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THE METHOD OF SIMILAR SOLUTIONS IN THE TIME OPTIMAL CONTROL PROBLEMS WITH DELAY AND STATE CONSTRAINTS

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Abstract. In the paper the time optimal control problem for the systems with delays and state constraints is investigated. By the method of similar solutions, the Pontryagin's maximum principle is obtained. The form of the obtained principle allows one to select the regular optimal trajectory.

Keywords: optimal control, state constraints, maximum principle, similar solutions.

AMS Subject Classification: 49J15.

1. Introduction

For the first time Pontryagin's maximum principle for the problems with state constraints was obtained by Gamkrelidze R.V. in 1959 [3], [12]. In 1963 another variant of the maximum principle [2] has been received. Then, this matter was a subject of many studies [4], [6], [8], [9]. This list of works is not exhaustive.

In the case, when the restriction is possed only on the control function without state constraints, necessary optimality conditions gives Pontryagin's maximum principle [12], [7]. These problems have been well studied because of the absolute continuity and non-triviality of the adjoint functions.

In the presence of the phase constraint optimal control problem turned out to be much more complicated due to the fact that the adjoint function to obtain the necessary conditions, in general, is a function of bounded variation, which is a rather complicated relationship with the optimal trajectory. Therefore, the optimal control problem with state constraints are outside the scope of the effective application of the Pontryagin's maximum principle [12], its language was complicated and more difficult to make an analytical study in specific situations. Questions arise: Are there any solutions of the optimal control problem for which the corresponding adjoint function is nontrivial and absolutely continuous, and if so, how to find them?

Applying the similar technique in [4] and [5] we try to answer these questions for the systems with delays and phase constraints.

This work is devoted to the elimination of the above noted deficiency by obtaining of the maximum principle for the regular solutions in the problems with delay and state constraints.

The optimal solution is called regular if the corresponding adjoint function is nontrivial absolutely continuous.

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2. Statement of the problem

Consider the optimal control problem

$$\dot{x} = f\left(x, x\left(t-\tau\right), u\right), \ t \in \left[0, T\right], \ T \in \left(\tau, +\infty\right), \ \tau > 0,$$

$$x\left(t\right) = l\left(t\right), t \in \left[-\tau, 0\right), x\left(0\right) = x_\alpha, x\left(T\right) = x_\beta, x_\alpha \neq x_\beta,$$

$$u\left(t\right) \in U, \ \ a.a. \ \ \ t \in \left[0, T\right], \tag{1}$$

$$x\left(t\right) \in X, \ t \in \left[0, T\right],$$

$$T \to \min.$$

Here, $x \in E^n$ is a state variable, $u \in E^m$ is a control parameter. Let $\Omega\left(E^n\right)$ be the set of all nonempty compacts and $conv\ \Omega\left(E^n\right)$- the set of all nonempty compact convex subsets of E^n.

Given function $l\left(t\right), t \in \left[-\tau, 0\right)$ is bounded and measurable. The functions $f\left(x, y, u\right), \ \frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}$ are continuous, $U \in \Omega\left(E^m\right)$ and $f\left(x, y, U\right) \in conv\ \Omega\left(E^n\right), \ x, y \in E^n$. X is a closed convex subset of E^n. $H\left(A, \psi\right) = \max\left\{\left(a, \psi\right): \ a \in A\right\}$ is a support function of $A \subset E^n$ in the direction of ψ, where $\left(a, \psi\right)$ denotes the inner product of vectors a and ψ. All finite-dimensional vectors are considered column vectors.

Function $u\left(t\right) \in U, t \in \left[0, T\right]$ is called an admissible control on the interval $\left[0, T\right]$, if it is measurable and the corresponding solution $x\left(t\right), t \in \left[0, T\right]$ of the given system of equations satisfies to the initial condition $x\left(t\right) = l\left(t\right), t \in \left[-\tau, 0\right), x\left(0\right) = x_\alpha$ and the inclusion $x\left(t\right) \in X, t \in \left[0, T\right]$.

The aim is to find an admissible control $u\left(t\right), t \in \left[0, T\right]$ for which $x\left(T\right) = x_\beta$ and T is minimal.

This work is dedicated to the derivation of the maximum principle for which the optimal solution is regular.

Let $\left(\bar{x}\left(t\right), \bar{u}\left(t\right)\right), t \in \left[0, T\right]$ be a solution of (1) and $U\left(\bar{x}\left(t\right)\right) \subset U$ a subset, for which

$$U\left(\bar{x}\left(t\right)\right) = \left\{u \in U f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), u\right) \in T\left(X, \bar{x}\left(t\right)\right)\right\}, \ \ a.\ a. \ \ t \in \left[0, T\right].$$

Here, $T\left(X, \bar{x}\left(t\right)\right)$ is the tangent cone to X at $\bar{x}\left(t\right) \in X, t \in \left[0, T\right]$, i.e. $T\left(X, \bar{x}\left(t\right)\right) = cl\left\{\lambda\left(y - \bar{x}\left(t\right)\right), \lambda \geq 0, y \in X\right\}$, where clA means the closure of A.

Let us consider the corresponding problem without phase constraints

$$\begin{cases} \dot{x}\left(t\right) = f\left(x\left(t\right), x\left(t-\tau\right), u\left(t\right)\right), \ u\left(t\right) \in U, \ t \in \left[0, T\right], \\ x\left(t\right) = \hat{l}\left(t\right), \ t \in \left[-\tau, 0\right), \ x\left(0\right) = \hat{x}_\alpha, \ x\left(T\right) = \hat{x}_\beta, \ \hat{x}_\alpha \neq \hat{x}_\beta, \\ \qquad T \to \min. \end{cases} \tag{2}$$

Definition 2.1. *If there exist a bounded and measurable initial function* $\hat{l}\left(t\right), \ t \in \left[-\tau, 0\right)$, *endpoints* $\hat{x}_\alpha, \ \hat{x}_\beta$ *and the corresponding solution* $\left(x_0\left(t\right), u_0\left(t\right)\right), \ \ t \in \left[0, T\right]$ *of the problem* (2), *for which the inclusion*

$$T\left(f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), U\left(\bar{x}\left(t\right)\right)\right), \dot{\bar{x}}\left(t\right)\right) \subset T\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), U\right), \dot{x}_0\left(t\right)\right), \ \ a.\ a. \ \ t \in \left[0, T\right]$$

holds, then $\left(x_0\left(t\right), u_0\left(t\right)\right), \ t \in \left[0, T\right]$ *is called similar to the solution* $\left(\bar{x}\left(t\right), \bar{u}\left(t\right)\right), \ t \in \left[0, T\right]$ *of* (1).

Note 1. The main requirement of this definition is the equality of the optimal time in the original (problem with state constraints) and minor (the problem without phase constraints) problems.

Note 2. There are many examples for which the class of similar solutions is nonempty.

Theorem 2.1. *Suppose there is a similar solution* $(x_0(t), u_0(t))$, $t \in [0,T]$ *of* (2) *to*$(\bar{x}(t), \bar{u}(t))$, $t \in [0,T]$. *Then there exists a nonzero absolutely continuous solution* $\psi(t), t \in [0,T]$ *of the adjoint system*

$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}\psi(t) - \frac{\partial f^*(x_0(t+\tau), x_0(t), u_0(t+\tau))}{\partial y}\psi(t+\tau),$$
$$0 \le t \le T-\tau$$
$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}\psi(t), \quad T-\tau \le t \le T,$$

for which the maximum condition

$$\max\{(f(\bar{x}(t), \bar{x}(t-\tau), u), \psi(t)) \ \ u \in U(\bar{x}(t))\} = (\dot{\bar{x}}(t), \psi(t)), \quad a.\ a. \quad t \in [0,T]$$

holds.

If the additional conditions

$$\left(\frac{\partial f^*(\bar{x}(t),\bar{x}(t-\tau),\bar{u}(t))}{\partial x} - \frac{\partial f^*(x_0(t),x_0(t-\tau),u_0(t))}{\partial x}\right)\psi(t) +$$
$$+\left(\frac{\partial f^*(\bar{x}(t+\tau),\bar{x}(t),\bar{u}(t+\tau))}{\partial y} - \frac{\partial f^*(x_0(t+\tau),x_0(t),u_0(t+\tau))}{\partial y}\right)\psi(t+\tau) = 0$$
$$a.\ a. \quad t \in [0, T-\tau]$$

and

$$\left(\frac{\partial f^*(\bar{x}(t), \bar{x}(t-\tau), \bar{u}(t))}{\partial x} - \frac{\partial f^*(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}\right)\psi(t) = 0 \ \ a.\ a. \ \ t \in [T-\tau, T],$$

hold, then the adjoint system of the differential equations has the form

$$\dot{\psi}(t) = -\frac{\partial f^*(\bar{x}(t), \bar{x}(t-\tau), \bar{u}(t))}{\partial x}\psi(t) - \frac{\partial f^*(\bar{x}(t+\tau), \bar{x}(t), \bar{u}(t+\tau))}{\partial y}\psi(t+\tau),$$
$$a.\ a. \ \ t \in [0, T-\tau],$$
$$\dot{\psi}(t) = -\frac{\partial f^*(\bar{x}(t), \bar{x}(t-\tau), \bar{u}(t))}{\partial x}\psi(t), \quad a.\ a. \ \ t \in [\,T-\tau, T].$$

Before the proof of the theorem we give the following lemma.

Lemma 2.1. *Let* $x(t)$, $t \in [0,T]$ *be absolutely continuous function such that* $x(t) \in X$, $\forall t \in [0,T]$, *where* $X \subset E^n$ *closed convex subset. Then*

$$\dot{x}(t) \in T(X, x(t)), \quad a.\ a. \ \ t \in [0,T].$$

The lemma is proved in [12].

Corollary 2.1. *If* $x(t)$, $t \in [0,T]$ *is an admissible trajectory of* (1), *then* $f(x(t), x(t-\tau), U) \bigcap \bigcap T(X, x(t)) \ne \oslash$, $a.\ a.$ $t \in [0,T]$. *The proof of the corollary follows immediately from the lemma.*

Now we give the proof of the theorem. Since $(x_0(t), u_0(t))$, $t \in [0,T]$ is a solution to the problem (2), which is a problem without phase constraints, we can apply the Pontryagin's maximum principle for this solution [7], [12]. In other words, there exists a nontrivial absolutely continuous function $\psi(t)$, $t \in [0,T]$, as a solution of the adjoint system of equations

$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}\psi(t) - \frac{\partial f^*(x_0(t+\tau), x_0(t), u_0(t+\tau))}{\partial y}\psi(t+\tau),$$
$$0 \le t \le T-\tau$$
$$\dot{\psi}(t) = -\frac{\partial f^*(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}\psi(t), \quad T-\tau \le t \le T$$

for which, the maximum condition

$$\max\left\{\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), u\right), \psi\left(t\right)\right) \ : \ u \in U\right\} = \left(\dot{x}_0\left(t\right), \psi\left(t\right)\right), \ \ a.\ a. \quad t \in [0, T]$$

holds.

The last equality means that

$$\psi\left(t\right) \in N\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), U\right), \dot{x}_0\left(t\right)\right) \ \ a.\ a.\ \ t \in [0, T], \tag{3}$$

where $N\left(A, a\right)$ is a normal cone of the set $A \in conv\, \Omega\left(E^n\right)$ at the point $a \in A$.

By the definition of the similar solutions and conditions of the theorem,

$$T\left(f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), U\left(\bar{x}\left(t\right)\right)\right), \dot{\bar{x}}\left(t\right)\right) \subset T\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), U\right), \dot{x}_0\left(t\right)\right), \ \ a.\ a.\ \ t \in [0, T],$$

therefore

$$N\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), U\right), \dot{x}_0\left(t\right)\right) \subset N\left(f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), U\left(\bar{x}\left(t\right)\right)\right), \dot{\bar{x}}\left(t\right)\right), \ \ a.\ a.\ \ t \in [0, T].$$

From the inclusion (3), we conclude that the absolutely continuous nontrivial one valued branch $\psi\left(t\right), \ t \in [0, T]$ of the multivalued maping $N\left(f\left(x_0\left(t\right), x_0\left(t-\tau\right), U\right), \dot{x}_0\left(t\right)\right)$ is a one valued branch of the multivalued mapping

$$N\left(f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), U\left(\bar{x}\left(t\right)\right)\right), \dot{\bar{x}}\left(t\right)\right), \ \ a.\ a.\ \ t \in [0, T].$$

At the same time this means that

$$\max\left\{\left(f\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), u\right), \psi\left(t\right)\right) \ : \ u \in U\left(\bar{x}\left(t\right)\right)\right\} = \left(\dot{\bar{x}}\left(t\right), \psi\left(t\right)\right), \ \ a.\ a. \quad t \in [0, T].$$

In the case of additional condition of the theorem, it is easy to show that the adjoint function $\psi\left(t\right), \ t \in [0, T]$ satisfies to the system of the differential equations

$$\dot{\psi}\left(t\right) = -\frac{\partial f^*\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), \bar{u}\left(t\right)\right)}{\partial x}\psi\left(t\right) - \frac{\partial f^*\left(\bar{x}\left(t+\tau\right), \bar{x}\left(t\right), \bar{u}\left(t+\tau\right)\right)}{\partial y}\psi\left(t+\tau\right),$$

$$a.\ a.\ \ t \in [0, T-\tau],$$

$$\dot{\psi}\left(t\right) = -\frac{\partial f^*\left(\bar{x}\left(t\right), \bar{x}\left(t-\tau\right), \bar{u}\left(t\right)\right)}{\partial x}\psi\left(t\right), \quad a.\ a.\ \ t \in [\,T-\tau, T].$$

The theorem is proved.

Consider the time optimal control problems

$$\begin{cases} \dot{x}\left(t\right) = Ax\left(t\right) + Bx\left(t-\tau\right) + \eta\left(u\right), \ t \in [0, T], \ T \in \left(\tau, +\infty\right), \ \tau > 0, \\ x\left(t\right) = l\left(t\right), \ t \in [-\tau, 0), \ x\left(0\right) = x_\alpha, \ x\left(T\right) = x_\beta, \ x_\alpha \neq x_\beta, \\ u\left(t\right) \in U, \ \ a.\ a.\ \ t \in [0, T], \\ x\left(t\right) \in X, \ t \in [0, T], \\ T \to \min \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x}\left(t\right) = Ax\left(t\right) + Bx\left(t-\tau\right) + \eta\left(u\right), \ t \in [0, T], \\ x\left(t\right) = \hat{l}\left(t\right), \ t \in [-\tau, 0), \ x\left(0\right) = \hat{x}_\alpha, \ x\left(T\right) = \hat{x}_\beta, \ \hat{x}_\alpha \neq \hat{x}_\beta, \\ u\left(t\right) \in U, \ \ a.\ a.\ \ t \in [0, T], \\ T \to \min, \end{cases} \tag{5}$$

where A and B are given constant $n \times n$ matrices, $U \in \Omega\left(E^m\right), \ \eta : E^m \to E^n$ is continuous and $\eta(U) \in conv\, \Omega\left(E^n\right)$.

Corollary 2.2. *Let $(\bar{x}(t), \bar{u}(t))$, $t \in [0,T]$ be a solution of (4) and there exists a similar solution $(x_0(t), u_0(t))$, $t \in [0,T]$ of (5). Then there exists a nontrivial absolutely continuous solution $\psi(t)$, $t \in [0,T]$ of the adjoint system of differential equations*

$$\dot{\psi}(t) = -A^*\psi(t) - B^*\psi(t+\tau),\ 0 \le t \le T-\tau$$
$$\dot{\psi}(t) = -A^*\psi(t),\ \ T-\tau \le t \le T$$

for which the maximum condition

$$\max\{(\eta(u),\ \psi(t)),\ u \in U(\bar{x}(t))\} = (\eta(\bar{u}(t)), \psi(t)),\ \ a.\ a.\quad t \in [0,T],$$

holds, where

$$U(\bar{x}(t)) = \{u \in U\ \ :\ \ \eta(u) \in (T(X, \bar{x}(t)) - A\bar{x}(t) - B\bar{x}(t-\tau))\},\ \ a.\ a.\ t \in [0,T].$$

Note 3. For the linear systems with respect to x, an additional condition is always true, as in this case

$$\frac{\partial f(\bar{x}(t), \bar{x}(t-\tau), \bar{u}(t))}{\partial x} = A = \frac{\partial f(x_0(t), x_0(t-\tau), u_0(t))}{\partial x}$$

and

$$\frac{\partial f(\bar{x}(t+\tau), \bar{x}(t), \bar{u}(t+\tau))}{\partial y} = B = \frac{\partial f(x_0(t+\tau), x_0(t), u_0(t+\tau))}{\partial y}.$$

Thus, in linear cases, we need only existence condition for the similar solution of (5).

Example 2.1.

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = u(t),\ T > \tau,\ \tau > 0, \end{cases}$$

$$x(t) = l(t) = (0,0),\ t \in [-\tau, 0],\ |u(t)| \le 1,\ x_2(t) \le 2\tau,\ x_\beta = \left(9\tau^2, -\tau\right),\ t \in [0,T],$$
$$T \to \min.$$

Solution. Indeed we consider the linear time optimal control problem with delay of transferring the system from initial point $x_0 = 0$ at time $t = 0$ to the end point $x_\beta = \left(9\tau^2, -\tau\right)$ in minimum time T, when the control u is subject to the constraint $|u| \le 1$ and the state x to the constraint $x_2 \le 2\tau$.

In the absence of the state constraint the time optimal control problem with delay has the form

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = u(t), \end{cases}$$

$$x(t) = l(t) = (0,0),\ t \in [-\tau, 0],\ |u(t)| \le 1,\ x_\beta = \left(9\tau^2, -\tau\right),\ t \in [0,T],$$
$$T \to \min.$$

As we know in this case the time optimal control function takes only the values ± 1. At first we solve the problem.

If $u(t) = 1,\ t \ge 0$, then

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = 1, \end{cases}$$

$$x(t) = l(t) = (0,0),\ t \in [-\tau, 0].$$

Solving the system we get that $x_1(t) = 0,\ x_2(t) = t,\ t \in [0,\tau]$ and $x_1(t) = \frac{(t-\tau)^2}{2},\ x_2(t) = t,\ t \in [\tau, +\infty)$ (Fig. 1).

If $u(t) = -1$, then

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = -1, \end{cases}$$

$$x(t) = l(t) = (0,0), \ t \in [-\tau, 0].$$

Thus the solution of the system is $x_1(t) = 0, \ x_2(t) = -t, \ t \in [0, \tau]$ and $x_1(t) = -\frac{(t-\tau)^2}{2}, \ x_2(t) = -t, \ t \in [\tau, +\infty)$ (Fig.1).

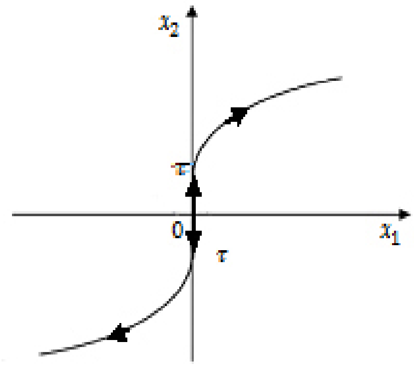

Figure 1.

Let us

$$u(t) = \begin{cases} +1, & if \ \ t \in [0, 3\tau), \\ -1, & if \ \ t \geq 3\tau, \end{cases}$$

then $x_1(t) = \frac{(t-\tau)^2}{2}, \ x_2(t) = -t + 6\tau, \ 3\tau \leq t \leq 4\tau$ or $x_1 = \frac{(x_2 - 5\tau)^2}{2}, \ x_2 \in [2\tau, 3\tau]$.

On the interval $[4\tau, 7\tau]$ the solution of the system is

$$x_1(t) = -\frac{(t-7\tau)^2}{2} + 9\tau^2, \ x_2(t) = -t + 6\tau, \ \ 4\tau \leq t \leq 7\tau$$

or

$$x_1 = -\frac{1}{2}(x_2 + \tau)^2 + 9\tau^2, \ t - 6\tau = x_2 \in [-\tau, 2\tau].$$

Thus we have the time optimal trajectory that transfers the system from the initial state $(0,0)$ to the endpoint $\left(9\tau^2, -\tau\right)$ and $T = 7\tau$ (Fig. 2).

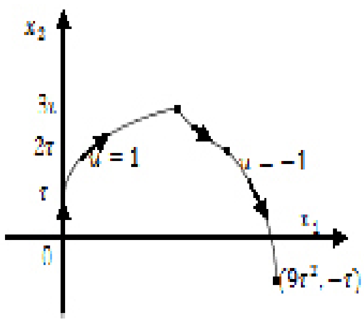

Figure 2.

In the case of the state constraint $x_2(t) \leq 2\tau, \ t \in [0, \ 7,5\tau]$ for the control function $\bar{u}(t), \ t \in [0, \ 7,5\tau]$ (Fig. 4)

$$\bar{u}(t) = \begin{cases} +1, & 0 \le t \le 2\tau, \\ 0, & 2\tau < t \le 4,5\tau, \\ -1, & 4,5 < t \le 7,5\tau \end{cases}$$

we construct the corresponding trajectory.

In the case $2\tau \le t \le 3\tau$ we have

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = 0, \end{cases}$$

hence

$$\bar{x}(t) = \begin{pmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{(t-\tau)^2}{2} \\ 2\tau \end{pmatrix}, \ 2\tau \le t \le 3\tau \text{ (Fig. 3)}.$$

If $3\tau \le t \le t^*$, then $\bar{x}_1(t) = 2\tau t - 4\tau^2$ and $\bar{x}_2(t) = 2\tau$ (Fig. 3).

In order to find the moment of the exit off of the boundary we solve the equation $2\tau t^* - 4\tau^2 = \bar{x}_1(t^*) = 5\tau^2 \Rightarrow t^* = 4,5\tau$.

On the interval $[4,5\tau,\ 5,5\tau]$ the trajectory is: $\bar{x}_1(t) = 2\tau t - 4\tau^2$ and $\bar{x}_2(t) = -t + 6,5\tau$ or $\bar{x}_1(t) = -2\tau\bar{x}_2(t) + 9\tau^2$ (Fig. 3).

On the interval $[5,5\tau,\ 7,5\tau]$ the trajectory is: $\bar{x}_1(t) = -\frac{(\bar{x}_2(t)+\tau)^2}{2}$ (Fig. 3).

It is clear that $U(\bar{x}(t) = U \bigcap (T(X, \bar{x}(t)) - B\bar{x}(t-\tau)))$ is a subset of $U = (0, [-1, +1])$ as in (Fig. 4).

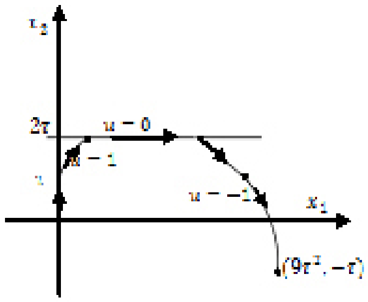

Figure 3.

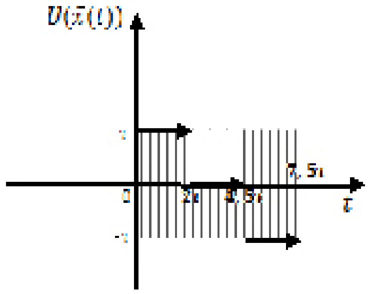

Figure 4.

In other words

$$U(\bar{x}(t)) = \begin{cases} [-1, +1], & t \in [0,\ 2\tau], \\ [-1, 0], & t \in (2\tau,\ 4,5\tau], \\ [-1, +1], & t \in (4,5\tau,\ 7,5\tau], \end{cases}$$

hence

$$T(U(\bar{x}(t)), \bar{u}(t)) = \begin{cases} (-\infty, 0], & 0 \le t \le 4,5\tau, \\ [0, +\infty), & 4,5\tau \le t \le 7,5\tau. \end{cases}$$

The corresponding time optimal control problem without state constraint

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau), \\ \dot{x}_2(t) = u(t), \end{cases}$$

$$x(t) = l(t) = (0,0),\ t \in [-\tau, 0],\ |u(t)| \le 1,\ \hat{x}_\beta = x_0(T),\ t \in [0, T]$$

$$T \to \min$$

has an optimal control

$$u_0(t) = \begin{cases} +1, & 0 \le t \le 4,5\tau \\ -1, & 4,5\tau < t \le 7,5\tau, \end{cases}$$

which is similar to $\bar{u}(t)$, $t \in [0,\ 7,5\tau]$, because

$$T(U, u_0(t)) = T(U(\bar{x}(t)), \bar{u}(t)) = \begin{cases} (-\infty, 0], & 0 \le t \le 4,5\tau, \\ [0, +\infty), & 4,5 \le t \le 7,5\tau. \end{cases}$$

Thus the condition of existence of the similar solution holds. Then there exists the nontrivial absolutely continuous adjoint function $\psi(t)$, $t \in [0,\ 7,5\tau]$ as a solution of the adjoint system of differential equations

$$\begin{cases} \dot{\psi}_1 = 0, \\ \psi_2 = -\psi_1(t+\tau), \ 0 \le t \le 6,5 \end{cases}$$

and

$$\begin{cases} \dot{\psi}_1 = 0, \\ \dot{\psi}_2 = 0, \quad 6,5 \le t \le 7,5\tau. \end{cases}$$

The refore the adjoint function $\psi(t)$, $t \in [0,\ 7,5\tau]$ is (Fig. 5)

$$\psi(t) = \begin{cases} (1, -t + 4,5\tau), & 0 \le t \le 6,5\tau, \\ (1, -2,5\tau), & 6,5\tau \le t \le 7,5\tau. \end{cases}$$

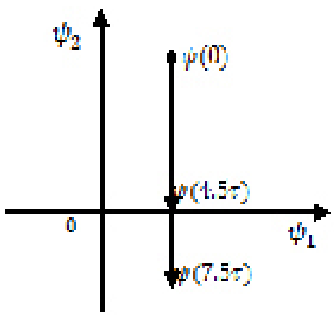

Figure 5.

3. CONCLUSION

Note that, the results obtained in this study include the entire regular optimal trajectory, i. e. the optimal trajectory is investigated as a whole, not dividing it to boundary or interior parts.

The advantage of this result is that the adjoint equation is much simpler and has the same form as in optimal control problems without state constraints [7], [12] and regular trajectories in this case may be irregular for the whole set U.

A specialty of this work is also that the maximum condition is not taken on the set U, but the subset $U(\bar{x}(t)) \subset U$, as done in [5], [9].

The work [6] deals with the derivation of the general necessary conditions, with the maximum condition also on a subset $U(\bar{x}(t))$, however, in this paper the problem of regularity in the sense that the above definition is not considered.

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