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GENERALIZATION OF SOME DETERMINANTAL IDENTITIES FOR NON-SQUARE MATRICES BASED ON RADIC'S DEFINITION

A. AMIRI [1], M. FATHY [1], M. BAYAT [2]

Abstract. In this paper, we focus on Radic's definition for the determinant of non-square matrices. We develop some important properties of this determinant. We generalize several classical important determinant identities, including Dodgson's condensation, Cauchy-Binet, and Trahan for non-square matrices. Also, we propose an efficient algorithm with $\Theta((mn)^2)$ time complexity for computing Radic's determinant based on Dodgson algorithms and dynamic programming technique.

Keywords: Radic's determinant, non-square matrix, determinantal identities, Binet-Cauchy formula, Dodgson's algorithm, Trahan formula, dynamic programing.

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1. Introduction

The determinant is a fundamental concept in linear algebra and applied statistics, and has significant applications in various branches of mathematics and engineering. Originally determinants are defined only for square matrices. Therefore, if one can generalize the definition to non-square matrices, then, we will be able to extract some of the important properties and identities of non-square matrices that can be used in defining many other important concepts such as eigenvalues, eigenvectors and efficient matrix decompositions for these matrices.

To date, many definitions have been proposed for the determinant of non-square matrices. Earlier works have been mainly focused on utilizing the determinant of square blocks to define the determinant of the non-square matrix. Joshi [6] defined the determinant of non-square matrices as the sum of the determinants of square blocks. They studied many useful properties of this determinant. Radic [9] proposed an efficient definition that has some of the major properties of the determinants of square matrices. Also, some other properties of Radic's determinant and its geometrical interpretations, involving polygons in the plan $\mathbb{R}^2$ and polyhedra in $\mathbb{R}^3$ are given in [8]-[13]. Also, Amiri et al. [1] defined generalized eigenvalue and eigenvector based on Radic's determinant. They applied it on video retrieval and video shot boundary detection applications [1]-[3]. Very recently, Y. Haruo and T. Yoshio [5] defined non-negative determinants for non-square matrices that have many valuable applications in multivariate analysis. They studied a broad spectrum of properties of this determinant.

Successful as they are, existing works on the determinants of non-square matrices have not focused on generalizing many useful square determinantal identities, such as Dodgson, Cauchy-Binet and Trahan, for non-square matirces. Also, there is no efficient algorithm for computing these determinants. With this in mind, in this paper we focus on Radic's definition [9] and we set out to develop some important properties of this determinant and to generalize some

[1] Iran University of Science and technology, Computer Engineering Department, Tehran, Iran and Computer Engineering Department, Zanjan University, Zanjan, Iran,
e-mail: amiri@iust.ac.ir, Mahfathy@iust.ac.ir
[2] Computer Engineering Department, Zanjan University, Zanjan, Iran,
e-mail: Bayat@iasbs.ac.ir
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of important determinantal identities for non-square matrices based on this definition. We will develop an efficient algorithm for computing this determinant based on generalized Dodgson identity and a dynamic programing technique that has polynomial time complexity.

The remainder of this paper is organized as follows. Section 2 reviews Radic's definition for determinant of non-square matrices. Section 3 studies some properties of this determinant. We extend the Dodgson identity for non-square matrices. We also design an efficient and straightforward algorithm for calculating this determinant with polynomial order time complexity in Section 4. Section 5 presents the generalized Cauchy-Binet and Trahan formulas for non-square matrices. Section 6 summarizes the paper.

2. Radic's definition

In the following, we first present Radic's definition [9] for determinant of non-square matrices.

Definition 2.1. *Let $A = (a_{i,j})$ be an $m \times n$ matrix with $m \leq n$. The determinant of A, is defined as:*

$$\det(A) = \sum_{1 \leq j_1 < \ldots < j_m \leq n} (-1)^{r+s} \det \begin{bmatrix} a_{1j_1} & \ldots & a_{1j_m} \\ \vdots & \ddots & \vdots \\ a_{mj_1} & \ldots & a_{mj_m} \end{bmatrix}, \tag{1}$$

where $j_1, j_2, \cdots, j_m \in \mathbb{N}$, $r = 1 + 2 + \cdots + m$ and $s = j_1 + \cdots + j_m$. If $m > n$, then we define $\det(A) = 0$.

Example 2.2.

$$\begin{aligned} \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} &= (-1)^{(1+2)+(1+2)} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} + (-1)^{(1+2)+(1+3)} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \\ &+ (-1)^{(1+2)+(2+3)} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} = \\ &= \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} - \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}. \end{aligned}$$

The determinant (1) is a skew-symmetric multilinear functional with respect to rows (only multiaffine with respect to columns) and, therefore, it has many well-known standard properties. The main properties are listed in the following theorem:

Theorem 2.3. *Suppose that A is an $m \times n$ matrix with $m \leq n$, then*

(1) *If $A = [a_1, a_2, \ldots, a_n]$, then* $\det(A) = a_1 - a_2 + \ldots + (-1)^{n-1} a_n$.
(2) *If a row of a matrix A is multiplied by λ, then the determinant of the new matrix is* $\lambda \det(A)$.
(3) *If the matrix A has two identical rows, then* $\det(A) = 0$.
(4) *If a row of matrix A is a linear combination of some other rows, then* $\det(A) = 0$.
(5) *The interchange of any two rows in A will only change the sign of* $\det(A)$.
(6) *The Laplace expansion theorem is valid.*

Proof. See [1-3],[8],[10],[12].

In addition, some of the other properties of Radic's determinant and its several excellent geometrical interpretations involving polygons in the plan $\mathbb{R}^2$ and polyhedra in $\mathbb{R}^3$ are studied in [8]-[10-12].

3. Other properties of Radic's determinant

The following lemmas and theorems exhibit some of the important properties of the Radic's determinant. They are essential in studying of the theorems of the subsequent sections.

Lemma 3.1. *Let A be an $m \times n$ matrix, $1 \leq m < n$, and $m + n$ odd, then:*

$$\det_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left[(a_{i,j}) \right] = 0,$$

where $a_{1,j} = 1$ *for all* $j, 1 \leq j \leq n$.

Proof. We proceed by induction on even integer n for all odd integer numbers $m, 1 \leq m < n$. If $n = 2$ then $m = 1$, we have

$$\det\Big([1,1]\Big) = 1 - 1 = 0.$$

Now, we assume that it is true for even n and all odd m, $1 \leq m < n$. We will show that the identity holds for $n+2$ (which is even number) and all odd m, $1 \leq m < n+2$. By expanding the determinant with respect to row m, we get

$$\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})\Big] = \sum_{k=1}^{n+2}(-1)^{k+m}a_{m,k}\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})_{\substack{i\neq m\\j\neq k}}\Big],$$

Next, by expanding the right determinant with respect to the row $m-1$, we obtain

$$\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})_{\substack{i\neq m\\j\neq k}}\Big] = \sum_{\substack{\acute{k}=1\\\acute{k}\neq k}}^{n+2}(-1)^{\acute{k}+m-1}a_{m-1,\acute{k}}\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})_{\substack{i\neq m-1,m\\j\neq k,\acute{k}}}\Big],$$

therefore, we get

$$\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})\Big] = \sum_{k=1}^{n+2}\sum_{\substack{\acute{k}=1\\\acute{k}\neq k}}^{n+2}(-1)^{k+\acute{k}-1}a_{m,k}a_{m-1,\acute{k}}\det_{\substack{1\leq i\leq m\\1\leq j\leq n+2}}\Big[(a_{i,j})_{\substack{i\neq m-1,m\\j\neq k,\acute{k}}}\Big].$$

On the other hand, all determinants on the right hand side of the above equality are of size m by n, with the first row entries equal to one. Thus, by induction hypothesis, those determinants are zero and the proof is completed. Another case can be established in the same way.

Lemma 3.2.

$$\det[A_1,\cdots,A_n,0_m] = \det[A_1,\cdots,A_n],$$

and,

$$\det[A_1,\cdots,A_{j-1},0_m,A_{j+1},\cdots,A_n] = \det[A_1,\cdots,A_{j-1},-A_{j+1},\cdots,-A_n],$$

where $m \leq n$, $A_k = [a_{1,k},\cdots,a_{m,k}]^T$ *for* $k \in \{1,\cdots,n\} - \{j\}$, *and* 0_m *is an* m *by* 1 *zero vector.*

Proof. The proof of the first formula is straightforward by Definition 1.1. We prove the second formula by induction on n. Set

$$A_j = \begin{bmatrix} a_{1,j} \\ B_j \end{bmatrix}, \quad B_j = \begin{bmatrix} a_{2,j} \\ \vdots \\ a_{m,j} \end{bmatrix}, \quad (1 \leq j \leq n).$$

Expanding the determinant with respect to the first row yields

$$\begin{aligned} D = \det[A_1,\cdots,A_{j-1},0_m,A_{j+1},\cdots,A_n] &= \det\begin{bmatrix} a_{1,1}\cdots a_{1,j-1} & 0 & a_{1,j+1}\cdots a_{1,n} \\ B_1\ \cdots\ B_{j-1} & 0_{m-1} & B_{j+1}\ \cdots\ B_n\end{bmatrix} = \\ &= a_{1,1}\det[B_2,\cdots,B_{j-1},0_{m-1},B_{j+1},\cdots,B_n] - \cdots + \\ &+(-1)^{j-1+1}a_{1,j-1}\det[B_1,\cdots,B_{j-2},0_{m-1},B_{j+1},\cdots,B_n] + \\ &+(-1)^{j+1+1}a_{1,j+1}\det[B_1,\cdots,B_{j-1},0_{m-1},B_{j+2},\cdots,B_n] + \cdots + \\ &+(-1)^{n+1}a_{1,n}\det[B_1,\cdots,B_{j-1},0_{m-1},B_{j+1},\cdots,B_{n-1}], \end{aligned}$$

next, by induction hypothesis

$$\begin{aligned} D = a_{1,1} \det[B_2, \cdots, B_{j-1}, -B_{j+1}, \cdots, -B_n] - \cdots + \\ +(-1)^{j-1+1} a_{1,j-1} \det[B_1, \cdots, B_{j-2}, -B_{j+1}, \cdots, -B_n] + \\ +(-1)^{j+1+1} a_{1,j+1} \det[B_1, \cdots, B_{j-1}, -B_{j+2}, \cdots, -B_n] + \cdots + \\ +(-1)^{n+1} a_{1,n} \det[B_1, \cdots, B_{j-1}, -B_{j+1}, \cdots, -B_{n-1}] \end{aligned}$$

or, equivalently

$$\begin{aligned} D = \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & -a_{1,j+1} & \cdots & -a_{1,n} \\ B_1 & \cdots & B_{j-1} & -B_{j+1} & \cdots & -B_n \end{bmatrix} = \\ = \det[A_1, \cdots, A_{j-1}, -A_{j+1}, \cdots, -A_n]. \end{aligned}$$

Theorem 3.3. *Suppose* $1 \le m < n$, *and* $m + n$ *be an odd integer,* $A = (a_{i,j}) = [A_1, \cdots, A_n]$ *be an* m *by* n *matrix, and* X *be an arbitrary* m *by* 1 *column vector, then:*

$$\det[A_1 + X, \cdots, A_n + X] = \det[A_1, \cdots, A_n].$$

Proof. We will prove the assertion by mathematical induction on n even and all m odd $1 \le m < n$. For $n = 2, m = 1$, we have

$$\det[a_{1,1} + x, a_{1,2} + x] = \det[a_{1,1}, a_{1,2}] = a_{1,1} - a_{1,2}.$$

Now, assume the assertion is true for all n even and m odd with $1 \le m < n$. We prove it is also true for $n+2$ even and all m odd with $1 \le m < n+2$. By setting

$$A_j = \begin{bmatrix} a_{1,j} \\ B_j \end{bmatrix}, \quad B_j = \begin{bmatrix} a_{2,j} \\ \vdots \\ a_{m,j} \end{bmatrix} \quad (1 \le j \le n+2)$$

$$X = \begin{bmatrix} x_1 \\ X' \end{bmatrix}, \quad X' = \begin{bmatrix} x_2 \\ \vdots \\ x_m \end{bmatrix},$$

we get

$$\begin{aligned} D = \det[A_1 + X, \cdots, A_{n+2} + X] = \det \begin{bmatrix} a_{1,1} + x_1 & \cdots & a_{1,n+2} + x_1 \\ B_1 + X' & \cdots & B_{n+2} + X' \end{bmatrix} = \\ = (-1)^{1+1} (a_{1,1} + x_1) \det\left[B_2 + X', \cdots, B_{n+2} + X'\right] + \cdots + \\ + (-1)^{1+n+2} (a_{1,n+2} + x_1) \det\left[B_1 + X', \cdots, B_{n+1} + X'\right], \end{aligned}$$

now, by inductive hypothesis

$$\begin{aligned} D = (-1)^{1+1}(a_{1,1} + x_1) \det[B_2, \cdots, B_{n+2}] + \\ + \cdots + (-1)^{1+n+2}(a_{1,n+2} + x_1) \det[B_1, \cdots, B_{n+1}] = \\ = \Big\{(-1)^{1+1} (a_{1,1}) \det[B_2, \cdots, B_{n+2}] + \cdots + (-1)^{1+n+2} (a_{1,n+2}) \det[B_1, \cdots, B_{n+1}]\Big\} - \\ -c_1 \Big\{(-1)^{1+1} \det[B_2, \cdots, B_{n+2}] + \cdots + (-1)^{1+n+2} \det[B_1, \cdots, B_{n+1}]\Big\} = \\ = \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,n+2} \\ B_1 & \cdots & B_n \end{bmatrix} + x_1 \begin{bmatrix} 1 & \cdots & 1 \\ B_1 & \cdots & B_{n+2} \end{bmatrix}, \end{aligned}$$

considering Lemma 3.1, the last determinant is zero and the proof is completed. The second case can be treated similarly.

Corollary 3.4. *For* $m + n$ *odd,* $1 \le m < n$ *and for all* $k, 1 \le k \le n$, *we have*

$$\det[A_1, \cdots, A_{k-1}, A_k, A_{k+1}, \cdots, A_n] = \det[A_k - A_1, \cdots, A_k - A_{k-1}, A_{k+1} - A_k, \cdots, A_n - A_k].$$

*Proof.*Applying Theorem 3.3 with $X = -A_k$ and Lemma 3.2 the result is obtained.
Theorem 3.5. [Cyclic] *If* $1 \leq m < n$, *and* $m + n$ *is an odd integer, then for all* $i \in \{1, \cdots, n\}$, *we have*

$$(-1)^{(i+1)m} \det[A_i, \cdots, A_n, \cdots, A_{i-1}] = \det[A_1, \cdots, A_n].$$

Proof. It is sufficient to prove

$$(-1)^m \det[A_n, A_1, \cdots, A_{n-1}] = \det[A_1, \cdots, A_n].$$

Applying Theorem 3.3 with $X = -A_n$ and Lemma 3.2 we have

$$\begin{aligned} (-1)^m \det[A_n, A_1, \cdots, A_{n-1}] &= (-1)^m \det[0_m, A_1 - A_n, \cdots, A_{n-1} - A_n] = \\ &= (-1)^m (-1)^m \det[A_1 - A_n, \cdots, A_{n-1} - A_n] = \\ &= \det[A_1 - A_n, \cdots, A_{n-1} - An, A_n - A_n] = \det[A_1, \cdots, A_n]. \end{aligned}$$

Theorem 3.6. [Semi-Cyclic] *If* $1 \leq m < n$, *and* $m + n$ *is even, then for all* $i \in \{1, \cdots, n\}$

$$(-1)^{(n-i)m} \det[A_i, \cdots, A_n, -A_1, \cdots, -A_{i-1}] = \det[A_1, \cdots, A_n].$$

Proof. It is sufficient to prove that

$$\det[A_n, -A_1, \cdots, -A_{n-1}] = \det[A_1, \cdots, A_n].$$

Applying Theorem 3.3 with $X = -A_n$ and Lemma 3.2 we have

$$\begin{aligned} \det[A_n, -A_1, \cdots, -A_{n-1}] &= \det[A_n, -A_1, \cdots, -A_{n-1}, 0_m] = \\ &= (-1)^m \det[-A_1, \cdots, -A_{n-1}, 0_m, A_n] = \\ &= (-1)^m \det[-A_1, \cdots, -A_{n-1}, -A_n, 0_m] = \\ &= (-1)^m (-1)^m \det[A_1, \cdots, A_n] = \det[A_1, \cdots, A_n]. \end{aligned}$$

The next theorem involves the product property of the determinant of non-square matrices.
Theorem 3.7. *Let* $1 \leq m \leq n$, *and* A *be an* $m \times m$ *matrix, and* B *be an* $m \times n$ *matrix, then*

$$\det(A.B) = \det(A). \det(B).$$

Proof. Let $B = [B_1, \cdots, B_n]$, then

$$\begin{aligned} \det(A.B) &= \det\Big(A.[B_1, \cdots, B_n]\Big) = \det\Big([A.B_1, \cdots, A.B_n]\Big) = \\ &= \sum_{1 \leq j_1 < \cdots < j_m \leq n} (-1)^{r+s} \det\Big([A.B_{j_1}, \cdots, A.B_{j_m}]\Big) = \\ &= \sum_{1 \leq j_1 < \cdots < j_m \leq n} (-1)^{r+s} \det(A). \det\Big([B_{j_1}, \cdots, B_{j_m}]\Big) = \\ &= \det(A) \sum_{1 \leq j_1 < \cdots < j_m \leq n} (-1)^{r+s} \det\Big([B_{j_1}, \cdots, B_{j_m}]\Big) = \det(A). \det(B). \end{aligned}$$

4. Dodgson algorithm for non-square matrices

Here, we are in a position to prove the main theorem of this paper by using the previous results. To this end, we need to introduce the following notation:
Notation 1. The $(n-k) \times (n-l)$ matrix obtained from A by removing the i_1^{th}, $i_2^{th}, \cdots, i_k^{th}$ rows and the j_1^{th}, $j_2^{th}, \cdots, j_l^{th}$ columns is denoted by $(a_{i,j})_{\substack{i \neq i_1, i_2, \cdots, i_k \\ j \neq j_1, j_2, \cdots, j_k}}$.

The well-known Dodgson's algorithm for a square matrix $A=(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}$, is as follows (see [4] and [15]):

$$\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{i\neq k,k'\\j\neq l,l'}}\right]=\det\begin{bmatrix}\det\left[(a_{i,j})_{\substack{i\neq k\\j\neq l}}\right] & \det\left[(a_{i,j})_{\substack{i\neq k\\j\neq l'}}\right]\\ \det\left[(a_{i,j})_{\substack{i\neq k'\\j\neq l}}\right] & \det\left[(a_{i,j})_{\substack{i\neq k'\\j\neq l'}}\right]\end{bmatrix},\qquad(2)$$

for all $k,k',l,l'\in\{1,2,\cdots,n\}$ with $k\neq k'$ and $l\neq l'$.
Here, we will generalize Dodgson Condensation Formula for the determinant of non-square matrices.

Theorem 4.1 [Dodgson Condensation Formula for Non-Square Determinants] *Let $A=(a_{i,j})$ be an m by n matrix, $2\le m\le n$, then:*

$$\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m-1,m\\j\neq n-1,n}}\right]=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m\\j\neq n}}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m-1\\j\neq n-1}}\right]+$$

$$-\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m\\j\neq n-1}}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m-1\\j\neq n}}\right]+\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{i\neq m-1,m}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{j\neq n-1,n}\right].$$

Proof. We distinguish between two cases:

Case I. Assume that $m+n$ is odd, and $2\le m<n$. Now, we proceed by induction on n. Considering Corollary 3.4, for $k=1$, we have,

$$\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})\right]=\det\left[A_1,\cdots,A_n\right]=\det\left[A_1-A_2,\cdots,A_1-A_n\right]=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{j\neq 1}\right],$$

and,

$$\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m-1,m\\j\neq n-1,n}}\right]=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m-1,m\\j\neq 1,n-1,n}}\right],$$

therefore, we conclude

$$D=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq m-1,m\\j\neq n-1,n}}\right]=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{j\neq 1}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m-1,m\\j\neq 1,n-1,n}}\right]$$

Now, by induction hypothesis

$$D=\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m\\j\neq 1,n}}\right]\cdot\det_{\substack{1\le i\ leqm\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m-1\\j\neq 1,n-1}}\right]-$$

$$-\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m\\j\neq 1,n-1}}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m-1\\j\neq 1,n}}\right]+$$

$$+\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{\substack{i\neq m-1,m\\j\neq 1}}\right]\cdot\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{1,j}-a_{i,j})_{j\neq 1,n-1,n}\right],$$

Here, by simplifying the above expression, and using Theorem 3.3 in reverse direction, the proof is completed.

Case II. We assume that $m+n$ is even. By applying Lemma 3.2, we get

$$\det_{\substack{1\le i\le m\\1\le j\le n}}\left[(a_{i,j})\right]=\det\left[A_1,\cdots,A_n\right]=\det\left[A_1,\cdots,A_n,0_m\right]=$$

$$=\det\left[A_1,\cdots,A_{n-2},0_m,-A_{n-1},-A_n\right].$$

Since the last determinant is determinant of a matrix of size $m\times(n+1)$ and $m+n+1$ is odd, the result follows from the case I.

Corollary 4.2. [Dodgson Condensation Formula] *Suppose $A = (a_{i,j})$ be an n by n square matrix, then:*

$$\det_{1\le i,j\le n}\left[(a_{i,j})\right]\cdot \det_{1\le i,j\le n}\left[(a_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right] =$$

$$= \det_{1\le i,j\le n}\left[(a_{i,j})_{\substack{i\neq n\\ j\neq n}}\right]\cdot \det_{1\le i,j\le n}\left[(a_{i,j})_{i\neq n-1\ atop j\neq n-1}\right] - \det_{1\le i,j\le n}\left[(a_{i,j})_{\substack{i\neq n\\ j\neq n-1}}\right]\cdot \det_{1\le i,j\le n}\left[(a_{i,j})_{\substack{i\neq n-1\\ j\neq n}}\right].$$

Next, we give a generalization of Theorem 4.1.

Theorem 4.3. *Let $A = (a_{i,j})$ be an m by n matrix, $2 \le m \le n$,, then:*

$$\det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})\right]\cdot \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq r',s'\\ j\neq r,s}}\right] = \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq s'\\ j\neq s}}\right] \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq r'\\ j\neq r}}\right] +$$

$$- \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq s'\\ j\neq r}}\right] \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{\substack{i\neq r'\\ j\neq s}}\right] + \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{i\neq r',s'}\right] \det_{\substack{1\le i\le m\\ 1\le j\le n}}\left[(a_{i,j})_{j\neq r,s}\right],$$

where $r, s \in \{1, \cdots, n\}, r', s' \in \{1, \cdots, m\}, r \neq s, r' \neq s'$.

Proof. In the proof of Theorem 4.1, instead of column one, we choose column k such that $k \neq r, s$. The rest of proof can be treated in the same way.

Dodgson's algorithm explains a recursive method for determinant computation. Here, we demonstrate an efficient iterative algorithm for computing determinant of non-square matrices based on Dodgson's algorithm. Before describing the proposed algorithm, first we review the simplest recursive algorithm which follows directly from Dodgson's method. The pseudo code of the algorithm was shown in Figure 1.

```
int Dodgson-Recursive(Mat, m, n)
inputs:
     Mat:  a non-square matrix,
     n:  Number of rows in Mat matrix,
     m:  number of columns in Mat matrix,
outputs:
     Determinant of Mat matrix,
Begin
     if (m == 1)
          Det =0, Sign=1
          for i = 1 to n do
               Det = Det + Sign*Mat(1,i), Sign=-Sign
          End for
     else
          DetM(1) = Dodgson-Recursive(Mat(1:m-1, 1:n-1), m-1, n-1);
          DetM(2) = Dodgson-Recursive(Mat(2:m, 2:n), m-1, n-1);
          DetM(3) = Dodgson-Recursive(Mat(1:m-1, 2:n), m-1, n-1);
          DetM(4) = Dodgson-Recursive(Mat(2:m-1, 2:n-1), m-1, n-1);
          DetM(5) = Dodgson-Recursive(Mat(1:m, 2:n-1),m, n-2);
          DetM(6) = Dodgson-Recursive(Mat(2:m-1, 1:n),m-2, n);
          DetM(7) = Dodgson-Recursive(Mat(2:m-1, 2:n-1),m-2, n-2);
          Det = DetM(1)*DetM(2)-DetM(3)*DetM(4)+DetM(5)*DetM(6)/DetM(7);
     end
     returen(Det);
End
```

Figure 1: The simplest version of Non-square Dodgson's algorithm.

Although this algorithm was easy to create and understandable, it is extremely inefficient. For time complexity analysis, we will consider the comparison instruction as a basic-operation, and m and n, the size of matrix, as input size. If $T(m,n)$ be the time complexity of Dodgson-Recursive algorithm, then:

$$T(m,n) = \begin{cases} 1 & \text{if} \quad m = 1, \\ 4T(m-1,n-1) + T(m-2,n-2) + \\ T(m,n-2) + T(m-2,n) + 1 & \text{if} \quad n \geq m \geq 2. \end{cases}$$

It is obvious that this recursive relation has exponential solution. Consequently, the time complexity of Dodgson-Recursive algorithm is very inefficient. A more efficient algorithm with the time cost of polynomial order is developed next, using the dynamic programming technique. For $1 \leq i, i+l \leq m, 1 \leq j, j+k \leq n$ and $l, k \geq 0$, let $\mathrm{Det}(i, i+l, j, j+k)$ be the determinant of matrix $\mathrm{Mat} = [m_{p,q}]_{\substack{i \leq p \leq i+l \\ j \leq q \leq j+k}}$. The pseudo code of the dynamic programming algorithm is shown in Figure 2.

```
int Dodgson-Dynamic-Programming(Ma, m, n)
inputs:
     Mat:  a square matrix,
     n:  Number of rows in Mat matrix,
     m:  number of columns in Mat matrix,
outputs:
     Determinant of Mat matrix
Begin
      Compute directly all 1 × i determinants with 1 ≤ i ≤ n.
     for l=1 to m do
      for i=1 to m-l do
       for k=l to n do
        for j=l to j-k do
          Det(i,i+l,j,j+k) = [Det(i,i+l-1,j,j+k-1) * Det(i+1,i+l,j+1,j+k)-
          Det(i,i+l-1,j+1,j+k)* Det(i+1,i+j,j,j+l-1)
          +Det(i,i+l,j,j+k-2)*Det(i,i+l-2,j,j+k)] / Det(i,i+l,j,j+k);
  End
```

Figure 2: Dodgson's algorithm using dynamic programing design.

According to this pseudo code, it can be shown simply that the time complexity of the proposed algorithm is of order of $\Theta((mn)^2)$. Thus, the proposed algorithms has polynomial order time cost. This result confirms the importance of the proposed algorithm.

5. Binet-Cauchy and Trahan formulas for non-square matrices

In this section, we will extend two well-known determinant identities for non-square matrices, Binet-Cauchy and Trahan formulas (see [7] and [14]).
First, we need the following lemma.

Lemma 5.1. *Let A be an $m \times n$ matrix with $m \leq n$, then:*

$$\det(A) = \sum_{1 \leq j_1 < \ldots < j_m \leq n} a_{1j_1} \ldots a_{mj_m} \det \begin{bmatrix} \delta_{j_1} \\ \vdots \\ \delta_{j_m} \end{bmatrix},$$

where δ_{j_k} is an $1\times n$ column vector with entry j_k equals 1 and the others equals 0.
Proof. Let $A=(a_{ij})$, then:

$$A=\begin{bmatrix} \sum_{j_1=1}^{n} a_{1j_1}\delta_{j_1} \\ \vdots \\ \sum_{j_m=1}^{n} a_{mj_m}\delta_{j_m} \end{bmatrix}.$$

Since $\det(A)$ is linear with respect to each row,

$$\det(A)=\sum_{j_1=1}^{n}\dots\sum_{j_m=1}^{n}\det\begin{bmatrix} a_{1j_1}\delta_{j_1} \\ \vdots \\ a_{mj_m}\delta_{j_m}\end{bmatrix}=\sum_{j_1,\dots,j_m=1}^{n} a_{1j_1}\dots a_{mj_m}\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}.$$

Note that $\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}=0$, unless $j_1,\dots,j_m$ are distinct integer numbers. Furthermore when $j_1,\dots,j_m$ are distinct, obviously $\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}=\pm 1$.

Theorem 5.2. [Binet-Cauchy Formula for Non-Square Matrices] *Let A and B be $m\times p$ and $p\times n$ matrices respectively, and $m\le n$, $m\le p$, then:*

$$\det(AB)=\sum_{1\le k_1<\dots<k_m\le p}\det\Big(A_{k_1,\dots k_m}\Big)\det\Big(B^{k_1,\dots,k_m}\Big),$$

where $\det\Big(A_{k_1,\dots k_m}\Big)$ is determinant obtained from the columns of A whose numbers are $k_1,\dots,k_m$ and $\det\Big(B^{k_1,\dots,k_m}\Big)$ is determinant obtained from the rows of B whose numbers are $k_1,\dots,k_m$.
Proof. Suppose that $D=AB$, that $d_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$. According to Lemma 5, we have

$$\begin{aligned}\det(D) &= \sum_{1\le j_1<\dots<j_m\le p} d_{1j}\dots d_{mj_m}\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}= \\ &= \sum_{1\le j_1<\dots<j_m\le p}\Big(\sum_{k_1=1}^{p}a_{1k_1}b_{k_1j_1}\Big)\dots\Big(\sum_{k_m=1}^{p}a_{mk_m}b_{k_mj_m}\Big)\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}= \\ &= \sum_{k_1,\dots,k_m=1}^{p}(a_{1k_1}\dots a_{mk_m})\sum_{1\le j_1<\dots<j_m\le p}(b_{k_1j_1}\dots b_{k_mj_m})\det\begin{bmatrix}\delta_{j_1}\\ \vdots \\ \delta_{j_m}\end{bmatrix}= \\ &= \sum_{k_1,\dots,k_m=1}^{p}(a_{1k_1}\dots a_{mk_m})\det\Big(B^{k_1,\dots,k_m}\Big),\end{aligned}$$

if $k_1,\dots,k_m$ not be distinct, $\det\Big(B^{k_1,\dots,k_m}\Big)=0$, then in summation we consider only distinct $k_1,\dots,k_m$. Since $\det\Big(B^{\tau(k_1),\dots,\tau(k_m)}\Big)=\operatorname{sign}(\tau)\det\Big(B^{k_1,\dots,k_m}\Big)$, then for each permutation τ of

$k_1, \dots, k_m$ with $k_1 < \dots < k_m$, we have

$$\begin{aligned}\det(D) &= \sum_{k_1,\dots,k_m=1}^{p} (a_{1k_1}\dots a_{mk_m}) \det\Big(B^{k_1,\dots,k_m}\Big) = \\ &= \sum_{1\le k_1<\dots<k_m\le p} \left(\sum_{\tau} \operatorname{sign}(\tau) a_{1\tau(1)}\dots a_{m\tau(m)}\right) \det\Big(B^{k_1,\dots,k_m}\Big) = \\ &= \sum_{1\le k_1<\dots<k_m\le p} \det\Big(A_{k_1,\dots,k_m}\Big)\det\Big(B^{k_1,\dots,k_m}\Big).\end{aligned}$$

Theorem 5.3. [Trahan Formula for Non-Square Matrices] *Assume* $A = (a_{i,j})$ *is an* $m \times n$ *matrix, where* $1 \le m \le n$. *Let*

$$s_j = \sum_{i=1}^{m} a_{i,j}, \qquad S_j = \sum_{i=1}^{m} A_{i,j}, \qquad (j = 1, \cdots, n),$$

where

$$A_{r,s} = (-1)^{r+s} \det_{\substack{1\le i\le m\\1\le j\le n}} \left[(a_{i,j})_{\substack{i\ne r\\ j\ne s}}\right],$$

then:

$$\det_{\substack{1\le i\le m\\1\le j\le n}} [(a_{i,j})] = \frac{1}{m}\sum_{j=1}^{n} s_j S_j.$$

Proof. The proof is proceed by induction on n. We distinguish between two cases:
Case I. Assume that $m+n$ is odd. Considering Theorem 3 with $X = -A_n$, and Lemma 3.2, we have

$$D = \det_{\substack{1\le i\le m\\1\le j\le n}} [(a_{i,j})] = \det_{\substack{1\le i\le m\\1\le j\le n}} \Big[(a_{i,j} - a_{i,n})_{j\ne n}\Big].$$

Now, by induction hypothesis

$$D = \frac{1}{m}\sum_{j=1}^{n-1} s'_j S'_j,$$

where

$$s'_j = \sum_{i=1}^{m}(a_{i,j} - a_{i,n}) = \sum_{i=1}^{m} a_{i,j} - \sum_{i=1}^{m} a_{i,n} = s_j - s_n,$$

$$\begin{aligned}S'_j = \sum_{i=1}^{m} A'_{i,j} &= \sum_{i=1}^{m} (-1)^{i+j} \det_{\substack{1\le r\le m\\1\le s\le n}} \left[(a_{r,s} - a_{r,n})_{\substack{r\ne i\\ s\ne j,n}}\right] = \\ &= \sum_{i=1}^{m} (-1)^{i+s} \det_{\substack{1\le r\le m\\1\le s\le n}} \left[(a_{r,s})_{\substack{r\ne i\\ s\ne j}}\right] = \sum_{i=1}^{m} A_{i,j} = S_j,\end{aligned}$$

therefore

$$\begin{aligned}D &= \frac{1}{m}\sum_{j=1}^{n-1}(s_j - s_n)S_j = \frac{1}{m}\sum_{j=1}^{n-1} s_j S_j - \frac{s_n}{m}\sum_{j=1}^{n-1} S_j = \\ &= \frac{1}{m}\sum_{j=1}^{n} s_j S_j - \frac{s_n}{m}\sum_{j=1}^{n} S_j.\end{aligned}$$

Now, it is sufficient to prove that $\sum_{j=1}^{n} S_j = 0$,

$$\sum_{j=1}^{n} S_j = \sum_{j=1}^{n}\sum_{i=1}^{m}(-1)^{i+j} \det_{\substack{1\le r\le m\\1\le s\le n}} \left[(a_{r,s})_{\substack{r\ne i\\ s\ne j}}\right] =$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j} \det_{\substack{1\le r\le m\\1\le s\le n}} \left[(a_{r,s})_{\substack{r\ne i\\ s\ne j}}\right] = 0.$$

Using Lemma 3.1, we get

$$\sum_{j=1}^{n}(-1)^{i+j} \det_{\substack{1\le r\le m\\1\le s\le n}} \left[(a_{r,s})_{\substack{r\ne i\\ s\ne j}}\right] = \det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ 1 & 1 & \cdots & 1 \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = 0.$$

Case II. Now, we assume that $m + n$ is even. By applying Lemma 3.2, we get

$$\det[A_1, \cdots, A_n] = \det[A_1, \cdots, A_n, 0_m],$$

then $m + n$ is odd and the result follows from the Case I.

6. Conclusion

An efficient and effective determinant definition for non-square matrices is of great interest in linear algebra and applied statistic applications. In this paper, we focus on Radic's definition for the determinant of non-square matrices, because, it has some of the major properties of the determinants of square matrices . Unlike previous works, we extract some useful properties of this determinant in order to generalize several determinantal identities such as Dodgson, Cauchy-Binet and Trahan, for non-square matrices. Also, we present an efficient and straightforward algorithm for computing Radic's determinant based on the generalized Dodgson algorithm and a dynamic programing technique that has polynomial complexity.

More work for defining eigenvalue and eigenvector for non-square matrices and extending some matrix decompositions based on these results will be reported in the near future.

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Ali Amiri was born in Zanjan, Iran, in 1982. He received the M.S. degree in computer engineering from Iran University of Science and Technology (IUST in 2006, and Ph.D. degree in Artificial Intelligence in 2010 from IUST, Iran. His research interests include video segmentation, video retrieval and summarization, matrix computation, pattern recognition and moving object detection and tracking.

Mahmood Fathy received B.S. degree in electronics from Iran University of Science and Technology in 1985, and M.S. in 1987 from Bradford University, UK, in computer architecture, and the Ph.D. degree in image processing computer architecture in 1991 from UMIST, UK. Since 1991, he has worked in the Computer Engineering Department of IUST as an Associate Professor. His research interests include real-time image processing in particular in traffic engineering and QoS in computer networks including video and image transmission over Internet.

Morteza Bayat received M.S. degree in 2002 from Institute for Advance Studies in Basic Sciences (IASBS), Zanjan, Iran, in Applied Mathematics, and the Ph.D. degree in Differential Equations in 2008 from IASBS. Since 2008, he has worked in the Computer Engineering Department of Zanjan University as an Professor Assistant. His research interests include Matrix Computations and Differential Equations.