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NUMERICAL SOLUTION OF GENERAL SINGULAR PERTURBATION BOUNDARY VALUE PROBLEMS BASED ON ADAPTIVE CUBIC SPLINE

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Abstract. We use adaptive cubic spline functions to develop a numerical method for solving a class of singular perturbation two-point boundary value problems. The scheme derived in this method is second order accurate. Convergence of the method is shown. The resulting linear system of equations has been solved by using a tri-diagonal solver. Numerical examples are given to show the applicability and efficiency of our method.

Keywords: second-order boundary value problem, adaptive cubic spline, convergence analysis, singular perturbation problems

AMS Subject Classification: 65L10.

1. Introduction

We consider a class of general singularly perturbed two-point boundary value problems of the form

$$c_\epsilon y^{''}(x) + a(x)y^{'}(x) + b(x)y(x) = f(x), 0 \le x \le 1, \tag{1}$$

subject to the boundary conditions

$$y(0) = \eta_0, \quad y(1) = \eta_1, \tag{2}$$

where η_0, η_1 are given constants and $c_\epsilon = +\epsilon$ or $-\epsilon$ with $0 < \epsilon < 1$. Further, $f(x)$, $a(x)$ and $b(x)$ are sufficiently smooth functions satisfying the conditions

$$a(x) \ge \alpha > 0, \quad b(x) \ge \beta > 0.$$

The approximate solution of boundary-value problems with a small parameter affecting highest derivative of the differential equation is described. Parameter dependent differential equations (such as (1)) has recently gained importance in the literature for two main reasons. Firstly, they occur naturally in various fields of science and engineering, for example, combustion, nuclear engineering, control theory, elasticity, fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics, etc. A few notable examples are boundary-layer problems, WKB Theory, the modeling of steady and unsteady viscous flow problems with large Reynolds number and convective heat transport problems with large Peclet number. Secondly, the occurrence of sharp boundary-layers as ϵ, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes. Different numerical methods have been proposed by various authors for solving singular perturbation problems (see [1-10]). The numerical solution of two point boundary-value problems using splines has been considered by many authors; see for example, [3,7,11-17] and references therein.

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It is known that most of the methods fail when ϵ is small relative to the mesh with h, that is used for discretization of the equation (1). In general, classical numerical methods fail to produce good approximations for these equations. In this paper, We present a uniformly convergent uniform mesh difference scheme using adaptive cubic splines for solution of (1) which don't fail when ϵ is small relative to the mesh with h. Kadalbajoo and Bawa [3] give a second order method which becomes a special case of our method. ϵ-Uniformly convergent fitted mesh finite difference methods for this problem derived by Kadalbajoo and Patidar [9] which in some examples our results are better than their results. The use of cubic splines for the solution of (regular) linear boundary-value problems was suggested by Bickley [18]. Later, Fyfe [14] discussed the application of deferred corrections to the method suggested by Bickley, by considering the case of (regular) linear boundary-value problems. Our scheme for the corresponding problem (i.e. $\epsilon = 1$, $a \equiv 0$) reduces to the Bickley scheme. Tariq Aziz and Arshad Khan [17] derived cubic spline in compression method for the solution of (1).

In this article, we present a direct method based on adaptive cubic splines for (1). In section 2, we give a brief derivation of this adaptive cubic spline. We present the spline relations to be used for discretization of the given problem (1). In section 3, we present the formulation of our method. In section 4, convergence of the methods have been discussed. Finally, in section 5, numerical evidence is included to demonstrate the efficiency of the method. At the end a brief conclusion is given.

2. Adaptive cubic spline functions

Let us consider a mesh with nodal points x_i on $[a, b]$ such that:

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_N = b,$$

where $h = x_i - x_{i-1}$ for $i = 1(1)N$. We also denote the function value $y(x_i)$ by y_i.
A function $S_\Delta(x, \theta)$ of class $C^2[a, b]$ which interpolates $y(x)$ at the mesh points x_i, depends on a parameter θ, reduces to cubic spline $S_\Delta(x)$, in $[a, b]$ as $\theta \to 0$ is termed an adaptive spline function. Following [16] it satisfies the following differential equation:

$$aS''_\Delta(x, \theta) - bS'_\Delta(x, \theta) = (aM_i - bm_i)\frac{x - x_{i-1}}{h} + (aM_{i-1} - bm_{i-1})\frac{x_i - x}{h}, \tag{3}$$

where a and b are constants, $S'_\Delta(x_i, \theta) = m_i$, $S''_\Delta(x_i, \theta) = M_i$ and $x \in [x_{i-1}, x_i]$.
Solving (3) and using the interpolatory constraints $S_\Delta(x_{i-1}, \theta) = y_{i-1}$, $S_\Delta(x_i, \theta) = y_i$, we have

$$\begin{aligned} S_\Delta(x, \theta) = A_i + B_i e^{\theta Z} - \frac{h^2}{\theta^3}[\frac{1}{2}\theta^2 Z^2 + \theta Z + 1](M_i - \frac{\theta}{h}m_i) + \\ + \frac{h^2}{\theta^3}[\frac{1}{2}\theta^2(1 - Z)^2 + \theta(1 - Z) + 1](M_{i-1} - \frac{\theta}{h}m_{i-1}), \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_i(e^\theta - 1) = -y_i + y_{i-1}e^\theta - \frac{h^2}{\theta^3}[(\frac{\theta^2}{2} + \theta + 1) - \theta e^\theta](M_i - \frac{\theta}{h}m_i) - \\ - \frac{h^2}{\theta^3}[(\frac{\theta^2}{2} - \theta + 1) - \theta](M_{i-1} - \frac{\theta}{h}m_{i-1}), \\ B_i(e^\theta - 1) = y_i - y_{i-1} + \frac{h^2}{\theta^3}[(\frac{\theta}{2} + 1)(M_i - \frac{\theta}{h}m_i)] + \\ + (\frac{\theta}{2} - 1)(M_{i-1} - \frac{\theta}{h}m_{i-1}), \end{aligned}$$

$\theta = \frac{bh}{a}$ and $Z = \frac{x - x_{i-1}}{h}$.

The function $S_\Delta(x,\theta)$ on the interval $[x_i, x_{i+1}]$ is obtained with $(i+1)$ replacing i in (4), the condition of continuity of the first or second derivative of $S_\Delta(x,\theta)$ at $x = x_i$ yields the following equation:

$$(M_{i+1} - \frac{\theta}{h}m_{i+1})[e^{-\theta}(\frac{\theta^2}{2} + \theta + 1) - 1] + (M_i - \frac{\theta}{h}m_i)[e^{-\theta}(\frac{\theta^2}{2} - \theta - 2)+$$
$$+(-\frac{\theta^2}{2} - \theta + 2) + (M_{i-1} - \frac{\theta}{h}m_{i-1})[e^{-\theta} - 1 + \theta - \frac{\theta^2}{2}] =$$
$$= -\frac{\theta^2}{h^2}[e^{-\theta}y_{i+1} - (1 + e^{-\theta})y_i + y_{i-1}]. \tag{5}$$

Some additional relations for the adaptive spline are listed below:

$$(i) \qquad m_{i-1} = -h(A_1M_{i-1} + A_2M_i) + \frac{1}{h}(y_i - y_{i-1}),$$
$$(ii) \qquad m_i = h(A_3M_{i-1} + A_4M_i) + \frac{1}{h}(y_i - y_{i-1}),$$
$$(iii) \qquad \frac{Ch}{2\theta}M_{i-1} = -(A_4m_{i-1} + A_2m_i) + B_1\frac{(y_i - y_{i-1})}{h},$$
$$(iv) \qquad \frac{Ch}{2\theta}M_i = (A_3m_{i-1} + A_1m_i) + B_2\frac{(y_i - y_{i-1})}{h}. \tag{6}$$

Where

$A_1 = \frac{1}{4}(1 + C) + \frac{C}{2\theta}$, $A_2 = \frac{1}{4}(1 - C) - \frac{C}{2\theta}$,

$A_3 = \frac{1}{4}(1 + C) - \frac{C}{2\theta}$, $A_4 = \frac{1}{4}(1 - C) + \frac{C}{2\theta}$,

$B_1 = \frac{1}{2}(1 - C)$, $B_2 = -\frac{1}{2}(1 + C)$ and $C = \coth\frac{\theta}{2} - \frac{2}{\theta}$.

We also obtain,

$$A_2M_{i+1} + (A_1 + A_4)M_i + A_3M_{i-1} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}]. \tag{7}$$

In the limiting case when $\theta \to 0$, i.e ($\frac{bh}{a} \to 0$), then we have:

$C = 0$, $\frac{C}{\theta} = \frac{1}{6}$, $A_1 = A_4 = \frac{1}{3}$,

$A_2 = A_3 = \frac{1}{6}$, $B_1 = \frac{1}{2}$,, $B_2 = -\frac{1}{2}$

and the spline function given by (4) reduces into cubic spline.

3. Description of the method

In this section, we shall mostly consider the problem

$$-\epsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), 0 \le x \le 1,$$
$$y(0) = \eta_0, \quad y(1) = \eta_1. \tag{8}$$

At the gird points x_i, the proposed differential equation (8) may be discretized by

$$-\epsilon y_i'' + a_iy_i' + b_iy_i = f_i, \tag{9}$$

where $a_i = a(x_i)$, $b_i = b(x_i)$ and $f_i = f(x_i)$.

By using moment of spline in (9) we obtain

$$-\epsilon M_i + a_i y_i' + b_i y_i = f_i. \tag{10}$$

Following [20] now we use the following approximations for first derivative of y:

$$y_i^{'} \cong \frac{y_{i+1} - y_{i-1}}{2h}, \tag{11}$$

$$y_{i+1}^{'} \cong \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \tag{12}$$

$$y_{i-1}^{'} \cong \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}. \tag{13}$$

By substituting (10)-(13) in (7) and simplifying we get the following tri-diagonal system which gives the approximations $y_1, y_2, ..., y_{N-1}$ of the solution $y(x)$ at $x_1, x_2, ..., x_{N-1}$.

$$\begin{gathered}
[\epsilon + \frac{3}{2}A_3 h a_{i-1} - A_3 h^2 b_{i-1} + \frac{1}{2}(A_1 + A_4) h a_i - \frac{1}{2}A_2 h a_{i+1}] y_{i-1} + [-2\epsilon - \\
-2A_3 h a_{i-1} - (A_1 + A_4) h^2 b_i + 2A_2 h a_{i+1}] y_i + [\epsilon + \frac{1}{2}A_3 h a_{i-1} - A_2 h^2 b_{i-1} - \\
-\frac{1}{2}(A_1 + A_4) h a_i - \frac{3}{2}A_2 h a_{i+1}] y_{i-1} = -h^2 [A_3 f_{i-1} + (A_1 + A_4) f_i + A_2 f_{i+1}] \\
\text{with} \quad y(0) = \eta_0, \quad y(1) = \eta_1, \quad i = 1(1)N-1.
\end{gathered} \tag{14}$$

By expanding (14) in Taylor series about x_i, and using (9) we obtain the following local truncation error

$$\begin{gathered}
T_i(h) = [-\epsilon(A_1 + A_2 + A_3 + A_4 - 1)] h^2 y''(\xi_i) + [\epsilon(A_3 - A_2)] h^3 y'''(\xi_i) + \frac{1}{6}[2A_2 a_{i+1} - \\
-(A_1 + A_4) a_i + 2A_3 a_{i-1}] h^4 y'''(\xi_i) + [\frac{1}{2}\epsilon(A_2 + A_3 - \frac{1}{6})] h^4 y^4(\xi_i) + \\
+O(h^5), x_{i-1} < \xi_i < x_{i+1}.
\end{gathered} \tag{15}$$

If we choose $A_1 + A_2 + A_3 + A_4 = 1$ and $A_3 = A_2$ then we have

$$\begin{gathered}
T_i(h) = \frac{1}{6}[2A_2 a_{i+1} - (A_1 + A_4) a_i + 2A_3 a_{i-1}] h^4 y'''(\xi_i) + O(h^5), \\
x_{i-1} < \xi_i < x_{i+1}.
\end{gathered} \tag{16}$$

4. Convergence of the method

Putting the tri-diagonal system (14) in matrix-vector form, we have

$$MY + h^2 BF = D, \tag{17}$$

where $M = (m_{i,j})$ is tri-diagonal, diagonally dominant matrix of order N-1, with

$m_{i,i\pm l}$ = coefficient of $y_{i\pm l}$ in (14), $l = 0, 1$,

$F = (f_1, f_2, ..., f_{N-1})^T$,

$Y = (y_1, y_2, ..., y_{N-1})^T$,

$$B = \begin{pmatrix} (A_1+A_4) & A_2 & & & & & \\ A_3 & (A_1+A_4) & A_2 & & & & \\ \ddots & \ddots & \ddots & & & & \\ & & A_3 & (A_1+A_4) & A_2 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_3 & (A_1+A_4) & A_2 \\ & & & & A_3 & (A_1+A_4) \end{pmatrix}$$

and where

$$D = (d_1, 0, ..., 0, d_{N-1})^T,$$

with

$d_1 = -h^2 A_3 f_0 - \eta_0[\epsilon + \frac{3}{2}A_3 h a_0 - A_3 h^2 b_0 + \frac{1}{2}(A_1 + A_4)h a_1 - \frac{1}{2}A_2 h a_2],$

$d_i = 0$ for $i = 2(1)N-2,$

$d_{N-1} = -h^2 A_2 f_N - \eta_1[\epsilon - \frac{1}{2}A_3 h a_{N-2} - A_2 h^2 b_N - \frac{1}{2}(A_1 + A_4)h a_{N-1} - \frac{3}{2}A_2 h a_N].$

Also, we have

$$M\bar{Y} + h^2 BF = T(h) + D, \tag{18}$$

where

$$\bar{Y} = (y(x_1), y(x_2), ..., y(x_{N-1}))^T$$

denotes the exact solution and

$$T(h) = (T_1(h), T_2(h), ..., T_{N-1}(h))^T,$$

is the truncation error vector defined in relation (16). From (17) and (18), we have

$$M(\bar{Y} - Y) = AE = T(h), \tag{19}$$

where

$$E = \bar{Y} - Y = (e_1, e_2, ..., e_{N-1})^T.$$

Clearly, the row sums $U_1, U_2, ..., U_{N-1}$ of M are

$U_1 = \sum_{j=1}^{N-1} m_{1,j} = -\epsilon - \frac{3}{2}A_3 h a_0 - A_2 h^2 b_2 - \frac{1}{2}(A_1 + A_4)h a_1 + \frac{1}{2}A_2 h a_2 - (A_1 + A_4)h^2 b_1,$

$U_i = \sum_{j=1}^{N-1} m_{i,j} = -h^2[A_3 b_{i-1} + (A_1 + A_4)b_i + A_2 b_{i+1}),\ i = 2(1)N-2,$

$$U_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j} = -\epsilon + \tfrac{3}{2}A_2 h a_0 - A_3 h^2 b_{N-2} + \tfrac{1}{2}(A_1 + A_4) h a_{N-1} -$$

$$-\tfrac{1}{2}A_3 h a_{N-2} - (A_1 + A_4) h^2 b_{N-1}.$$

If we choose h to be adequately small so that the matrix M becomes irreducible and monotone [19]. It follows that M^{-1} exists and its elements are nonnegative.
Hence from Eq.(19), we have

$$E = M^{-1} T(h). \tag{20}$$

By the definition of multiplication of matrices with its inverse we have

$$\sum_{i=1}^{N-1} m^*_{j,i} U_i = 1, j = 1(1)N-1, \tag{21}$$

where $m^*_{j,i}$ is the (j,i)th element of the matrix M^{-1}. Therefore,

$$\sum_{i=1}^{N-1} m^*_{j,i} \leq \frac{1}{\min_{1 \leq i \leq N-1} U_i} = \frac{1}{h^2 B_{i_\mu}} \leq \frac{1}{h^2 \|B_{i_\mu}\|}, \tag{22}$$

for some i_μ between 1 and $N-1$.
From (15), (21) and (22) we have

$$e_j = \sum_{i=1}^{N-1} m^*_{j,i} T_i(h), j = 1(1)N-1, \tag{23}$$

and therefore

$$|e_j| \leq \frac{\eta h^2}{|B_{i_\mu}|}, j = 1(1)N-1, \tag{24}$$

where η is constant independent of h.

It follows that

$$\|E\| = O(h^2). \tag{25}$$

We summarize the above results in the following theorem:

Theorem 4.1. *The method given by Eq.(14) for solving the boundary value problem (1-2) for adequately small h gives a second-order convergent solution for arbitrary A_1, A_2, A_3 and A_4 with $A_1 + A_2 + A_3 + A_4 = 1$ and $A_2 = A_3$.*

Remark 4.1. *In some especial cases it may possible that some eigenvalues appear close to zero. This may steal from the power of discretization.*

5. Numerical illustrations

To illustrate our method and to demonstrate its convergence and applicability of our presented method computationally, we have solved four singularly perturbed two-point boundary value problems from [9,10] whose exact solutions are known to us.
We solve this problems with $N = 64, 128, 256, 512, 1024$ by our method for different values of h, ϵ, A_1, A_2, A_3 and A_4. The maximum absolute errors at the nodal points, $\max|y(x_i) - y_i|$ are tabulated in Tables 1-8 for different values of the parameters.
Problem 1. Consider the following singularly perturbed boundary value problem [9]:

$$-\epsilon y''(x) + \frac{1}{x+1} y'(x) + \frac{1}{x+2} y(x) = f(x), \quad y(0) = 1 + 2^{-\frac{1}{\epsilon}}, \quad y(1) = exp(1) + 2,$$

$$f(x) = (-\epsilon + \frac{1}{x+1} + \frac{1}{x+2})exp(x) + \frac{1}{x+2}2^{-\frac{1}{\epsilon}}(x+1)^{1+\frac{1}{\epsilon}}.$$

The exact solution of this problem is $y(x) = exp(x)2^{-\frac{1}{\epsilon}}(x+1)^{1+\frac{1}{\epsilon}}$. The results obtained by our method are shown in Tables 1-2.

TABLE 1. The maximum absolute errors in solution of problem 1 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	3.74(−6)	9.36(−7)	2.34(−7)	5.85(−8)	1.46(−8)
2^{-2}	2.47(−5)	6.19(−6)	1.54(−6)	3.87(−7)	9.67(−8)
2^{-3}	1.71(−4)	4.28(−5)	1.07(−5)	2.67(−6)	6.69(−7)
2^{-4}	8.12(−4)	2.03(−4)	5.07(−5)	1.26(−5)	3.17(−6)
2^{-5}	3.53(−3)	8.79(−4)	2.19(−4)	5.48(−5)	1.37(−5)
2^{-6}	1.50(−2)	3.68(−3)	9.17(−4)	2.29(−4)	5.72(−5)
2^{-7}	6.75(−2)	1.54(−2)	3.77(−3)	9.37(−4)	2.34(−4)
2^{-8}	2.66(−1)	6.83(−2)	1.55(−2)	3.81(−3)	9.48(−4)
2^{-9}	6.92(−1)	2.68(−1)	6.87(−2)	1.56(−2)	3.83(−3)

TABLE 2. The maximum absolute errors in solution of problem 1 for $A_1 = A_4 = \frac{4}{9}$ and $A_2 = A_3 = \frac{1}{18}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	1.22(−5)	3.08(−6)	7.71(−7)	1.92(−7)	4.82(−8)
2^{-2}	6.49(−5)	1.61(−5)	4.05(−6)	1.00(−6)	2.52(−7)
2^{-3}	2.69(−4)	6.76(−5)	1.68(−5)	4.22(−6)	1.04(−6)
2^{-4}	1.00(−3)	2.50(−4)	6.27(−5)	1.56(−5)	3.91(−6)
2^{-5}	3.90(−3)	9.71(−4)	2.41(−4)	6.06(−5)	1.50(−5)
2^{-6}	1.57(−2)	3.86(−3)	9.62(−4)	2.39(−4)	6.00(−5)
2^{-7}	6.91(−2)	1.56(−2)	3.85(−3)	9.59(−4)	2.39(−4)
2^{-8}	2.69(−1)	6.90(−2)	1.57(−2)	3.85(−3)	9.58(−4)
2^{-9}	6.97(−1)	2.70(−1)	6.90(−2)	1.57(−2)	3.85(−3)

Problem 2. Consider the following singularly perturbed singular boundary value problem [9]:

$$-\epsilon y''(x) + y'(x)) = \exp(x), \qquad y(0) = y(1) = 0,$$

The exact solution of this problem is

$$y(x) = \frac{1}{1-\epsilon}[\exp(x) - \frac{1 - \exp[1-(1/\epsilon)] + [\exp(x) - 1]\exp[(x-1)/\epsilon]}{1-\exp(-1/\epsilon)}].$$

The results obtained by our method are shown in Tables 3-4.

Problem 3. Consider the problem [10]:

$$-\epsilon y''(x) - y'(x) = 0, \qquad y(0) = 1, \qquad y(1) = exp(-1/\epsilon).$$

TABLE 3. The maximum absolute errors in solution of problem 2 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	5.34(−5)	1.33(−5)	3.34(−6)	8.35(−7)	2.08(−7)
2^{-4}	3.53(−3)	8.79(−4)	2.19(−4)	5.48(−5)	1.37(−5)
2^{-8}	6.06(−1)	2.33(−1)	5.95(−2)	1.35(−2)	3.32(−3)
10^{-5}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−1)	1.64(−2)
10^{-6}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−1)	1.64(−2)
10^{-7}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−1)	1.64(−2)
10^{-8}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−1)	1.64(−2)

TABLE 4. The maximum absolute errors in solution of problem 2 for $A_1 = A_4 = \frac{4}{9}$ and $A_2 = A_3 = \frac{1}{18}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	1.71(−5)	4.28(−5)	1.06(−5)	2.67(−6)	2.08(−7)
2^{-4}	1.43(−3)	3.50(−3)	8.72(−4)	2.17(−4)	1.37(−5)
2^{-8}	1.03(−1)	6.06(−2)	2.33(−2)	6.95(−3)	3.32(−4)
10^{-5}	3.09(−1)	2.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-6}	3.09(−1)	2.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-7}	3.09(−1)	2.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-8}	3.09(−1)	2.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)

Its exact solution is given by

$$y(x) = exp(-x/\epsilon).$$

The results obtained by our method are shown in Tables 5-6.

TABLE 5. The maximum absolute errors in solution of problem 3 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	1.54(−5)	3.86(−6)	9.66(−7)	2.41(−7)	6.04(−8)
2^{-2}	1.04(−4)	2.61(−5)	6.54(−6)	1.63(−6)	4.09(−7)
2^{-3}	4.77(−4)	1.19(−4)	2.98(−5)	7.45(−6)	1.86(−6)
2^{-4}	1.92(−3)	4.79(−4)	1.19(−4)	2.99(−5)	7.48(−6)
2^{-5}	7.87(−3)	1.92(−3)	4.79(−4)	1.19(−4)	2.99(−5)
2^{-6}	3.45(−2)	7.87(−3)	1.92(−3)	4.79(−4)	1.19(−4)
2^{-7}	1.35(−1)	3.45(−2)	7.87(−3)	1.92(−3)	4.79(−4)
2^{-8}	3.51(−1)	1.35(−1)	3.45(−2)	7.87(−3)	1.92(−3)
2^{-9}	6.00(−1)	3.51(−1)	1.35(−1)	3.45(−2)	7.87(−3)

Problem 4. Consider the problem [9]:

$$\epsilon y''(x) + (1+x)^2 y'(x) + 2(1+x)y(x) = 0, \quad y(0) = 0, \quad y(1) = exp(-1/2) - exp(-7/3\epsilon).$$

Its exact solution is given by

$$y(x) = exp(-x/2) - exp[-x(x^2 + 3x + 3)/\epsilon].$$

TABLE 6. The maximum absolute errors in solution of problem 3 for $A_1 = A_4 = \frac{4}{9}$ and $A_2 = A_3 = \frac{1}{18}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	1.51(−5)	3.86(−6)	9.65(−7)	2.40(−7)	6.00(−8)
2^{-2}	1.04(−4)	2.60(−5)	6.52(−6)	1.61(−6)	4.04(−7)
2^{-3}	4.75(−4)	1.15(−4)	2.97(−5)	7.40(−6)	1.81(−6)
2^{-4}	1.90(−3)	4.79(−4)	1.16(−4)	2.93(−5)	7.43(−6)
2^{-5}	7.87(−3)	1.90(−3)	4.77(−4)	1.17(−4)	2.96(−5)
2^{-6}	3.44(−2)	7.85(−3)	1.90(−3)	4.75(−4)	1.15(−4)
2^{-7}	1.35(−1)	3.45(−2)	7.84(−3)	1.90(−3)	4.72(−4)
2^{-8}	3.49(−1)	1.33(−1)	3.43(−2)	7.83(−3)	1.90(−3)
2^{-9}	5.98(−1)	3.50(−1)	1.30(−1)	3.41(−2)	7.81(−3)

The results obtained by our method are shown in Tables 7-8.

TABLE 7. The maximum absolute errors in solution of problem 4 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	5.13(−5)	1.28(−5)	3.20(−6)	8.02(−7)	2.00(−7)
2^{-4}	1.45(−3)	3.64(−4)	9.09(−5)	2.27(−5)	5.68(−6)
2^{-8}	3.43(−1)	1.33(−1)	3.39(−2)	7.75(−3)	1.89(−3)
10^{-4}	2.45(00)	1.01(00)	8.95(−1)	8.11(−1)	6.59(−1)
10^{-5}	2.36(+1)	6.04(00)	1.71(00)	9.86(−1)	9.58(−1)
10^{-6}	2.36(+2)	6.00(+1)	1.51(+1)	3.88(00)	1.27(00)
10^{-7}	2.36(+3)	6.00(+2)	1.51(+2)	3.88(+1)	1.27(+1)

TABLE 8. The maximum absolute errors in solution of problem 4 for $A_1 = A_4 = \frac{4}{9}$ and $A_2 = A_3 = \frac{1}{18}$.

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	6.48(−5)	1.62(−5)	4.05(−6)	1.01(−6)	2.53(−7)
2^{-4}	2.44(−3)	6.08(−4)	1.52(−4)	3.80(−5)	9.50(−6)
2^{-8}	3.62(−1)	1.38(−1)	3.51(−2)	8.00(−3)	1.95(−3)
10^{-4}	2.05(00)	1.56(00)	9.05(−1)	8.15(−1)	6.60(−1)
10^{-5}	1.37(00)	1.53(00)	3.77(00)	1.04(00)	7.74(−1)
10^{-6}	1.34(00)	5.37(00)	4.42(00)	1.71(00)	8.05(−1)
10^{-7}	2.36(+3)	6.23(00)	5.81(00)	2.93(00)	9.11(−1)

6. CONCLUSION

A numerical method based on adaptive spline is developed to solve the singularly perturbed two-point boundary value problems. High order perturbations of the problem are generated and applied to derive higher orders of accuracy. The spline relations are used for discretization of the given problem (1). Convergence analysis of the method has been discussed. It has been found that the proposed algorithm gives highly accurate numerical results. Furthermore these results show that for very large values of subintervals the method remain stable and the round off errors can not be dominate but when the number of subintervals goes to infinity (larger than 1024) the round off errors are dominated and the method doesn't remain stable.

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