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LIFTS OF $(1,1)$-TENSOR FIELDS ON PURE CROSS-SECTIONS OF (p,q) -TENSOR BUNDLES

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Abstract. In this paper we show that the tensor bundles $T_q^p(M_n)$ admit an almost algebraic $\tilde{\Pi}$-structure if the base manifold M_n admits an integrable almost algebraic Π-structure.

Keywords: algebraic structure, complete lifts, pure cross-section, Tachibana operator, tensor bundles.

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1. Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_q^p(M_n) = \bigcup_{P\in M_n} T_q^p(P)$ is, by definition, the tensor bundle of type (p,q) over M_n, where $\bigcup$ denotes the disjoint union of the tensor spaces $T_q^p(P)$ for all $P \in M_n$. For any point $\tilde{P}$ of $T_q^p(M_n)$ such that $\tilde{P} \in T_q^p(M_n)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T_q^p(M_n) \to M_n$. The projection π defines the natural differentiable manifold structure of $T_q^p(M_n)$, that is, $T_q^p(M_n)$ is a C^∞-manifold of dimension $n + n^{p+q}$. If x^j are local coordinates in a neighborhood U of $P \in M_n$, then a tensor t at P which is an element of $T_q^p(M_n)$ is expressible in the form $(x^j, t_{j_1\dots j_q}^{i_1\dots i_p})$, where $t_{j_1\dots j_q}^{i_1\dots i_p}$ are components of t with respect to natural base. We may consider $(x^j, t_{j_1\dots j_q}^{i_1\dots i_p}) = (x^j, x^{\bar{j}}) = x^J$, $j = 1,...,n$, $\bar{j} = n+1,...,n+n^{p+q}$, $J = 1,...,n+n^{p+q}$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

We denote by $\Im_s^r(M_n)$ the $F(M_n)$ module of all tensor fields of class C^∞ and of type (r,s) on M_n, where $F(M_n)$ is the ring of C^∞-functions on M_n. If $\alpha \in \Im_p^q(M_n)$, it is regarded, by contraction, as a function in $T_q^p(M_n)$, which we denote by $\imath\alpha$. If α has the local expression

$$\alpha = \alpha_{i_1\dots i_p}^{j_1\dots j_q} \partial_{j_1} \otimes \dots \otimes \partial_{j_q} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

in a coordinate neighborhood $U(x^j) \subset M_n$, then $\imath\alpha = \alpha(t)$ has the local expression

$$\imath\alpha = \alpha_{i_1\dots i_p}^{j_1\dots j_q} t_{j_1\dots j_q}^{i_1\dots i_p}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $\pi^{-1}(U)$.
Suppose that $A \in \Im_q^p(M_n)$. Then there is a unique vector field ${}^VA \in \Im_0^1(T_q^p(M_n))$ (*vertical lift* of A) such that for all $\alpha \in \Im_p^q(M_n)$ [4]

$${}^VA(\imath\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A)),$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \Im_0^0(M_n)$. We call ${}^VA$ the vertical lift of $A \in \Im_q^p(M_n)$ to $T_q^p(M_n)$.The vertical lift ${}^VA$ has components of the form

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$$^{V}A = \left(\ ^{V}A^{j}, \quad ^{V}A^{\bar{j}} \ \right) = \left(\ 0, \quad A^{i_1 \ldots i_p}_{j_1 \ldots j_q} \ \right) \tag{1}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T^p_q(M_n)$.
We define the complete lift ^{C}V of V to $T^p_q(M_n)$ (see [4]) by $^{C}V(\imath\alpha) = \imath(L_V\alpha)$, for all $\alpha \in \Im^q_p(M_n)$.
The complete lift ^{C}V of $V \in \Im^1_0(M_n)$ to $T^p_q(M_n)$ has components of the form

$$^{C}V = \left(\ V^j, \quad \sum_{\lambda=1}^{P} t^{i_1 \ldots m \ldots i_p}_{j_1 \ldots j_q} \partial_m V^{i_\lambda} - \sum_{\mu=1}^{q} t^{i_1 \ldots i_p}_{j_1 \ldots m \ldots j_q} \partial_{j_\mu} V^m \ \right) \tag{2}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T^p_q(M_n)$, where Γ^k_{ij} are local components of ∇ in M_n.

2. Cross-section in the tensor bundle

Suppose that there is given a tensor field $\xi \in \Im^p_q(M_n)$. Then the correspondence $x \rightarrow \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a mapping $\sigma_\xi : M_n \rightarrow T^p_q(M_n)$, such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T^p_q(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local component $\xi^{h_1 \ldots h_p}_{k_1 \ldots k_q}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\begin{cases} x^k = x^k \\ x^{\bar{k}} = \xi^{h_1 \ldots h_p}_{k_1 \ldots k_q}(x^k) \end{cases} \tag{3}$$

with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T^p_q(M_n)$. Differentiating (3) by x^j, we see that n tangent vector fields B_j to $\sigma_\xi(M_n)$ have components

$$(B^K_j) = \left(\frac{\partial x^K}{\partial x^j}\right) = \left(\ \delta^k_j, \quad \partial_j \xi^{h_1 \ldots h_p}_{k_1 \ldots k_q} \ \right) \tag{4}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^p_q(M_n)$.
On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k \quad = const, \\ t^{h_1 \ldots h_p}_{k_1 \ldots k_q} = t^{h_1 \ldots h_p}_{k_1 \ldots k_q}, \end{cases}$$

where $t^{h_1 \ldots h_p}_{k_1 \ldots k_q}$ being considered as parameters. Thus, on differentiating with respect to $x^{\bar{j}} = t^{i_1 \ldots \ldots i_p}_{j_1 \ldots j_q}$, we see that n^{p+q} tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$(C^K_{\bar{j}}) = \left(\frac{\partial x^K}{\partial x^{\bar{j}}}\right) = \left(\ 0, \quad \delta^{j_1}_{k_1} \ldots \delta^{j_q}_{k_q} \delta^{h_1}_{i_1} \ldots \delta^{h_p}_{i_p} \ \right) \tag{5}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^p_q(M_n)$, where δ is the Kronecker symbol.

Definition 2.1. *A vector field X along a cross-section $\sigma_\xi : M_n \rightarrow T^p_q(M_n)$ is mapping $X : M_n \rightarrow T(T^p_q(M_n))$ ($T(T^p_q(M_n))$- tangent bundle over the manifold $T^p_q(M_n)$) such that $\tilde{\pi} \circ x = \sigma_\xi$, where $\tilde{\pi}$ is the projection $\tilde{\pi} : T(T^p_q(M_n)) \rightarrow T^p_q(M_n)$.*

The vector field X assigns to each point $x \in M_n$ a tangent vector to $T^p_q(M_n)$ at $\sigma_\xi(x)$ and therefore $n + n^{p+q}$ local vector fields B_j and $C_{\bar{j}}$ in $\tilde{\pi}^{-1}(U) \subset T^p_q(M_n)$ are vector fields along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_{\bar{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B, C)- frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. From $^{C}V = {}^{C}V^h \partial_h + {}^{C}V^{\bar{h}} \partial_{\bar{h}}$ and $^{C}V = {}^{C}V^j B_j + {}^{C}V^{\bar{j}} C_{\bar{j}}$, We easily obtain $^{C}V^k = {}^{C}V^j B^k_j + {}^{C}V^{\bar{j}} C^k_{\bar{j}}$, $^{c}V^{\bar{k}} = {}^{C}V^j B^{\bar{k}}_j + {}^{C}V^{\bar{j}} C^{\bar{k}}_{\bar{j}}$. Now, taking account of (2) on the cross-section $\sigma_\xi(M_n)$, and also (4) and (5), we have $^{C}\tilde{V}^k = V^k$, $^{C}\tilde{V}^{\bar{k}} = -L_V \xi^{h_1 \ldots h_p}_{k_1 \ldots k_q}$. Thus, the complete lift ^{C}V has along $\sigma_\xi(M_n)$ components of the form

$$^{C}V = \left(\begin{array}{cc} V^{k}, & -L_{V}\xi^{h_1...h_p}_{k_1...k_q} \end{array} \right) \tag{6}$$

with respect to the adapted (B, C)- frame. From (1), (4) and (5), by using similar way the vertical lift ^{V}A also has components of the form

$$^{V}A = \left(\begin{array}{cc} 0, & A^{h_1...h_p}_{k_1...k_q} \end{array} \right) \tag{7}$$

with respect to the adapted (B, C)- frame.

3. The vertical-vector lift of a tensor field of type (1,1)

Let $\varphi \in \Im^1_1(M_n)$. Using the Jacobian matrix of the coordinate transformation in $T^p_q(M_n)$

$$\begin{cases} x^{j'} = x^{j'}(x^{j}), \\ x^{\bar{j}'} = t^{i'_1...i'_p}_{j'_1...j'_q} = A^{i'_1}_{i_1}...A^{i'_p}_{i_p}A^{j_1}_{j'_1}...A^{j_q}_{j'_q}t^{i_1...i_p}_{j_1...j_q} = A^{(i')}_{(i)}A^{(j)}_{(j')}x^{\bar{j}}, \end{cases}$$

where $A^{(i')}_{(i)}A^{(j)}_{(j')} = A^{i'_1}_{i_1}...A^{i'_p}_{i_p}A^{j_1}_{j'_1}...A^{j_q}_{j'_q}$, $A^{i'_1}_{i_1} = \frac{\partial x^{i'}}{\partial x^{i}}$, $A^{j_1}_{j'_1} = \frac{\partial x^{j}}{\partial x^{j'}}$ we can define a vector field $\gamma\varphi \in \Im^1_0(T^p_q(M_n))$, $p \geq 1, q \geq 0$[1]:

$$\gamma\varphi = ((\gamma\varphi)^{J}) = \left(\begin{array}{cc} 0, & t^{li_2...i_p}_{j_1...j_q}\varphi^{i_1}_{l} \end{array} \right),$$

where $\varphi^{i_1}_{l}$ are local components of φ in M_n. Clearly, we have $\gamma\varphi(^{V}f) = 0$ for any $f \in F(M_n)$. Thus $\gamma\varphi$ is a vertical-vector lift of the tensor field $\varphi \in \Im^1_1(M_n)$ to $T^p_q(M_n)$. We can easily verify that the vertical-vector lift $\gamma\varphi$ has along $\sigma_\xi(M_n)$ components

$$\gamma\varphi = ((\widetilde{\gamma\varphi})^{K}) = \left(\begin{array}{cc} 0, & \xi^{lh_2...h_p}_{k_1...k_q}\varphi^{h_1}_{l} \end{array} \right) \tag{8}$$

with respect to the adapted (B, C)-frame, where $\xi^{h_1...h_p}_{k_1...k_q}$ are local components of ξ in M_n.

4. Tachibana operator and complete lifts of affinor fields on a pure cross-section

A tensor field $\xi \in \Im^p_q(M_n)$ is called pure with respect to $\varphi \in \Im^1_1(M_n)$, if [8-11]:

$$\xi(\varphi X_1, X_2, ..., X_q, \alpha_1, \alpha_2, ..., \alpha_p) = \xi(X_1, \varphi X_2, ..., X_q, \alpha_1, \alpha_2, ..., \alpha_p) = ... =$$

$$= \xi(X_1, X_2, ..., \varphi X_q, \alpha_1, \alpha_2, ..., \alpha_p) = \xi(X_1, X_2, ..., X_q, \varphi' \alpha_1, \alpha_2, ..., \alpha_p) = \tag{9}$$

$$= \xi(X_1, X_2, ..., X_q, \alpha_1, \varphi' \alpha_2, ..., \alpha_p) = ... = \xi(X_1, X_2, ..., X_q, \alpha_1, \alpha_2, ..., \varphi' \alpha_p)$$

for any $X_1, X_2, ..., X_q \in \Im^1_0(M_n)$, $\alpha_1, \alpha_2, ..., \alpha_p \in \Im^0_1(M_n)$, where $(\varphi'\alpha)(X) = \alpha(\varphi X)$ $X \in \Im^1_0(M_n)$, $\alpha \in \Im^0_1(M_n)$. In particular, vector and covector fields will be considered to be pure. We shall now derive explicit expressions for ϕ_φ-operator (or Tachibana operator) which applied to an arbitrary pure tensor field of type (p, q). Explicit formulae of ϕ_φ-operator for pure tensor fields of types $(1, q)$ and $(0, q)$ are given in [9]. Also in [9] derives relations between the geometry of hyperholomorphic B-manifolds (Norden manifolds) and ϕ_φ-operator. We note that, ϕ_φ-operator is *extension* of the operator of Lie derivation L_X, $X \in \Im^1_0(M_n)$ to affinor fields $\varphi \in \Im^1_1(M_n)$. We denote by $\overset{*}{\Im}{}^r_s(M)$ the module of all pure tensor fields of type (r, s) on M with respect to the $(1, 1)$-tensor field φ. We now fix a positive integer λ. If K and L are pure tensor fields of types (p_1, q_1) and (p_2, q_2) respectively, then the tensor product of K and L with contraction $K \overset{C}{\otimes} L = (K^{i_1...m_\lambda...i_{p_1}}_{j_1...j_{q_1}} L^{r_1...r_{p_2}}_{s_1...m_\lambda...s_{q_2}})$ is also a pure tensor field.

We shall now make the direct sum $\overset{*}{\Im}(M) = \sum_{r,s=0}^{\infty} \overset{*}{\Im}{}^r_s(M)$ into the algebra over the real number $\mathbb{R}$ by defining the pure product (denoted by $\overset{C}{\otimes}$ or " $\circ$ ") of $K \in \overset{*}{\Im}{}^{p_1}_{q_1}(M)$ and $L \in \overset{*}{\Im}{}^{p_2}_{q_2}(M)$ as follows:

$$
\begin{aligned}
\overset{C}{\otimes} \;\; &: \;\; (K, L) \to (K \overset{C}{\otimes} L) = \\
&= \begin{cases} K^{i_1...m_\lambda...i_{p1}}_{j_1...j_{q_1}} L^{r_1...r_{p_2}}_{s_1...m_\lambda...s_{q_2}} \text{ for } \lambda \le p_1, q_2 \text{ } (\lambda \text{ is a fixed positive integer}), \\ K^{i_1...i_{p1}}_{j_1...m_\mu...j_{q_1}} L^{r_1...m_\mu...r_{p_2}}_{s_1...s_{q_2}} \text{ for } \mu \le p_2, q_1 \text{ } (\mu \text{ is a fixed positive integer}), \\ 0 \qquad\qquad\qquad \text{for } p_1 = 0, \; \mathrm{p}_2 = 0, \\ 0 \qquad\qquad\qquad \text{for } \mathrm{q}_1 = 0, \; q_2 = 0. \end{cases}
\end{aligned}
$$

In particular, let $K = X \in \Im^1_0(M)$, and $L \in \Lambda_q(M)$ be a $q-$form. Then the pure product $X \overset{C}{\otimes} L$ coincides with the interior product $\iota_X L$.

Definition 4.1. *[8,9] Let $\varphi \in \Im^1_1(M)$, and $\Im(M) = \sum_{r,s=0}^{\infty} \Im^r_s(M)$ be a tensor algebra over $\mathbb{R}$. A map $\phi_\varphi : \overset{*}{\Im}(M) \to \Im(M)$ is called a Tachibana operator or $\phi_\varphi-$operator on M if*
(a) ϕ_φ is linear with respect to constant coefficients,
(b) $\phi_\varphi : \overset{*}{\Im}{}^r_s(M) \to \Im^r_{s+1}(M)$ for all r, s,
(c) $\phi_\varphi(K \overset{C}{\otimes} L) = (\varphi_\phi K) \overset{C}{\otimes} L + K \overset{C}{\otimes} \varphi_\phi L$ for all $K, L \in \overset{*}{\Im}(M)$.
(d) $\phi_{\varphi X} Y = -(L_Y \varphi) X$ for all $X, Y \in \Im^1_0(M)$, where L_Y is the Lie derivation with respect to Y.
(e) $\phi_{\varphi X}(\iota_Y \omega) = (d(\imath_Y \omega))(\varphi X) - (d(\imath_Y(\omega \circ \varphi)))(X) = (\varphi X)(\iota_Y \omega) - X(\iota_{\varphi Y} \omega)$ for all $\omega \in \Im^0_1(M)$ and $X, Y \in \Im^1_0(M)$, where $\iota_Y \omega = \omega(Y) = \omega \overset{C}{\otimes} Y$.

Theorem 4.1. *Let $\omega \in \overset{*}{\Im}{}^0_s(M)$. Then*

$$\phi_{\varphi X}\left(\omega\left(Y_1, ..., Y_s\right)\right) = \left(\varphi X\right)\left(\omega\left(Y_1, ..., Y_s\right)\right) - X\left(\omega\left(\varphi Y_1, ..., Y_s\right)\right).$$

Proof. (see [8]). □

Let $t \in \overset{*}{\Im}{}^r_s(M)$, $\mathrm{r} > 1$, $\mathrm{s} \ge 1$. We now define a pure tensor field of type (0,s) $t_{\xi^1,\xi^2,...,\xi^r} \in \overset{*}{\Im}{}^0_s(M)$ by $t_{\xi^1,\xi^2,...,\xi^r}\left(Y_1, Y_2, ..., Y_s\right) = t\left(Y_1, Y_2, ..., Y_s, \xi^1, \xi^2, ..., \xi^r\right)$, where $t_{\xi^1,\xi^2,...,\xi^r}$ has components of the form:

$$\left(t_{\xi^1,\xi^2,...,\xi^r}\right)_{j_1 j_2 ... j_s} = t^{i_1...i_r}_{j_1...j_s} \xi^1_{i_1} \xi^2_{i_2} ... \xi^r_{i_r}.$$

According to Theorem 4.1, we find

$$
\begin{aligned}
& \phi_{\varphi X} t\left(Y_1, ..., Y_s, \xi^1, \xi^2, ..., \xi^r\right) = \\
= \;& \phi_{\varphi X} t_{\xi^1,...,\xi^r}\left(Y_1, ..., Y_s\right) = \\
= \;& \left(\varphi X\right) t_{\xi^1,...,\xi^r}\left(Y_1, ..., Y_s\right) - X t_{\xi^1,...,\xi^r}\left(\varphi Y_1, ..., Y_s\right) = \\
= \;& \left(\varphi X\right) t\left(Y_1, ..., Y_s, \xi^1, ..., \xi^r\right) - X t\left(\varphi Y_1, ..., Y_s, \xi^1, ..., \xi^r\right).
\end{aligned}
$$

Then, using $\phi_{\varphi X}\xi^{\mu} = L_{\varphi X}\xi^{\mu} - L_X(\xi^{\mu}\circ\varphi)$, we see that $\phi_{\varphi}t$ for $t \in \overset{*}{\Im}{}^r_s(M)$, r > 1, s ≥ 1, is by definition, a tensor field of type $(r, s+1)$ given by

$$\begin{aligned} & (\phi_{\varphi}t)\left(X, Y_1, ..., Y_s, \xi^1, ..., \xi^r\right) = \\ = \; & (\phi_{\varphi X}t)\left(Y_1, ..., Y_s, \xi^1, ..., \xi^r\right) = \\ = \; & \phi_{\varphi X}t\left(Y_1, ..., Y_s, \xi^1, ..., \xi^r\right) - \sum_{\lambda=1}^{s} t\left(Y_1, ..., \phi_{\varphi X}Y_{\lambda}, ..., Y_s, \xi^1, ..., \xi^r\right) - \\ & -\sum_{\mu=1}^{r} t\left(Y_1, ..., Y_s, \xi^1, ..., \phi_{\varphi X}\xi^{\mu}, ..., \xi^r\right) - \\ & -\sum_{\mu=1}^{r} t\left(Y_1, ..., Y_s, \xi^1, ..., L_{\varphi X}\xi^{\mu} - L_X(\xi^{\mu}\circ\varphi), ..., \xi^r\right). \end{aligned} \tag{10}$$

By setting $X = \partial_k,\ Y_{\lambda} = \partial_{j_{\lambda}},\ \xi^{\mu} = dx^{i_{\mu}},\ \lambda = 1, ..., s;\ \mu = 1, ..., r$ in the equation (10), we see that the components $(\phi_{\varphi}t)^{i_1...i_r}_{j_1...j_s}$ of $\phi_{\varphi}t$ with respect to local coordinate system $x^1, ..., x^n$ may be expressed as follows:

$$\begin{aligned} (\phi_{\varphi}t)^{i_1...i_r}_{kj_1...j_s} &= \varphi^m_k \partial_m t^{i_1...i_r}_{j_1...j_s} - \partial_k (t\circ\varphi)^{i_1...i_r}_{j_1...j_s} + \sum_{\lambda=1}^{s} \left(\partial_{j_{\lambda}}\varphi^m_k\right)\ t^{i_1...i_r}_{j_1...m...j_s} + \\ & + \sum_{\mu=1}^{r}\left(\partial_k\varphi^{i_{\mu}}_m - \partial_m\varphi^{i_{\mu}}_k\right)\ t^{i_1...m...i_r}_{j_1...j_s}, \end{aligned} \tag{11}$$

where $(t\circ\varphi)^{i_1...i_r}_{j_1...j_s} = t^{i_1...i_r}_{m...j_s}\varphi^m_{j_s} = ... = t^{i_1...i_r}_{j_1...m}\varphi^m_{j_s} = t^{m...i_r}_{j_1...j_s}\varphi^{i_1}_m = ... = t^{i_1...m}_{j_1...j_s}\varphi^{i_r}_m$.

Let $\overset{*}{\Im}{}^p_q(M_n)$ denotes a module of all the tensor fields $\xi \in \Im^p_q(M_n)$ which are pure with respect to φ. Now, we consider a pure cross-section $\sigma^{\varphi}_{\xi}(M_n)$ determined by $\xi \in \overset{*}{\Im}{}^p_q(M_n)$, $p \geq 1,\ q \geq 0$. We observe that the local vector fields

$$^{C}X_{(j)} = {}^{C}\left(\frac{\partial}{\partial x^j}\right) = {}^{C}\left(\delta^h_j \frac{\partial}{\partial x^h}\right) = \begin{pmatrix} \delta^h_j, & 0 \end{pmatrix}$$

and

$$\begin{aligned} {}^{V}X^{(\bar{j})} &= {}^{V}(\partial_{j_1}\otimes ... \otimes \partial_{j_p} dx^{i_1}\otimes ... \otimes dx^{i_q}) = \\ &= {}^{V}(\delta^{i_1}_{h_1} ... \delta^{i_q}_{h_q}\delta^{k_1}_{j_1} ... \delta^{k_p}_{j_p}\partial_{k_1}\otimes ... \otimes \partial_{k_p}\otimes dx^{h_1}\otimes ... \otimes dx^{h_q}) = \\ &= \begin{pmatrix} 0, & \delta^{i_1}_{h_1} ... \delta^{i_q}_{h_q}\delta^{k_1}_{j_1} ... \delta^{k_p}_{j_p} \end{pmatrix}, \end{aligned}$$

$j = 1, ..., n,\ \bar{j} = n+1, ..., n+n^{p+q}$ span the module of vector fields in $\pi^{-1}(U)$. Hence any tensor field is determined in $\pi^{-1}(U)$ by its action of ${}^{C}X_{(j)}$ and ${}^{V}X^{(\bar{j})}$. Then we define a tensor field ${}^{C}\varphi \in \Im^1_1(T^p_q(M_n))$ along the pure cross-section $\sigma^{\varphi}_{\xi}(M_n)$ by

$$\begin{cases} {}^{C}\varphi({}^{C}V) = {}^{C}(\varphi(V)) - \gamma(L_V\varphi) + {}^{V}((L_V\varphi)\circ\xi),\ \forall V \in \Im^1_0(M_n),\ (i) \\ {}^{C}\varphi({}^{V}A) = {}^{V}(\varphi(A)),\ \forall A \in \Im^p_q(M_n),\ (ii) \end{cases} \tag{12}$$

where $\varphi(A) \in \Im^p_q(M_n)$, $((L_V\varphi)o\xi)(X_1, ..., X_q; \alpha_1, ..., \alpha_p) = \xi(X_1, ..., X_q; (L_V\varphi)'\alpha_1, ..., \alpha_p)$ and call ${}^{C}\varphi$ the complete lift of $\varphi \in \Im^1_1(M_n)$ to $T^p_q(M_n)$, $p \geq 1$, $q \geq 0$ along $\sigma^{\varphi}_{\xi}(M_n)$[4]. In particular, if we assume that $p = 1$, $q > 0$ then we get

$$\gamma(L_V\varphi) = {}^{V}((L_V\varphi)\circ\xi),$$

substituting this into (12), we find (see [6])

$$^C\varphi(^CV) = {}^C(\varphi(V)),\ {}^C\varphi(^VA) = {}^V(\varphi(A)).$$

Remark 4.1. *The equation (*12*) is useful extension of the equation* $^CL(\imath\alpha) = \imath(L_V\alpha)$, $\alpha \in \Im^q_p(M_n)$ *(see [4]) to affinor fields along the pure cross-section* $\sigma^\varphi_\xi(M_n)$.

Let $^C\tilde{\varphi}^K_L$ be components of $^C\varphi$ with respect to the adapted (B,C)- frame of the pure cross-section $\sigma^\varphi_\xi(M_n)$. Then, from (7) and (12) we have

$$\begin{cases} {}^C\tilde{\varphi}^{KC}_L\tilde{V}^L = {}^C(\tilde{\varphi(V)})^K - (\gamma(\tilde{L}_V\varphi))^K + {}^V((\tilde{L_V\varphi})\circ\xi)^K,\ (i) \\ {}^C\tilde{\varphi}^{KV}_L\tilde{A}^L = {}^V(\tilde{\varphi(A)})^K,\ (ii) \end{cases} \tag{13}$$

where $({}^V(\tilde{\varphi(A)})^K) = \begin{pmatrix} 0, & \varphi^{h_1}_m A^{mh_2\ldots h_p}_{k_1\ldots k_q} \end{pmatrix}$, ${}^V((\widetilde{L_V\varphi})\circ\xi)^K = \begin{pmatrix} 0, & (L_V\varphi^{h_\lambda}_m)\xi^{h_1\ldots m\ldots h_p}_{k_1\ldots k_q} \end{pmatrix}$, $\gamma(\widetilde{L_V\varphi})^K = \begin{pmatrix} 0, & ((L_V\varphi)^{h_1}_m)\xi^{mh_2\ldots h_p}_{k_1\ldots k_q} \end{pmatrix}$, $L_V\varphi^{h_\lambda}_m$ are local component of $L_V\varphi$ in M_n.

Straightforward computations using the local expression (11) of Tachibana operator and the expressions (13), (6), (8), we obtain that the complete lift $^C\varphi \in \Im^1_1(T^p_q(M_n))$ of φ has along the pure cross-section $\sigma^\varphi_\xi(M_n)$ components

$$\begin{cases} {}^C\tilde{\varphi}^k_l = \varphi^k_l,\ {}^C\tilde{\varphi}^k_{\bar{l}} = 0,\ {}^C\tilde{\varphi}^{\bar{k}}_l = -(\phi_\varphi\xi)^{h_1\ldots h_p}_{lk_1\ldots k_q}, \\ {}^C\tilde{\varphi}^{\bar{k}}_{\bar{l}} = \varphi^{h_1}_{s_1}\delta^{h_2}_{s_2}\ldots\delta^{h_p}_{s_p}\delta^{r_1}_{k_1}\ldots\delta^{r_q}_{k_q} \end{cases} \tag{14}$$

with respect to the adapted (B,C)- frame of $\sigma^\varphi_\xi(M_n)$, where $\phi_\varphi\xi$ is the Tachibana operator and $x^{\bar{k}} = t^{h_1\ldots h_p}_{k_1\ldots k_q}$, $x^{\bar{l}} = t^{s_1\ldots s_p}_{r_1\ldots r_q}$(for details, see [2]).

Remark 4.2. $^C\varphi$ *in the form (*14*) is a unique solution of (*13*). Therefore, if an* $\overset{*}{\varphi}$ *is element of* $\Im^1_1(T^p_q(M_n))$*, such that* $\overset{*}{\varphi}(^CV) = {}^C\varphi(^CV) = {}^C(\varphi(V)) - \gamma(L_V\varphi) + {}^V((L_V\varphi)\circ\xi)$, $\overset{*}{\varphi}(^VA) = {}^C\varphi(^VA) = {}^V(\varphi(A))$*, then* $\overset{*}{\varphi} = {}^C\varphi$.

Remark 4.3. *Taking into account the formula (*14*), and specializing to the case* $p = 1$, $q = 0$*, one has the formula of the complete lift of affinor fields to tangent bundle along the cross-section* $\sigma_\xi(M_n)$ *(for details, see [12, p.126]).*

Remark 4.4. *In the case of* $\partial_m\xi^{h_1\ldots h_p}_{k_1\ldots k_q} = 0$, $(B,C)-$*frame is considered as a natural frame* $\{\partial_h, \partial_{\bar{h}}\}$ *of* $\sigma^\varphi_\xi(M_n)$*. Then, from (*14*) we obtain components of* $^C\varphi$ *along the pure cross-section*

$$\begin{aligned} {}^C\varphi^k_l &= \varphi^k_l,\ {}^C\varphi^k_{\bar{l}} = 0, \\ {}^C\varphi^{\bar{k}}_{\bar{l}} &= \varphi^{h_1}_{s_1}\delta^{h_2}_{s_2}\ldots\delta^{h_p}_{s_p}\delta^{r_1}_{k_1}\ldots\delta^{r_q}_{k_q}, \\ {}^C\varphi^{\bar{k}}_l &= (\partial_l\varphi^{h_1}_m)\xi^{mh_2\ldots h_p}_{k_1\ldots k_q} - \sum_{\mu=1}^{q}(\partial_{k_\mu}\varphi^m_l)\xi^{h_1\ldots h_p}_{k_1\ldots m\ldots k_q} - \\ &\quad - \sum_{\lambda=1}^{p}(\partial_l\varphi^{h_\lambda}_m - \partial_m\varphi^{h_\lambda}_l)\xi^{h_1\ldots m\ldots h_p}_{k_1\ldots k_q} \end{aligned} \tag{15}$$

with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ *of* $\sigma^\varphi_\xi(M_n)$ *in* $\pi^{-1}(U)$ *[7].*

5. Algebraic $\Pi-$ structures on a pure cross-section in $T_q^p(M_n)$

Let $\mathfrak{A}_{\mathrm{m}}$ be an associative commutative unital algebra of finite dimension m over the field R of real numbers. An algebraic $\Pi-$structure in M_n is a collection $\Pi = \left\{ \underset{\alpha}{\varphi} \right\}$, $\alpha = 1, ..., m$ of tensor fields of type $(1,1)$ such that $\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} = C_{\alpha\beta}^{\gamma} \underset{\gamma}{\varphi}$, where $C_{\alpha\beta}^{\gamma}$ are the structure constants of the algebra $\mathfrak{A}_{\mathrm{m}}$.

Theorem 5.1. *If* $\Pi = \left\{ \underset{\alpha}{\varphi} \right\}$ *is an integrable almost algebraic* $\Pi-$*structure in* M_n*, then the complete lift* ${}^{C}\Pi = \left\{ {}^{C} \underset{\alpha}{\varphi} \right\}$ *of* Π *to* $T_q^p(M_n)$ *along the pure cross-section* $\sigma_{\xi}^{\Pi}(M_n)$ *is an almost algebraic* ${}^{C}\Pi-$*structure in* $T_q^p(M_n)$.

Proof. Let $\underset{\alpha}{\varphi},\ \underset{\beta}{\varphi} \in \Pi$ $(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} = \underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi})$ and $S \in \Im_2^1(M_n)$. Then using (6), (8) and (12), we have

$$\begin{aligned} \gamma(\underset{\alpha}{\varphi} \pm \underset{\beta}{\varphi}) &= \gamma \underset{\alpha}{\varphi} \pm \gamma \underset{\beta}{\varphi},\ {}^{C} \underset{\alpha}{\varphi}(\gamma \underset{\beta}{\varphi}) = \gamma(\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}), \\ {}^{C} \underset{\alpha}{\varphi}({}^{V}(\underset{\beta}{\varphi} \circ \xi)) &= {}^{V}((\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}) \circ \xi), \\ (\gamma S)^{C} V &= \gamma S_V, {}^{V}(S \circ \xi)({}^{C}V) = {}^{V}(S_V \circ \xi), \end{aligned} \tag{16}$$

where S_V is the tensor field of type (1.1) in M_n defined by $S_V(W) = S(V,W)$, for any $W \in \Im_0^1(M_n)$. If $V \in \Im_0^1(M_n)$, from (12) and (16), we have

$$\begin{gathered} ({}^{C} \underset{\alpha}{\varphi} \circ {}^{C} \underset{\beta}{\varphi})^{C} V = {}^{C} \underset{\alpha}{\varphi}({}^{C} \underset{\beta}{\varphi}({}^{C}V)) = {}^{C} \underset{\alpha}{\varphi}({}^{C}(\underset{\beta}{\varphi}(V)) - \gamma(L_V \underset{\beta}{\varphi}) + {}^{V}((L_V \underset{\beta}{\varphi}) \circ \xi))) = \\ = {}^{C} \underset{\alpha}{\varphi}({}^{C}(\underset{\beta}{\varphi}(V))) - {}^{C} \underset{\alpha}{\varphi}(\gamma(L_V \underset{\beta}{\varphi})) + {}^{C} \underset{\alpha}{\varphi}({}^{V}((L_V \underset{\beta}{\varphi}) \circ \xi)) = \\ = {}^{C}(\underset{\alpha}{\varphi}(\underset{\beta}{\varphi}(V))) - \gamma(L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) + {}^{V}((L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) \circ \xi) - (\gamma(L_V \underset{\beta}{\varphi}) \circ \underset{\alpha}{\varphi}) + {}^{V}(((L_V \varphi_{\beta}) \circ \underset{\alpha}{\varphi}) \circ \xi)) = \\ = {}^{C}((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})(V)) - \gamma(L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) - \gamma((L_V \underset{\beta}{\varphi}) \circ \underset{\alpha}{\varphi})) + {}^{V}(((L_V \underset{\beta}{\varphi}) \circ \underset{\alpha}{\varphi} + (L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi})) \circ \xi) = \\ = {}^{C}((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})(V)) - \gamma((L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) + L_V(\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}) - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})) + \\ + {}^{V}((L_V(\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}) - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi}) + (L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi})) \circ \xi) = \\ = {}^{C}((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})(V)) - \gamma(L_V(\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi})) + {}^{V}((L_V(\underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi})) \circ \xi)_{\alpha}) - \\ - \gamma((L_{\underset{\beta}{\varphi}(V)} \varphi - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})) + {}^{V}(((L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})) \circ \xi) = \\ = {}^{C}((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})(V)) - \gamma(L_V(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})) + {}^{V}((L_V(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})) \circ \xi)_{\alpha}) - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})) - \\ - \gamma((L_{\underset{\beta}{\varphi}(V)} \varphi + {}^{V}(((L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi}) - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})) \circ \xi) = \\ = {}^{C}(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})({}^{C}V) - \gamma(\underset{\alpha,\beta}{N}\ {}_V) + {}^{V}(\underset{\alpha,\beta}{N}\ {}_V \circ \xi) = {}^{C}(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})({}^{C}V) - \gamma(\underset{\alpha,\beta}{N})({}^{C}V) + {}^{V}(\underset{\alpha,\beta}{N} \circ \xi)({}^{C}V) = \\ = {}^{C}((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi}) - \gamma(\underset{\alpha,\beta}{N}) + {}^{V}(\underset{\alpha,\beta}{N} \circ \xi))({}^{C}V), \end{gathered} \tag{17}$$

where $\underset{\alpha,\beta}{N}\ {}_V = L_{\underset{\beta}{\varphi}(V)} - \underset{\beta}{\varphi} \circ (L_V \underset{\alpha}{\varphi})$. Since $\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} = \underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}$, $(\phi_{\underset{\alpha}{\varphi}} \underset{\beta}{\varphi})(V,W) = (L_{\underset{\beta}{\varphi}(V)} \underset{\alpha}{\varphi} - \underset{\beta}{\varphi} \circ (L_V \varphi))W = [\underset{\alpha}{\varphi} V, \underset{\beta}{\varphi} W] - \underset{\alpha}{\varphi}[V, \underset{\beta}{\varphi} W] - \underset{\beta}{\varphi}[\underset{\alpha}{\varphi} V, W] + \underset{\beta}{\varphi} \circ \underset{\alpha}{\varphi}[V,W] = \underset{\alpha,\beta}{N}\ {}_V W$ is nothing but the Tachibana operator or the Nijenhuis-Shirokov tensor $\underset{\alpha,\beta}{N}(V,W) \in \Im_2^1(M_n)$ constructed from $\underset{\alpha}{\varphi}$ and $\underset{\beta}{\varphi}$ [2].

Similarly, if $A \in \Im_q^p(M_n)$, then by (12), we have

$$\begin{aligned}({}^C\underset{\alpha}{\varphi} \circ {}^C\underset{\beta}{\varphi})^V A &= {}^C\underset{\alpha}{\varphi}({}^C\underset{\beta}{\varphi}\,{}^V A) = {}^C\underset{\alpha}{\varphi}({}^V(\underset{\beta}{\varphi}(A))) = \\ &= {}^V(\underset{\alpha}{\varphi}(\underset{\beta}{\varphi}(A))) = {}^V((\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})(A)) = {}^C(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi})^V A.\end{aligned} \tag{18}$$

Suppose now that ∇ is linear connection (with zero torsion) on M_n. If $\Pi = \left\{\underset{\alpha}{\varphi}\right\}$ is an almost integrable algebraic $\Pi-$structure with respect to ∇, i.e. $\nabla \underset{\alpha}{\varphi} = 0$, $\alpha = 1, ..., m$, then $\underset{\alpha,\beta}{N} = 0$ [2]. If we take $\underset{\alpha,\beta}{N} = 0$, then by the Remark 4.1 made in §4, (17), (18) and the linearity of the complete lift, we have

$${}^C\underset{\alpha}{\varphi} \circ {}^C\underset{\beta}{\varphi} = {}^C(\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi}) = {}^C(C_{\alpha\beta}^{\gamma}\underset{\gamma}{\varphi}) = C_{\alpha\beta}^{\gamma}\,{}^C\underset{\gamma}{\varphi} \quad .$$

□

Let M_n and N_m be two manifolds with algebraic structures $\Pi = \left\{\underset{\alpha}{\varphi}\right\}$ and $\tilde{\Pi} = \left\{\underset{\alpha}{\psi}\right\}$, $\alpha = 1, ..., m$ determined by the same associative commutative unital algebra $\mathfrak{A}_{\mathrm{m}}$. A differentiable mapping $f : M_n \to N_m$ is called a quasi-$\mathfrak{A}$-holomorphic mapping with respect to $(\Pi, \tilde{\Pi})$ (see [5]), if at each point $P \in M_n$

$$df_p \circ \underset{\alpha}{\varphi}\ {}_p = \underset{\alpha}{\psi}\ {}_{f(p)} \circ df_p,\ \alpha = 1, ..., m. \tag{19}$$

As the mapping $f : M_n \to N_m (m = n + n^{p+q})$ we take a cross-section $\sigma_\xi^\Pi : M_n \to T_q^p(M_n)$ determined by the pure tensor field $\xi \in \Im_q^p(M_n)$ with respect to $\Pi-$structure. The pure cross-section $\sigma_\xi^\Pi : M_n \to T_q^p(M_n)$ can be locally expressed by (3). In (19), if $\tilde{\Pi} = \left\{\underset{\alpha}{\psi}\right\}$ is the almost algebraic ${}^C\Pi-$structure (see Theorem 5.1), the condition that the pure cross-section $\sigma_\xi^\Pi : M_n \to T_q^p(M_n)$ be quasi-$\mathfrak{A}$-holomorphic tensor field with respect to $(\Pi, {}^C\Pi)$ is locally given by

$$\underset{\alpha}{\varphi}\ {}_l^m \partial_m x^K = {}^c\underset{\alpha}{\varphi}\ {}_M^K \partial_l x^M, \alpha = 1, ..., m, \tag{20}$$

where ${}^C\underset{\alpha}{\varphi}\ {}_M^K$ are components of ${}^C\underset{\alpha}{\varphi}$ along the pure cross-section $\sigma_\xi^\Pi(M_n)$ with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$. In the case $K = k$, by virtue of (3) and (15) we get the identity $\varphi_l^k = \varphi_l^k$. When $K = \bar{k}$, by virtue of (3), (9) and (15), (20) reduces to

$$\begin{aligned}(\phi_\varphi \xi)_{lk_1...k_q}^{h_1...h_p} &= \varphi_l^m \partial_m \xi_{k_1...k_q}^{h_1...h_p} - \partial_l \overset{*}{\xi}{}_{k_1...k_q}^{h_1...h_p} + \textstyle\sum_{a=1}^q (\partial_{k_a}\varphi_l^m)\, \xi_{k_1...m...k_q}^{h_1...h_p} + \\ &+2\textstyle\sum_{\lambda=1}^p \partial_{[l}\, \varphi_{m]}^{h_\lambda} \xi_{k_1...k_q}^{h_1...m...h_p} = 0,\end{aligned} \tag{21}$$

where $\phi_\varphi \xi$ is the Tachibana operator. Thus, a quasi-$\mathfrak{A}$-holomorphic tensor field with respect to $(\Pi, {}^C\Pi)$ is given by (21). The equation $\phi_\varphi \xi = 0$ is the equation characterizing the usual almost holomorphic tensor field [3], [10]. Thus, if $\Pi-$structure is almost integrable, then our quasi-$\mathfrak{A}$-holomorphic tensor field with respect to $(\Pi, {}^C\Pi)$ coincides with the usual almost holomorphic tensor field.

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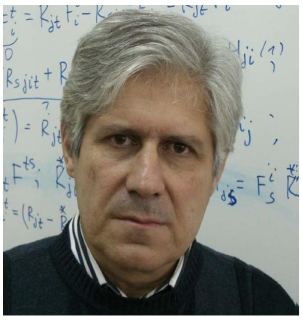

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