geq 0$ we denote by $W_p^{(l)}(G)$ the isotropic Sobolev space of functions $f \in L_p(G)$ on G that possess generalized (in the sense of Sobolev) lth order partial derivatives $D^{(l)}(\mathbf{e})f(D^{(0)}(\mathbf{e})f \equiv f)$ in the direction of each n-dimensional unit vector $\mathbf{e}$; this space is equipped by the finite quasi-seminorm (see [6]):

$$|f|_{W_p^{(l)}(G)} = \sup\left\{||D^{(l)}(\mathbf{e})f||_{p,G} : \mathbf{e} \in \mathbb{R}^n, ||\mathbf{e}|| = 1\right\}.$$

For $l \in \mathbb{Z}_+$, by $C^{(l)}(G)$ we denote the set of all functions f having for each $\mathbf{e}$ the derivative $D^{(l)}(\mathbf{e})f$ which is continuous on G.

For integer $k \geq 0$, the symbol $\Delta^{(k)}(\varepsilon\mathbf{e})f$ denotes the kth difference of the function f with the step $\varepsilon \geq 0$ in the direction of the unit vector $\mathbf{e}$:

$$\Delta^{(0)}(\varepsilon\mathbf{e})f(\mathbf{x}) \equiv f(\mathbf{x}),\ \Delta^{(k)}(\varepsilon\mathbf{e})f(\mathbf{x}) := \sum_{i=0}^{k}(-1)^{k-i}C_k^i f(\mathbf{x}+i\varepsilon\mathbf{e})\ (k \geq 1).$$

If for each $\mathbf{e}$ and some integer $l \geq 0$ the derivative $D^{(l)}(\mathbf{e})f \in L_p(G)(C(G))$ exists on the set G, then for $\delta \geq 0$ we define the isotropic kth order moduli of smoothness for the lth order derivatives as follows

$$\omega_k^{(l)}(f,\delta)_p = \sup\{\sup\{||\Delta^{(k)}(\varepsilon\mathbf{e})D^{(l)}(\mathbf{e})f||_{p,G_{k\varepsilon\mathbf{e}}} : 0 \leq \varepsilon \leq \delta\} :$$
$$\mathbf{e} \in \mathbb{R}^n,\ ||\mathbf{e}|| = 1\},$$
$$\Big(\omega_k^{(l)}(f,\delta) = \sup\{\sup\{||\Delta^{(k)}(\varepsilon\mathbf{e})D^{(l)}(\mathbf{e})f||_{C(G_{k\varepsilon\mathbf{e}})} : 0 \leq \varepsilon \leq \delta\} :$$
$$\mathbf{e} \in \mathbb{R}^n,\ ||\mathbf{e}|| = 1\}\Big),$$

where $G_{k\varepsilon\mathbf{e}}$ is the subset of G consisting of all points $\mathbf{x} \in G$ such that the intervals $[\mathbf{x},\ \mathbf{x}+k\varepsilon\mathbf{e}]$ of the lines entirely lie in G. The superscript (subscript) on $\omega_k^{(l)}(\cdot)$ is omitted if this variable is zero (one).

If $D^{(l)}(\mathbf{e})f \in C(G)$ for any $\mathbf{e}$ and for some positive numbers K, α we have the inequality $\omega^{(l)}(f,\ \delta) \leq K\delta^{\alpha}$; then we write $f \in W^l KH^{\alpha}(G)$. We denote by $Conv_n^{(l)}(G)$ the set of all functions f, such that for any vector $\mathbf{e}$ $D^{(l)}(\mathbf{e})f$ is continuous on G and convex (up or down) on the nonempty intersection of each parallel to vector $\mathbf{e}$ straight line $L \subset \mathbb{R}^n$ with domain G. Let

$$M^{(l)}(f,G) := \sup\left\{||D^{(l)}(\mathbf{e})f||_{C(G)} : \mathbf{e} \in \mathbb{R}^n,\ ||\mathbf{e}|| = 1\right\}.$$

For some $M = const > 0$ we denote by $Conv_n^{(l)}(M,G)$ the set of all functions $f \in Conv_n^{(l)}(G)$, which satisfy the inequality $M^{(l)}(f,G) \leq M$.

The set of all functions $f \in Conv_n^{(l)}(G)$ having the derivative $D^{(l)}(\mathbf{e})f \in KH^{\alpha}(G)$ for each vector $\mathbf{e}$, we denote by $Conv_n^{(l)}H^{\alpha}(K,G)$.

Let $V_n^{(l)}(M,G)$ be the set of all functions f having, for each vector $\mathbf{e}$, a finite derivative $D^{(l)}(\mathbf{e})f$ on the nonempty intersection of each parallel to vector $\mathbf{e}$ straight line $L \subset \mathbb{R}^n$ with the domain G the total variations of which on these intersections are bounded by the same number $M > 0$ for all vectors $\mathbf{e}$.

For integers $N \geq 0$ and $n \geq 1$, by $\mathbf{P}_{N,n}$ we denote the set of all algebraic polynomials $g(\mathbf{x}) = \sum C_{\lambda}\mathbf{x}^{\lambda}$ of degree $\leq N$ with respect to all n variables, where $\lambda = (\lambda_1,\ \ldots,\ \lambda_n)$ is a multi-index with integer nonnegative components λ_i, $\mathbf{x}^{\lambda} = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$, C_{λ} are real numbers, and the sum is taken over all λ such that $|\lambda| := \lambda_1 + \ldots + \lambda_n \leq N$. By $E_N(f,G)_p$ $(E_N^{(s)}(f,G)_p)$

we denote the least deviation of the function $f \in L_p(G)\,(f \in W_p^{(l)}(G))$ from the polynomials $g \in \mathbf{P}_{N,n}$ with respect to the quasi-norm (quasi-seminorm) of the space $L_p(G)\,(W_p^{(s)}(G))$ i. e.,

$$E_N(f,G)_p := \inf\{||f-g||_{p,G} : g \in \mathbf{P}_{N,n}\},$$

$$\left(E_N^{(s)}(f,G)_p := \inf\{|f-g|_{W_p^{(s)}(G)} : g \in \mathbf{P}_{N,n}\}\right).$$

We set

$$E_N^{(s)}(X)_p := \sup\{E_N^{(s)}(f,G)_p :\ f \in X\},$$

where X is some set of functions.

Throughout this paper, $C(\beta_1,\beta_2,\ldots)$, $C_1(\beta_1,\beta_2,\ldots)$ denote positive constants which depend only on the parameters indicated in parenthesis and on the subscripts.

3. Short history of the problems and the main results of the paper

Let Δ be a finite closed interval with the length $|\Delta|$. From the Jackson's theorem [2] we have the estimates

$$E_N(Conv_1^{(l)}(M,\Delta)) \le C(l)|\Delta|^l N^{-l} \sup_{f\in Conv_1^{(l)}(M,\Delta)} \omega^{(l)}(f,\frac{|\Delta|}{N}), \tag{1}$$

$$E_N(Conv_1^{(l)}H^\alpha(K,\Delta)) \le C(l)K|\Delta|^{l+\alpha}N^{-l-\alpha} \tag{2}$$

for all $l \in \mathbb{Z}_+$, $N \ge l$.

In the paper [1] by K. G. Ivanov the estimate

$$E_N(Conv_1^{(0)}(M,\Delta))_1 \le C_1 M|\Delta|N^{-2} \tag{3}$$

was proved for all $N \in \mathbb{N}$. In particular, this implies that for all $\alpha \in (0,1]$ we have the estimate

$$E_N(Conv_1^{(0)}H^\alpha(K,\Delta))_1 \le C_2K|\Delta|^{1+\alpha}N^{-2}. \tag{4}$$

In the paper [7] by M. P. Stojanova it was obtained an estimate which implies that for all $N \in \mathbb{N}$, $1 < p \le \infty$ and $\alpha \in (0,1]$ one has

$$E_N(Conv_1^{(0)}H^\alpha(K,\Delta))_p \le C_3K|\Delta|^{\alpha+1/p}N^{-\alpha-\frac{2-\alpha}{p}}. \tag{5}$$

The exactness of the estimates (2), (3), (4) and also (5) for $\alpha = 1$ in the sense of the order of smallness follows from results of the book of A. F. Timan [8], where it is shown that the exact orders of tending to zero as $n \to \infty$ of the value $E_N(x^q|x|^\beta,[-1,1])$ for $q = -1,0,1,\ldots$; $\beta \in (0,2)$ and of the value $E_N(|x|,[-1,1])_p$ for $1 \le p \le \infty$ are equal to $N^{-q-\beta}$ and $N^{-1-1/p}$ respectively.

The following result obtained by us in the paper [3] is a multidimensional analog of the estimates (2), (4) and (5) for $\alpha = 1$ and their generalizations on the best polynomial approximations in quasi-seminorm of the space $W_p^{(s)}(G)$ for all $p : 0 < p \le \infty$ and $s = 0,1,2,\ldots,l+1$.

Theorem 3.1. *Let G be a bounded closed convex domain in $\mathbb{R}^n$ $(n \ge 2)$, $l \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $K = const > 0$. Then at $s = 0,1,2,\ldots,l+1$ and $N \to \infty$ we have*

$$E_N^{(s)}(Conv_n^{(l)}H^1(K,G))_p \asymp N^{-l-1+s}\times$$
$$\times \sup_{f\in Conv_n^{(l)}H^1(K,G)} \omega_k^{(l+1)}(f,\frac{1}{N})_p \asymp N^{-l-1-1/p+s}$$

for $1 \le p \le \infty$ and

$$E_N^{(s)}(Conv_n^{(l)}H^1(K,G))_p \asymp N^{-l-1+s}\times$$

$$\times \sup_{f\in Conv_n^{(l)} H^1(K,G)} \omega_k^{(l+1)}(f,\frac{1}{N})_1 \asymp N^{-l-2+s}$$

for $0 < p < 1$.

The following theorem is a generalization of the upper bounds of Theorem 3.1. ([4]).

Theorem 3.2. *Let* G *be a bounded closed convex domain in* $\mathbb{R}^n$ $(n \ge 2)$, $l \in \mathbb{Z}_+$, $f \in Conv_n^{(l)}(G)$. *Then for any* $s = 0, 1, 2, \dots, l$, $1 \le p \le \infty$, *and natural numbers* $N \ge l+1$ *we have the estimate*

$$E_N^{(s)}(f,G)_p \le C(n,l,s,p,G)N^{-l-1/p+s}\omega^{(l)}(f,\frac{1}{N}).$$

Corollary 3.1. *If* G *is a bounded closed convex domain in* $\mathbb{R}^n$ $(n \ge 2)$, $l \in \mathbb{Z}_+$, $K = Const > 0$, $f \in Conv_n^{(l)} H^\alpha(K,G)$, *then for any* $s = 0, 1, \dots, l$, $1 \le p \le \infty$, *and natural number* $N \ge l+1$ *the inequality*

$$E_N^{(s)}(f,G)_p \le C(n,l,s,p,\alpha,G)KN^{-l-\alpha-1/p+s}$$

holds.

The following theorem is a generalization of Theorem 3.1 and the main result of the paper.

Theorem 3.3. *Let* G *be a bounded closed convex domain in* $\mathbb{R}^n$ $(n \ge 2)$, $l \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $M = Const > 0$. *Then for any* $s = 0, 1, 2, \dots, l$ *and for* $N \to \infty$ *we have the relations*

$$E_N^{(s)}(W_p^{(l)}(G) \cap V_n^{(l)}(M,G))_p \asymp N^{-l+s} \times$$

$$\times \sup_{f\in W_p^{(l)}(G)\cap V_n^{(l)}(M,G)} \omega_k^{(l)}(f,\frac{1}{N})_p \asymp N^{-l-1/p+s}$$

for $1 \le p \le \infty$ *and*

$$E_N^{(s)}(W_1^{(l)}(G) \cap V_n^{(l)}(M,G))_p \asymp N^{-l+s} \times$$

$$\times \sup_{f\in W_1^{(l)}(G)\cap V_n^{(l)}(M,G)} \omega_k^{(l)}(f,\frac{1}{N})_1 \asymp N^{-l-1+s}$$

for $0 < p < 1$.

Remark 3.1. *Since* $Conv_n^{(l)} H^1(K,G) \subset W_p^{(l+1)}(G) \cap V_n^{(l+1)}(2K,G)$ *the upper estimate of Theorem 3.1 follows from Theorem 3.3. The function realizing the lower estimate of Theorem 3.3 belong to the class* $Conv_n^{(l)} H^1(K,G)$ *for* $1 \le p \le \infty$ *after substituting* l *by* $l+1$. *The function mentioned above can be changed so that it also will belong to the class* $Conv_n^{(l)} H^1(K,G)$ *and gives the lower estimate of Theorem 3.1 for* $0 < p < 1$ *(see Remark 5.1 below). Thus Theorem 3.1 is a consequence of Theorem 3.3.*

Theorem 3.4. *If* G *is a bounded closed convex domain in* $\mathbb{R}^n$ $(n \ge 2)$, $l \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $M = Const > 0$, $1 < p \le \infty$ *and* $f \in C^{(l)}(G) \cap V_n^{(l)}(M,G)$, *then for all* $s = 0, 1, 2, \dots, l$ *and natural numbers* $N \ge \max\{1, l\}$ *the estimate*

$$E_N^{(s)}(f,G)_p \le C(n,l,k,s,p,G)M^{1/p}N^{-l-1/p+s}\left(\omega^{(l)}(f,\frac{1}{N})\right)^{1-1/p}$$

holds, i.e. for every function $f \in C^{(l)}(G) \cap V_n^{(l)}(M,G)$

$$E_N^{(s)}(f,G)_p = o(N^{-l-1/p+s}) \qquad as \qquad N \to \infty.$$

Corollary 3.2. *If G is a bounded closed convex domain in $\mathbb{R}^n$ $(n \geq 2)$, $l \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $M = Const > 0$, $1 < p \leq \infty$ and $f \in W^l KH^\alpha(G) \cap V_n^{(l)}(M, G)$, then for all $s = 0, 1, 2, \dots, l$ and natural numbers $N \geq \max\{1, l\}$ the estimate*

$$E_N^{(s)}(f, G)_p \leq C(n, l, k, s, p, G) M^{1/p} K^{1-1/p} N^{-l-1/p-\alpha(1-1/p)+s}$$

holds.

The proofs of Theorems 3.3 and 3.4 are based on the use of several auxiliary propositions. We will expound the schemes of proofs of the upper bounds of Theorems 3.3, 3.4 and of the lower bound of Theorem 3.3.

4. Scheme of proof of the upper bounds of the theorems 3.3 and 3.4

Proposition 4.1. *If G is a bounded closed domain in $\mathbb{R}^n$ $(n \geq 1)$, $l \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$ and $f \in W_p^{(l)}(G)$, then at $\forall \delta \in (0, \infty)$ and $s = 0, 1, \dots, l$ the estimate*

$$\omega_{l-s+k}^{(s)}(f, \delta)_p \leq \delta^{l-s} \omega_k^{(l)}(f, \delta)_p$$

holds.

Proof. From the definition of the isotropic module of smoothness of the $(l - s + k)$th order, the Lebegue theorem on representation of an absolutely continuous function as the indefinite integral of its derivative up to a constant summand and the generalized Minkowski inequality we get

$$\omega_{l-s+k}^{(s)}(f, \delta)_p := \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} ||\Delta^{(k)}(h\mathbf{e}) \Delta^{(l-s)}(h\mathbf{e}) D^{(s)}(\mathbf{e}) f(\mathbf{x})||_{p, G_{(l-s+k)h\mathbf{e}}} =$$

$$= \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} ||\Delta^{(k)}(h\mathbf{e}) \int\limits_{[0,\, h]^{l-s}} D^{(l)}(\mathbf{e}) f\left(\mathbf{x} + \sum_{j=1}^{l-s} t_j \mathbf{e}\right) \prod_{j=1}^{l-s} dt_j ||_{p, G_{(l-s+k)h\mathbf{e}}} \leq$$

$$\leq \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} \int\limits_{[0,\, h]^{l-s}} \left\{ \int\limits_{G_{(l-s+k)h\mathbf{e}}} \left| \Delta^{(k)}(h\mathbf{e}) D^{(l)}(\mathbf{e}) f\left(\mathbf{x} + \sum_{j=1}^{l-s} t_j \mathbf{e}\right) \right|^p \times \right.$$

$$\left. \times d\mathbf{x} \right\}^{1/p} \prod_{j=1}^{l-s} dt_j = \delta^{l-s} \omega_k^{(l)}(f, \delta)_p.$$

□

Proposition 4.2. *If the function f has the bounded variation $V(f, \Delta)$ on the finite segment $\Delta = [a, b]$, then for any $\delta = (0, |\Delta|/2)$ we have*

$$\omega(f, \delta)_p \leq C(p) \begin{cases} (V(f, \Delta))^{1/p} \delta^{1/p} (\omega(f, \delta))^{1-1/p}, & 1 \leq p \leq \infty \\ V(f, \Delta)\delta, & 0 < p < 1. \end{cases}$$

Proof. Step 1. It is sufficient to prove Proposition 4.2 for the segment $[0, 1]$ and a function φ with $V(\varphi, [0, 1]) = 1$. Indeed, if $y = a + (b - a)x$, $x \in [0, 1]$, and $f(y) = V(f, \Delta)\varphi(x)$, $x \in [0, 1]$, then

$$\omega(f, \delta)_p := \sup_{0 \leq h \leq \delta} \left[\int_a^{b-h} |f(y + h) - f(y)|^p dy \right]^{1/p} =$$

$$= V(f, \Delta) |\Delta|^{1/p} \sup_{0 \leq h \leq \delta} \left[\int_0^{1 - \frac{h}{|\Delta|}} |\varphi(x + \frac{h}{|\Delta|}) - \varphi(x)|^p dx \right]^{1/p} =$$

$$= V(f,\Delta)|\Delta|^{1/p}\omega(\varphi,\frac{\delta}{|\Delta|})_p \leq$$

$$\leq C(p)V(f,\Delta)\begin{cases}\delta^{1/p}(\omega(\varphi,\frac{\delta}{|\Delta|}))^{1-1/p}, & 1\leq p\leq\infty\\ \delta, \quad 0<p<1\end{cases} =$$

$$= C(p)\begin{cases}(V(f,\Delta))^{1/p}\delta^{1/p}(\omega(f,\delta))^{1-1/p}, & 1\leq p\leq\infty\\ V(f,\Delta)\delta, \quad 0<p<1.\end{cases}$$

Step 2. Let the function f have the total variation $V(f,[0,1])$ equal to 1. Further we need the following theorem.

Theorem 4.1. *Any function f with bounded variation can be represented as the difference of two monotone nondecreasing functions:*

$$f(x)=v(x)-u(x), \quad x\in[0,1], \quad v(x)=V(f,[0,x]).$$

Using this theorem for the function $\psi(x)=f(x)-f(0)=v(x)-u(x)$, where $v(x)=V(\psi,[0,x])$ and $u(x)=v(x)-\psi(x)$, we get the estimates

$$|v(x)|\leq V(\psi,[0,1])=V(f,[0,1])=1,$$
$$|u(x)|\leq|f(x)-f(0)|+|v(x)|\leq 2. \tag{6}$$

From the inequalities (6) we obtain

$$\int\limits_0^{1-h}|f(x+h)-f(x)|dx=\int\limits_0^{1-h}|\psi(x+h)-\psi(x)|dx\leq$$

$$\leq\int\limits_0^{1-h}|v(x+h)-v(x)|dx+\int\limits_0^{1-h}|u(x+h)-u(x)|dx=$$

$$=\int\limits_{1-h}^{1}v(x)dx-\int\limits_0^{h}v(x)dx+\int\limits_{1-h}^{1}u(x)dx-\int\limits_0^{h}u(x)dx\leq 5h. \tag{7}$$

The inequality (7) gives us the required estimate for all $p\geq 1$ and $0\leq h\leq\delta$:

$$\int\limits_0^{1-h}|f(x+h)-f(x)|^{1+p-1}dx\leq 5\delta(\omega(f,\delta))^{p-1}.$$

From the last estimate it follows that

$$\omega(f,\delta)_p\leq 5^{1/p}\delta^{1/p}(\omega(f,\delta))^{1-\frac{1}{p}}. \tag{8}$$

Now let $0<p<1$. Using the Holder's integral inequality and the inequality (7), we get

$$\Big(\int\limits_0^{1-h}|f(x+h)-f(x)|^p dx\Big)^{1/p}\leq\int\limits_0^{1-h}|f(x+h)-f(x)|dx\leq 5\delta$$

for $0\leq h\leq\delta$.

Using the last estimate, we obtain the inequality

$$(\omega(f,\delta))_p\leq 5\delta. \tag{9}$$

The inequalities (8) and (9) give us Proposition 4.2 for the function f under consideration. Proposition 4.2 is proved. □

Let $\mathbf{e}$ be any unit vector of the space $\mathbb{R}^n$ $(n \geq 2)$, let $\Pi(\mathbf{e})$ be the $n-1$-dimensional subspace orthogonal to the vector $\mathbf{e}$, let $G(\mathbf{e})$ be the orthogonal projection of the bounded closed convex domain $G \subset \mathbb{R}^n$ to the subspace $\Pi(\mathbf{e})$, and let $\Pi_{n-1}(G) := \sup\limits_{\mathbf{e}} V(G(\mathbf{e}))$ be the maximal volume of the $n-1$-dimensional projections of G. Let $\mathbf{y} \in G(\mathbf{e})$ be an arbitrary point, let $[a(\mathbf{y}), b(\mathbf{y})]$ be the segment obtained by intersection of the straight line $L \subset \mathbb{R}^n$ passing through $\mathbf{y}$ and parallel to the vector $\mathbf{e}$ with the domain G.

Proposition 4.3. *If G is a bounded closed convex domain in $\mathbb{R}^n\,(n \geq 2)$, $l \in \mathbb{Z}_+$, $f \in W_p^{(l)}(G) \cap V_n^{(l)}(M, G)$, then*

$$\omega^{(l)}(f,\delta)_p \leq C(p)\Pi_{n-1}^{1/p}(G) \begin{cases} M^{1/p}\delta^{1/p}(\omega^{(l)}(f,\delta))^{1-1/p}, & 1 \leq p \leq \infty \\ M\delta, \quad 0 < p < 1. \end{cases}$$

Proof. Taking into account the Fubini's theorem, for $0 < p < \infty$ we have

$$\omega^{(l)}(f,\delta)_p := \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} \left\{ \int\limits_{G_{h\mathbf{e}}} |\Delta^{(1)}(h\mathbf{e}) D^{(l)}(\mathbf{e}) f(\mathbf{x})|^p d\mathbf{x} \right\}^{1/p} =$$

$$= \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} \left\{ \int\limits_{G_{h\mathbf{e}}(\mathbf{e})} \int\limits_{a(\mathbf{y})}^{b(\mathbf{y})-h\mathbf{e}} |\Delta^{(1)}(h\mathbf{e}) D^{(l)}(\mathbf{e}) f(\xi)|^p d\xi d\mathbf{y} \right\}^{1/p} \leq$$

$$\leq \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} V^{1/p}(G_{h\mathbf{e}}(\mathbf{e})) \sup_{\mathbf{e}} \sup_{0 \leq h \leq \delta} \left\{ \int\limits_{a(\mathbf{y})}^{b(\mathbf{y})-h\mathbf{e}} |\Delta^{(1)}(h\mathbf{e}) D^{(l)}(\mathbf{e}) f(\xi)|^p d\xi \right\}^{1/p}.$$

Therefore, using Proposition 4.2 we get

$$\omega^{(l)}(f,\delta)_p \leq C(p)\Pi_{n-1}^{1/p}(G) \begin{cases} M^{1/p}\delta^{1/p}(\omega^{(l)}(f,\delta))^{1-1/p}, & 1 \leq p \leq \infty \\ M\delta, \quad 0 < p < 1. \end{cases}$$

□

Further we need the following theorem of V. N. Konovalov [5].

Theorem 4.2. ***(V. N. Konovalov)*** *If G is a bounded domain in $\mathbb{R}^n\,(n \geq 2)$ with Lipschitzian boundary, $1 \leq p \leq \infty$, l and k are nonnegative integers such that $l + k > 0$, and $f \in W_p^{(l)}(G)$, then for each natural number $N \geq l + k - 1$ there exists a polynomial $g \in \mathbf{P}_{N,n}$ such that the inequality*

$$|f - g|_{W_p^{(s)}(G)} \leq C(n,l,k,s,G)\omega_{l-s+k}^{(s)}\left(f, \frac{1}{N}\right)_p$$

holds for $s = 0, 1, \ldots, l$.

The upper bounds of Theorem 3.3 for $1 \leq p \leq \infty$ and Theorem 3.4 for $1 < p \leq \infty$ follow directly from Theorem 4.2 due to Propositions 4.1 and 4.3. For $0 < p < 1$ the upper bound of Theorem 3.3 we get using Theorem 3.3 for $p = 1$, and the Holder's integral inequality.

5. Scheme of proof of the lower bound of theorem 3.3.

Proposition 5.1. *For any $l \in \mathbb{Z}_+$ and natural numbers N, n the function $f(\mathbf{x}) := f_l(\mathbf{x}) := x_1^{l-1}|x_1|$ is such that*

$$f \in W_p^{(l)}([-1,\, 1]^n) \cap V_n^{(l)}(M,\, [-1,\, 1]^n)$$

with the constant $M = 2l!$ and for $1 \le p \le \infty$ the estimate

$$E_N^{(s)}\left(f,\, [-1,\, 1]^n\right)_p \ge C(n,p,l,s)N^{-l-1/p+s}$$

holds.

Proof. First it is easy to show that

$$f \in W_p^{(l)}([-1,\, 1]^n) \cap V_n^{(l)}(2l!,\, [-1,\, 1]^n).$$

Then using the Fubini's theorem it is also easy to show that

$$E_N^{(s)}\left(f,\, [-1,\, 1]^n\right)_p \ge 2^{(n-1)/p} E_N^{(s)}\left(f,\, [-1,\, 1]\right)_p.$$

Further, to complete the proof of the desired lower bounds, we use the well-known statement from the book of A.F. Timan [8] that for any $q \in \mathbb{Z}_+$ and $1 \le p \le \infty$ the exact order of decreasing of the value $E_N(x_1^{q-1}|x_1|,\, [-1,\, 1])_p$ as $N \to \infty$ is $N^{-q-1/p}$. □

Let $G \subset \mathbb{R}^n$ $(n \ge 1)$ be a bounded open set. By $Q = \{\mathbf{x} \in \mathbb{R}^n : a_i \le x_i \le b_i,\, i = 1, \dots, n\}$ we denote the maximal n-dimensional cube contained in $\overline{G}$ (in particular, $b_i - a_i = b_j - a_j$).

Proposition 5.2. *For any bounded open set $G \subset \mathbb{R}^n (n \ge 1)$, any $l \in \mathbb{Z}_+$ the function*

$$f(\mathbf{x}) := f_{l,G}(\mathbf{x}) := \left(\frac{b_1 - a_1}{2}\right)^l \left(\frac{2}{b_1 - a_1} x_1 - \frac{a_1 + b_1}{b_1 - a_1}\right)^{l-1} \times$$

$$\times \left|\frac{2}{b_1 - a_1} x_1 - \frac{a_1 + b_1}{b_1 - a_1}\right|$$

belongs to the class $W_p^{(l)}(\overline{G}) \cap V_n^{(l)}(2l!,\, \overline{G})$ and is such that for any $1 \le p \le \infty$, $s = 0, 1, \dots, l$ and a natural number N the inequality

$$E_N^{(s)}\left(f,\, \overline{G}\right)_p \ge C(n,p,l,s,G)N^{-l-1/p+s}$$

holds.

Proof. Obviously one has $f \in W_p^{(l)}(\overline{G}) \cap V_n^{(l)}(2l!,\, \overline{G})$. Then passing to the linear transformation $\mathbf{y} = L(\mathbf{x}) : Q \to [-1,\, 1]^n$ and using Proposition 4, we have

$$E_N^{(s)}\left(f,\, \overline{G}\right)_p \ge E_N^{(s)}\left(f,\, Q\right)_p =$$

$$= [(b_1 - a_1)/2]^{l-s+n/p} E_N^{(s)}\left(f_l,\, [-1,\, 1]^n\right)_p \ge C(n,p,l,s,G)N^{-l-1/p+s}.$$

Proposition 5.2 is proved. □

Proposition 5.3. *For any $l \in \mathbb{Z}_+$, $N, n \in \mathbb{N}$ the function*

$$f(\mathbf{x}) := f_{N,l}(\mathbf{x}) := (6\pi N)^{-l-1} \sin(6\pi N x_1)$$

belongs to the class $W_p^{(l)}\left([0,\, 1]^n\right) \cap V_n^{(l)}\left(\frac{2}{\pi},\, [0,\, 1]^n\right)$ for $0 < p \le \infty$ and is such that for all $s = \overline{0,\, l+1}$ the estimate

$$E_N^{(s)}\left(f,\, [0,\, 1]^n\right)_p \ge (6\pi)^{-l-1+s}[6(1+p)]^{-1/p} N^{-l-1+s}$$

holds.

Proof. It is easy to see that $f \in W_p^{(l)}([0,\,1]^n) \cap V_n^{(l)}\left(\frac{2}{\pi},\,[0,\,1]^n\right)$ for $0 < p \le \infty$. As in the proof of Proposition 5.1 using the Fubini's theorem we get the inequality

$$E_N^{(s)}(f,\,[0,\,1]^n)_p \ge E_N^{(s)}(f,\,[0,\,1])_p\,. \tag{10}$$

Further, to obtain lower bounds of the latter, we use the simple idea consisting in the following: if an approximated function has the oscillation large enough, then any polynomial of degree not exceeding N cannot approximate it and differs from it on a significant part of the segment Δ under consideration. The function in Proposition 5.3 has this property.

Let s be an even number. Then

$$f^{(s)}(\mathbf{x}) := \frac{\partial^s f(\mathbf{x})}{\partial x_1^s} = (-1)^{s/2}(6\pi N)^{-l-1+s}\sin(6\pi N x_1)$$

and $x_i^0 = i/(6N)$, $i = 0, 1, \ldots, 6N$ are the zeros of the function $f^{(s)}$, belonging to the segment $[0,1]$. Therefore, the function $f^{(s)}$ has, on the segment $[0,1]$ $6N+1$, pieces of monotony. Let $g_1 \in P_{N,1}$ be any polynomial. The number of zeros of the function $g^{(s+1)}$ does not exceed $N-s-1$. Thus the number of pieces of monotony of the function $g_1^{(s)}$ does not exceed $N-s$. Let

$$\Delta_j := \left[x_{3j}^0,\, x_{3j+3}^0\right],\, j = \overline{0, 2N-1};\;\; \delta_i := \left[x_{i-1}^0,\, x_i^0\right],\; i = \overline{1,\, 6N}.$$

Denote by Δ'_j the segments of Δ_j inside of which $g^{(s)}$ is monotone. The segments Δ_j inside of which the monotony of the function $g_1^{(s)}$ breaks we denote by Δ''_j . Obviously, the number of the segments Δ''_j does not exceed $N-s-1$. Thus the number of segments Δ'_j is not less than $N+s+1$. Since inside the segment Δ'_j the function $g_1^{(s)}$ is monotone, there is at least one segment $\delta_i \subset \Delta'_j$ such that at all points of it the inequality $f^{(s)}(x_1)g_1^{(s)}(x_1) \le 0$ holds.

Thus for any segment Δ'_j and $0 < p < \infty$ we have

$$\left\{\int_{\Delta'_j}\left|f^{(s)}(x_1) - g_1^{(s)}(x_1)\right|^p dx_1\right\}^{1/p} \ge \left\{\int_{x_{i-1}^0}^{x_i^0}\left|f^{(s)}(x_1)\right|^p dx_1\right\}^{1/p} \ge$$
$$\ge (6\pi)^{-l-1+s}[6(1+p)]^{-1/p}N^{-l-1-1/p+s}.$$

Hence

$$||f^{(s)} - g_1^{(s)}||_{p,\,[0,1]} \ge \left\{\sum_{\Delta'_j}\int_{\Delta'_j}\left|f^{(s)}(x_1) - g_1^{(s)}(x_1)\right|^p dx_1\right\}^{1/p} \ge$$
$$\ge (6\pi)^{-l-1+s}[6(1+p)]^{-1/p}N^{-l-1+s}. \tag{11}$$

From (10) and (11), for $0 < p < \infty$ we have the estimate

$$E_N^{(s)}(f,\,[0,\,1]^n)_p \ge (6\pi)^{-l-1+s}[6(1+p)]^{-1/p}N^{-l-1+s}.$$

Going to the limit as $p \to \infty$ in the last inequality, we get

$$E_N^{(s)}(f,\,[0,\,1]^n)_\infty \ge (6\pi)^{-l-1+s}N^{-l-1+s}.$$

If s is an odd number, we will have the same estimates. Proposition 5.3 is proved. $\square$

Proposition 5.4. *Let G be a closed bounded domain in $\mathbb{R}^n\,(n \ge 1)$, let $Q = \{\mathbf{x} \in \mathbb{R}^n : a_i \le x_i \le b_i,\, i = 1, \ldots, n\}$ be the maximal n-dimensional cube contained in G and let $Q' = \{\mathbf{x} \in \mathbb{R}^n : a_i' \le x_i \le b_i',\, i = 1, \ldots, n\}$ be the minimal n-dimensional cube containing G , $d = \max\{a_1 - a_1',\, b_1' - b_1\}$, $l \in \mathbb{Z}_+$, $N \ge l+1$ is a natural number. Then the function*

$$f(\mathbf{x}) = [(b_1 - a_1)/(6\pi N)]^{l+1}\sin[6\pi N(x_1 - a_1)/(b_1 - a_1)]$$

possesses the properties:

1) *for each unit vector* $\mathbf{e} \in \mathbb{R}^n$ *and any* $0 < p \leq \infty$

$$f \in W_p^{(l)}\left(Q'\right) \cap V_n^{(l)}\left(\frac{2(V(Q))^{1/n} + 4d}{\pi}, Q'\right);$$

2) *for all* $s = 0, 1, \ldots, l+1$ *and* $0 < p \leq \infty$ *the estimate*

$$E_N^{(s)}\left(f, G\right)_p \geq C(n, p, l, s, G) N^{-l-1+s}$$

holds.

Proof. Obviously, $f \in W_p^{(l)}(Q')$ and

$$V_n^{(l)}(f, Q') \leq 6N \frac{2(b_1 - a_1)}{6\pi N} + 2 \frac{6dN}{b_1 - a_1} \cdot \frac{2(b_1 - a_1)}{6\pi N} =$$

$$= \frac{2(b_1 - a_1)}{\pi} + \frac{4d}{\pi} = \frac{2(V(Q))^{1/n} + 4d}{\pi},$$

where $V(Q)$ is the volume of Q. Therefore, $f \in W_p^{(l)}(Q') \cap V_n^{(l)}(K, Q')$ with $K = [2(V(Q))^{1/n} + 4d]/\pi$. Further, passing to the linear transformation of the arguments of the function f and using Proposition 5.3, we get

$$E_N^{(s)}\left(f, G\right)_p \geq E_N^{(s)}\left(f, Q\right)_p = (b_1 - a_1)^{l+1-s+n/p} E_N^{(s)}\left(f, [0, 1]^n\right)_p \geq$$

$$\geq C(n, p, l, s, G) N^{-l-1+s}.$$

□

Remark 5.1. *On the base of Proposition 5.4 it is not difficult to notice that for the function*

$$f(\mathbf{x}) = (x_1 - a_1)^{l+2} + [(b_1 - a_1)/(6\pi N)]^{l+2} \sin[6\pi N (x_1 - a_1)/(b_1 - a_1)]$$

belonging to the class $Conv_n^{(l)} H^1(K, G)$ *with the constant*

$$K = [(l+2)! + 1/(6\pi N)]|G|,$$

where $|G|$ *is the diameter of* G*, the estimate*

$$E_N^{(s)}\left(f, G\right)_p \geq C(n, p, l, s, G) N^{-l-2+s}$$

holds for all $s = \overline{0, l+2}$, $0 < p \leq \infty$, *and any natural number* $N \geq l+2$.

The lower bound of Theorem 3.3 immediately follow from Propositions 5.2 and 5.4.

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