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S^*- PARABOLIC MANIFOLDS*

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Abstract. An open Riemann surface is called parabolic in case every bounded subharmonic function on it reduces to a constant. In literature were introduced seemingly different analogs of this notion for Stein manifolds of arbitrary dimension. In the present paper we give a different definitions of parabolisity and give some immediate relations among these different definitions.In particular,we discuss the following open problem:is the S-parabolic manifold S*-parabolic?

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1. Introduction

Definition 1.1. *A Stein manifold X is called $S-$ parabolic, if there exits special exhaustion function $\rho(z) \in psh(X)$ such, that*

a) $\{z \in X : \rho(z) \leq C\} \subset\subset X \forall C-$ const;

b) Monge - Ampere operator $(dd^c \rho)^n = 0$ off a compact $E \subset\subset X$. It means that ρ is maximal psh outside K.

We note, that without maximality condition b) a exhaustion function $\sigma(z) \in psh(X) \bigcap C^\infty(X)$ exist for any Stein manifold, because the Stein manifold can be properly embedded in $\mathbb{C}_w^{2n+1}$ and take for σ the restriction $ln|w|$ on X.

In the previous papers on parabolic manifolds (see an example [5]-[8]) usually required the condition of continuity or C^∞- smoothes of ρ. Here we distinguish the case of continuous exhaustion function $\rho(z) \in psh(X) \bigcap C(X)$ and will call this kind of manifold as S^* - parabolic.

The special exhaustion function $\rho(z)$ is a key object in the Nevanlinnas' value distribution theory of holomorphic maps $f : X \to P^m$, where $P^m - m$ dimensional projective manifold (see. [5, 6]).

On $S-$ parabolic manifolds any bounded above plurisubharmonic function is constant. In particular, there are no nonconstant bounded holomorphic functions on such manifolds.

The complex manifolds, on which every bounded above plurisubharmonic function reduces to a constant, a characteristic shared by affine-algebraic manifolds, play a big role in the study of the Fr'echet spaces of analytic functions on Stein manifolds, the bases on them and in finding continuous extension operators for analytic functions from complex submanifolds see, papers of first author [1, 2].

The parabolic manifolds (also the parabolic Stein spaces) and the structure of certain psh functions and currents on them were studied in detail by J.P.Demailly [3] and A.Zeriahi [10, 11]. Moreover on manifolds which have a special exhaustion function was defined extremal Green functions and applied it to the pluripotential theory on such manifolds; in particular introduced

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a class of polynomials: let $\pounds$ to be Lelon class of plurisubharmonic (psh) functions $u(z) \in psh(X),\ u(z) \leq const + \rho^+(z)$, where $\rho^+ = max(\rho, 0)$. For a compact $K \subset X$ let $\pounds(K)$ to be class $\{u(z) \in \pounds : u|_K \leq 0\}$.

Definition 1.2. *The upper regularization* $V^*(z, K) = \overline{\lim}_{w \to z} V(w, K)$ *of* $V(z, K) = sup\{u(z) : u \in \pounds(K)\}$ *is called Green function of the compact* K.

We note, that the Green function $V^*(z, K)$ is both, or $\equiv +\infty$ (K is pluripolar), or belong to class $\pounds$ (K is not pluripolar).

Correspondingly, a holomorphic on X function $p(z)$ is called polynom if for some integer d

$$\frac{1}{d} ln|p(z)| \leq const + \rho^+(z). \tag{1}$$

The minimal such that d is called degree of the polynom p and the space of all polynoms of degree p is denoted as $\wp^d_\rho(X)$. Then $\wp^d_\rho(X)$ is a finit dimensional linear space and $dim \wp^d_\rho(X) \leq \binom{n+d \cdot N}{n}$, where $N > 1$ is a integer independing of d. (see the paper A. Zeriahi [10], where this result proved for S^* - manifold and the proof is true also for $S-$ parabolic manifolds)

The aim of this paper is to establish a interrelation between S and S^* - -parabolisities. The equalency of these notions still does not proved. It is not clear also the fact, that is the Green function $V^*(z, K)$ continuous off a compact $E \subset X$ for pluriregular set $K \subset X$. In this direction we formulate the following result of A. Zeriahi [10]: if K is a pluriregular compact on the S^* - parabolic manifold X then the Green function $V^*(z, K)$ continuous on X.

If $\rho(z)$ is fixed exhaustion function on $S-$ parabolic manifold X, then it is important to control a jumps at discontinuous points of ρ. For a fixed point $z \in X$ we put

$$\rho_*(z) = \underline{\lim}_{w \to z} \rho(w).$$

We note,

$$\rho^*(z) = \overline{\lim}_{w \to z} \rho(w) = \rho(z)$$

that for psh function ρ and $\rho_*(z) < \rho(z)$ means that ρ has a jump at the point z.

The following theorem gives, that if the exhaustion function ρ is "continuous" at "∞" points of X, then X is S^*–parabolic.

2. Main results

Theorem 2.1. *Let X is to be a $S-$ parabolic manifold and ρ its exhaustion function.*

Then the followings are equivalent:

a) the function ρ satisfies the following continuity condition at ∞:

$$\lim_{\rho(z) \to \infty} \frac{\rho(z)}{\rho_*(z)} = 1; \tag{2}$$

b) the Green function $V^*(z, K)$ is continuous at ∞, i.e. V^* satisfies the condition (2) for arbitrary nonpluripolar compact set K;

c) the $V^*(z, K)$ is continuous at ∞ at least for one nonpluripolar compact set K;

d) the Green function $V^*(z, K)$ is continuous on X for arbitrary pluriregular compact K;

e) the manifold X is S^*- parabolic.

Proof. For arbitrary nonpluripolar set $K \subset X$ there are constants C_1, C_2 such, that

$$\rho(z) + C_1 \leq V^*(z, K) \leq \rho(z) + C_2. \tag{3}$$

This follows equality a), b) and c). Implication $d) \to e)$. Let $D \subset X$ to be a close domain with smooth boundary ∂D. Since D is pluriregular and the Green function $V^*(z, D)$ is maximal outside D, from d) follows e) : $V^*(z, K)$ is continuous exhaustion function on X.

Now since $e) \to a)$ is trivial it is enough to prove the implication $b) \to d)$. We fix a pluriregular compact $K \subset X$. Then the Green function $\nu(z) = V^*(z, K)$ is continuous at ∞, i.e. V^* satisfies the condition (2). By Fornaes-Narasimhan approximation theorem [4] (the theorem independed, later was proved also by second author, see [7]) we can find a sequence of psh and smooth functions $\nu_j(z) \in psh(X) \bigcap C^\infty(X), \nu_j(z) \downarrow \nu(z)$.

Since K is pluriregular, $\nu|_K = 0$. Therefore, for arbitrary fixed $\varepsilon > 0$ by Hartogs theorem we have $\nu_j(z) < \varepsilon$ uniformly on K, $j \geq j_0, z \in K$.

By (2) there exist $R > 0$ such, that

$$\nu(z) \leq \nu_*(z) + \varepsilon\nu_*(z) \ for \ z \notin B_R = \{z \in X : \nu(z) < R\}. \tag{4}$$

Then $\nu|_{\partial B_R} \leq (1 + 2\varepsilon)R$ and applying again the Hartog's theorem we have

$$\nu_j(z) \leq (1 + 2\varepsilon)R, j > j_1 \geq j_0, z \in \partial B_R.$$

For $j \geq j_1$ we put $w(z) = max\{\nu_j(z), (1+3\varepsilon)\nu(z) - \varepsilon R\}$, if $z \in B_R$ and $w(z) = (1+3\varepsilon)\nu(z) - \varepsilon R$, if $z \notin B_R$. Since for $z \in \partial B_R$ we have $w(z) = (1 + 3\varepsilon)\nu(z) - \varepsilon R \geq (1 + 2\varepsilon)R \geq \nu_j(z)$, then the $w(z)$ is psh on X. Hence, the function $\frac{1}{1+3\varepsilon}[w(z) - \varepsilon] \in \pounds(K)$ and $\frac{1}{1+3\varepsilon}[w(z) - \varepsilon] \geq V^*(z, K) = \nu(z)$. It follows, that $\nu_j(z) \leq (1 + 3\varepsilon)V^*(z, K) + \varepsilon$ and this with $\nu_j(z) \geq V^*(z, K)$ gives continuity of $V^*(z, K)$.Q.E.D. □

Corollary 2.1. *If the "jump" of the exhaustion function $\rho(z) - \rho_*(z) = o(\rho_*(z))$, then X is S^*-parabolic.*

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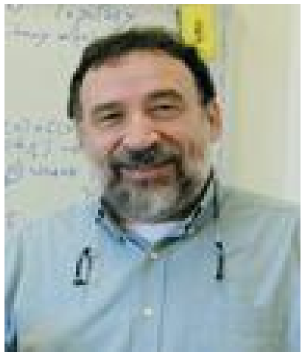

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