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LOG-CONCAVE MEASURES

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Abstract. We study the log-concave measures, their characterization via the Prékopa-Leindler property and also define a subset of it whose elements are called super log-concave measures which have the property of satisfying a logarithmic Sobolev inequality. We give some results about their stability. Certain relations with measure transportation are also indicated.

Keywords: Log-concave measures, logarithmic Sobolev inequality, measure transport, Prḱopa-Leindler inequality, H-convexity.

AMS Subject Classification: 60B, 60H; 28C

1. Introduction

The importance of logarithmically concave functions and measures has been discovered in the 70's (cf. [1, 9, 10, 11]) and they have found applications immediately in physics (cf. [13]). This notion has gained further importance when its close relations to Monge-Ampère equation and more generally to measure transportation has been realized (cf.[16] and the references there). In this work we give a general treatment of the subject beginning from the finite dimensional case and going towards to the infinite dimensions. A preliminary definition of log-concave measures is given using the Lebesgue measure then we extend this definition to the measures without density by using the Prékopa-Leindler property which characterizes them, which is also equivalent to Brunn-Minkowski property. We also introduce the notion of super log-concave measures, namely these are the measures which decrease very rapidly at infinity, in fact faster than some Gaussian measure and they always satisfy the logarithmic Sobolev inequality. Their definition uses the Euclidean structure of the underlying space on which they are defined, consequently, we need a special structure if we want to extend this notion to the infinite dimensional case. This is done by adjoining a rigged Hilbert space structure to the Fréchet space supporting the measure under question, whose typical example is an abstract Wiener space with its Cameron-Martin space. Afterwards we concentrate ourselves to the case of Wiener space, first we give some complementary results about the Jacobians corresponding to the image of the Wiener measure under a general perturbation of identity with lacking regularity and show that although each term of the Jacobian is not properly defined, their multiplication may create a renormalization, which is the typical case with monotone shifts. Even in this case, the log-concave character of the Jacobian is preserved and we use this observation to give another proof of the Prékopa-Leindler property for the Gaussian measure with "less log-concave" functions (called 1-log concave) and we give also another proof of this property using the reverse martingale convergence theorem, in particular we prove that the property of the functions used to test Prékopa-Leindler property is preserved under the conditional expectations and the Ornstein-Uhlenbeck semigroup.

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2. Preliminaries and notations

Generally each section contains its notational conventions with the exception of the Gaussian case; in fact we denote by (W, H, μ) an abstract Wiener space, namely, W is a separable Fréchet space, $H \subset W$ is a separable Hilbert space densely injected into W, μ is the unit Gauss measure supported by W which is quasi-invariant under the translation by the elements of H. One defines the usual Gateaux derivative of the nice functions on W along the subspace H, due to the quasi-invariance, this derivative has a unique (closed) extension to all the $L^p(\mu)$-spaces with $p \geq 1$ and it is denoted by ∇ and called Sobolev derivative (cf. [14] for instance). Hence, for a nice function f on W, ∇f defines a linear functional on H μ-almost surely, consequently it can be identified with an element of $H^\star = H$. Since the Sobolev derivative maps the scalar functions to H-valued functions, its adjoint, called the divergence operator, maps the vector valued functions to scalar ones and denoted by δ, in particular, we have

$$\int_W (\xi, \nabla f)_H d\mu = \int_W f\, \delta\xi d\mu\,,$$

for $\xi : W \to H$ cylindrical, well-known as the integration by parts formula. The very remarkable property of the divergence is that in the case of classical Wiener space $W = C([0,1], \mathbb{R}^n)$, if ξ has an adapted Lebesgue density $\xi'(t, w)$, then it holds that

$$\delta\xi(w) = \int_0^1 \xi'(t, w) dW_t(w)\,,$$

where the integral with dW denotes the Itô stochastic integral.

3. Log-concave and super log-concave measures in finite dimension

We begin by an initial concept:

Definition 1. *A (positive) measure ρ on $\mathbb{R}^d$ is said to satisfy the Prékopa-Leindler property if for any positive, continuous functions of compact support, say a, b, c such that*

$$a(sx + ty) \geq b(x)^s c(y)^t \tag{1}$$

for any $x, y \in \mathbb{R}^d$, $s + t = 1$, one has

$$\rho(a) \geq \rho(b)^s \rho(c)^t\,. \tag{2}$$

Theorem 1. *Assume that θ is a log-concave function, denote by ρ the measure $d\rho(x) = \theta(x)dx$. Then ρ satisfies the Prékopa-Leindler property.*

Here are two other well-known results which are due to Prékopa, [10], that we derive using the above considerations:

Corollary 1. *Let f and g be two integrable, log-concave functions, then their convolution $f \star g$ is log-concave. Moreover, for any $c \in \mathbb{R}$, the function $f \star_c g$ which is defined as*

$$f \star_c g(x) = \int f(cx - y) g(y) dy$$

is again a log-concave function.

Proof. Let $f_x(u) = f(x - u)$, then $(f_{sx+ty}g)(su + tv) \geq (f_x(u)g(u))^s (f_y(v)g(v))^t$, hence we can apply the theorem and obtain the inequality which characterizes the log-concavity. The proof of the second part is similar.

Corollary 2. *Let $f(x, \xi)$ be a log-concave function on $\mathbb{R}^d \times \mathbb{R}^m$, define $F(x)$ as to be*

$$F(x) = \int_{\mathbb{R}^m} f(x, \xi) d\xi\,.$$

Then F is log-concave on $\mathbb{R}^d$.

Proof. Let $f_x(\xi)$ denote $f(x,\xi)$, then for $s+t=1$, $x,y \in I\!R^d$, $\xi,\eta \in I\!R^m$, we have

$$f_{sx+ty}(s\xi+t\eta) \geq f_x(\xi)^s f_y(\eta)^t ,$$

hence

$$\begin{aligned} F(sx+ty) &= \int f_{sx+ty}(\xi)d\xi \geq \left(\int f_x(\xi)d\xi\right)^s \left(\int f_y(\xi)d\xi\right)^t = \\ &= F(x)^s F(y)^t . \end{aligned}$$

The results above indicate that the following definition is reasonable

Definition 2. *A (positive) measure on $I\!R^d$ is called log-concave if any of its convolutions with log-concave continuous functions of compact support has a log-concave density.*

Remark. Let us note that, for a measure ρ to be log-concave, it suffices the existence of just one continuous, log-concave function θ such that $\theta_\epsilon \star \rho$ has a log-concave density for any $\epsilon > 0$ where θ_ϵ denotes the d-dimensional rescaling of θ. In fact using the commutativity of convolutions, we obtain also that $\tilde{\theta} \star \rho$ has again a log-concave density, for any other continuous log-concave function of compact support $\tilde{\theta}$.

Proposition 1. *Let ρ_1 and ρ_2 be two log-concave measures on $I\!R^n$ and $I\!R^m$ respectively. Then the product measure $\rho_1 \otimes \rho_2$ is log-concave on $I\!R^n \times I\!R^m$.*

Proof. From Definition 2, it suffices to assume that $d\rho_1(x) = e^{-V_1(x)}dx$ and $d\rho_2(y) = e^{-V_2(y)}dy$, where V_i, $i=1,2$ are convex functions. Then $(x,y) \to V_1(x)+V_2(y)$ is convex on $I\!R^n \times I\!R^m$.

Proposition 2. *Let $(\rho_n, n \geq 1)$ be a sequence of log-concave measures converging weakly to ρ, then ρ is also log-concave.*

Proof. Let θ be a log-concave function as described above, then the density of $\theta \star \rho$ is the limit of the densities of $(\theta \star \rho_n)$.
The following theorem is the first pavement to extend the definition of log-concavity to the infinite dimensional case where the Lebesgue measure does not exist:

Theorem 2. *ρ is a log-concave measure if and only if it satisfies the Prékopa-Leindler property.*

Proof. Suppose that ρ has a continuous density F with respect to the Lebesgue measure of $I\!R^d$. Let $f = 1_A$, $g = 1_B$, where A and B are two balls whose centers are located at a and b respectively and whose diameter will tend to zero. Let $C = \frac{1}{2}(A+B)$ and $h = 1_C$ then $h(\alpha x + \beta y) \geq f(x)^\alpha g(y)^\beta$ for any $\alpha + \beta = 1$, $x,y \in I\!R^d$. Hence for $\alpha = \frac{1}{2}$, by taking the limit, we obtain

$$F(\frac{1}{2}(a+b)) \geq F(a)^{\frac{1}{2}} F(b)^{\frac{1}{2}} ,$$

which is a sufficient condition for the log-concavity of F. The general case follows by taking the convolution of ρ. The necessity follows from the transport argument that we have used in the proof of Theorem 1.

Corollary 3. *If a measure satisfies the relation (2) for the continuous functions satisfying the condition (1), then it also satisfies the same relation for Borel functions satisfying (1).*

Proof. If the relation (2) is satisfied by ρ, then it is also satisfied by $\rho \star \theta$ where θ is a log-concave function, since $\rho(\theta \star a) = (\rho \star \theta)(a)$.

Corollary 4. *Let ρ be a log-concave measure and let F be a convex function, then the measure ν defined as*

$$d\nu(x) = e^{-F(x)}d\rho(x)$$

is again log-concave.

Proof. The new measure obviously satisfies the Prékopa-Leindler property.
A very close characterization of the log-concave measures can be given by the Brunn-Minkowski inequality whose proof is similar to that of Theorem 2

Theorem 3. *The measure ρ is log-concave if and only if*

$$\rho(sA+tB) \geq \rho(A)^s \rho(B)^t$$

for any measurable A, B and $s+t=1$.

Definition 3. *A (positive) measure ρ on $\mathbb{R}^n$ is called α-super log-concave (α-s.l.c in short) if the measure*

$$e^{\frac{\alpha}{2}|x|^2}\rho(dx)$$

is a log-concave measure, where $\alpha \geq 0$ and $|\cdot|$ denotes the Euclidean norm.

Remark. Since $\exp -\frac{\alpha}{2}|x|^2$ is a log-concave function, any α-s.l.c. measure is log-concave.

Proposition 3. *Assume that a measure ρ can be represented as*

$$d\rho(x) = e^{-V(x)}dx$$

where V is a locally integrable, lower bounded function such that

$$\nabla^2 V \geq \alpha I_{\mathbb{R}^n}$$

in the sense of distributions, then ρ is α-s.l.c.

Proof. Evidently the condition implies the convexity of the function $x \to V(x) - \frac{\alpha}{2}|x|^2$.
Remark. If ρ_i are finitely many α_i-s.l.c. measures on $\mathbb{R}^{n_i}$, then their product is an $\min_i \alpha_i$-s.l.c. measure.
The proof of the following lemma follows from that of Theorem 2:

Lemma 1. *A measure ρ is an α-s.l.c. if and only if for any a, b, c continuous, positive functions of compact support such that, for any $x, y \in \mathbb{R}^n$, $s+t=1$, $a(sx+ty) \geq b(x)^s c(y)^t$, one has*

$$\rho(a_\alpha) \geq \rho(b_\alpha)^s \rho(c_\alpha)^t ,$$

where, for a given function f, f_α is defined as

$$f_\alpha(x) = \exp(\frac{\alpha}{2}|x|^2)f(x) .$$

The proof of the following is obvious:

Lemma 2. *Suppose that $(\rho_n, n \geq 1)$ is a sequence of measures converging weakly to ρ. Assume that ρ_n is α_n-s.l.c. for any $n \geq 1$, then ρ is $\alpha_0 = \inf_n \alpha_n$-s.l.c.*

Lemma 3. *Assume that $d\rho$ is an α-s.l.c. measure and denote by p_σ the Gaussian density $\exp -\frac{1}{2\sigma}|x|^2$. Then, for any $\delta > 0$ satisfying*

$$\frac{1}{\delta} - \frac{1}{\alpha} > \sigma ,$$

the measure $\rho \star p_\sigma$ is δ-s.l.c.

Proof. We want to determine the set of δ's for which the function

$$x \to e^{\frac{\delta}{2}|x|^2} (\rho \star p_\sigma)(x)$$

is log-concave. Let us denote by ρ_α the measure defined by

$$d\rho_\alpha(y) = \exp \frac{\alpha}{2}|y|^2 d\rho(y) .$$

We can write

$$
\begin{aligned}
e^{\frac{\delta}{2}|x|^2}(\rho\star p_\sigma)(x) &= \int e^{\frac{\delta}{2}|x|^2}p_\sigma(x-y)e^{-\frac{\alpha}{2}|y|^2}\rho_\alpha(dy) = \\
&= \int \exp\left[-\frac{1}{2}\left|(\frac{1}{\sigma}-\delta)^{1/2}x-\frac{y}{\sigma(\frac{1}{\sigma}-\delta)^{1/2}}\right|^2-\frac{|y|^2}{2}\left(\frac{\alpha-\delta-\delta\alpha\sigma}{1-\delta\sigma}\right)\right]\rho_\alpha(dy)\,.
\end{aligned}
$$

Let

$$p_{\sigma,\alpha}(y)=\exp\left[-\frac{|y|^2}{2}(\frac{\alpha-\delta-\delta\alpha\sigma}{1-\delta\sigma})\right]$$

which is a Gaussian kernel provided that

$$\frac{1}{\delta}-\frac{1}{\alpha}>\sigma$$

and

$$x\to e^{\frac{\delta}{2}|x|^2}(\rho\star p_\sigma)(x)$$

is a log-concave function from Corollary 2.

Lemma 4. *If ρ is an α-s.l.c. measure on $I\!R^m$, and if $F: I\!R^m\to I\!R^n$, $m\geq n$, is a linear map, then $F(\rho)=\rho_F$ is an α-s.l.c. measure.*

Proof. Assume first that ρ has a density w.r. to the Lebesgue measure l. We can write $I\!R^m=\mathrm{Im}(F)+\ker(F)$, then

$$\rho_F(f)=\int_{\mathrm{Im}(F)}f(y)\left(\int_{\ker(F)}l(y+y^\perp)dy^\perp\right)dy\,.$$

Since, by Corollary 1,

$$y\to\int_{\mathrm{Im}(F)}\exp\frac{\alpha}{2}(|y|^2+|y^\perp|^2)l(y+y^\perp)dy^\perp$$

is log-concave,

$$y\to\exp\frac{\alpha}{2}|y|^2\int_{\mathrm{Im}(F)}l(y+y^\perp)dy^\perp$$

is also log-concave.

Theorem 4. *Assume that ρ is an α-s.l.c. measure on $I\!R^n$, then it satisfies the logarithmic Sobolev inequality:*

$$\rho(f^2\log f^2)\leq\frac{2}{\alpha}\rho(|\nabla f|^2)$$

for any smooth function f with $\rho(f^2)=1$.

Proof. Assume first that $d\rho(x)=\rho'(x)dx$, denote by μ_α the Gauss measure with covariance $\sqrt{1/\alpha}I_{I\!R^n}$. Then $\rho\ll\mu_\alpha$ with

$$\frac{d\rho}{d\mu_\alpha}(x)=e^{\frac{\alpha}{2}|x|^2}\rho'(x)\,.$$

By the hypothesis, this Radon-Nikodym derivative is log-concave, consequently from a theorem of Caffarelli (cf. [4] and [7]), there exists a 1-Lipschitz map $T=I_{I\!R^n}+\nabla\varphi$ such that $\rho=T(\mu_\alpha)$. Consequently, applying the logarithmic Sobolev inequality for the Gaussian measure (cf.[8])

$$
\begin{aligned}
\rho(f^2\log f^2) &= \mu_\alpha(f^2\circ T\log f^2\circ T)\leq \\
&\leq \frac{2}{\alpha}\mu_\alpha(|\nabla f\circ T|^2\|I+\nabla^2\varphi\|^2)\leq \\
&\leq \frac{2}{\alpha}\mu_\alpha(|\nabla f\circ T|^2)= \\
&= \frac{2}{\alpha}\rho(|\nabla f|^2)\,.
\end{aligned}
$$

The general case now follows from Lemma 3 and a limit procedure.

4. Infinite dimensional case

The following is basic:

Theorem 5. *Let E be a separable Fréchet space and let ρ be a probability on $(E,\mathcal{E})$, where $\mathcal{E}$ denotes the Borel sigma algebra of E. Assume that the finite dimensional projections of ρ are log-concave. Assume that f,g,h are positive Borel functions satisfying*

$$h(su+tv) \geq f(u)^s g(v)^t$$

for any $u,v \in E$ and $s+t=1$. Then ρ satisfies the Prékopa-Leindler property:

$$\rho(h) \geq \rho(f)^s \rho(g)^t \,.$$

Proof. We can suppose that ρ has convex, compact support K and replace E by $I\!R^{\mathcal{N}}$ by injection. Then we can replace K by a product of compact intervals $J=\prod_1^\infty J_n$. If f and g are continuous, cylindrical functions on J, for $x \in J$, define

$$k(x) = \sup\left(f(u)^s g(v)^t : x = su+tv,\, u,v \in J\right) \,.$$

The function k is then continuous on J and we have, from the finite dimensional case,

$$\rho(k) \geq \rho(f)^s \rho(g)^t \,.$$

If f and g are upper semi-continuous on J, there exist two sequences of continuous and cylindrical functions (f_n) and (g_n), decreasing to f and g respectively. Hence, $(k_n,\, n \geq 1)$, where k_n is defined as above, converges to k as defined above and we again have

$$\rho(k) \geq \rho(f)^s \rho(g)^t \,.$$

Finally, if f and g are only Borel measurable, there exist two monotone, increasing sequences (f_n), (g_n) whose elements are upper semi-continuous such that $\lim_n \rho(f_n) = \rho(f)$ and $\lim_n \rho(g_n) = \rho(g)$. Since we have $h \geq k_n$, it follows that

$$\begin{aligned} \rho(h) &\geq \sup_n \rho(k_n) \geq \sup_n \rho(f_n)^s \rho(g_n)^t = \\ &= \rho(f)^s \rho(g)^t \,. \end{aligned}$$

The proof of the above theorem contains also the proof of the following

Lemma 5. *In order the Prékopa-Leindler to hold it is necessary and sufficient that it holds only for the continuous functions f,g,h such that $h(sx+ty) \geq f(x)^s g(y)^t$ for any $x,y \in E$ and $s+t=1$.*

The proof of the following theorem is quite similar to the proof of Theorem 5:

Theorem 6. *Assume that E and F are two separable Fréchet spaces with ρ and ν satisfying the Prékopa-Leindler property on E and F respectively. Then the product measure $\rho \otimes \nu$ satisfies also the Prékopa-Leindler property on $E \times F$.*

The following definition is now justified:

Definition 4. *A Radon measure on a locally convex space E is called log-concave if it satisfies the Prékopa-Leindler property.*

The following result is immediate:

Proposition 4. *The image of a log-concave measure under any linear, continuous map is again log-concave.*

From Lemma 5 we get at once

Corollary 5. *Let $(\rho_n, n \geq 1)$ be a sequence of log-concave measures converging weakly to a measure ρ, then ρ is also log-concave.*

Corollary 6. *Let ρ be a bounded measure on $I\!R^{\mathcal{N}}$, let us denote by $(\pi_n, n \geq 1)$ the canonical finite dimensional projections. The measures $(\pi_n(\rho), n \geq 1)$ are log-concave if and only if ρ is log-concave.*

Proof. We can write ρ as the weak limit of the sequence of measures $(\pi_n(\rho) \otimes \delta^n, n \geq 1)$ where δ^n is the image under $I - \pi_n$ of the Dirac δ_0 measure on $I\!R^{\mathcal{N}}$. Assume now that E is a separable Fréchet space and assume that H is a separable Hilbert space continuously and densely injected into E. We identify H with its continuous dual, hence $E^\star \subset H \subset E$. Choose a sequence $(\tilde{e}_i, i \geq 1)$ from $E^\star$ such that its image under the injection $E^\star \hookrightarrow H$, denoted as $(e_i, i \geq 1)$ is a complete, orthonormal base of H. Define π_n on E as

$$\pi_n(x) = \sum_{i \leq n} \langle x, \tilde{e}_i \rangle e_i \,.$$

The typical examples for this situation is the case of the Wiener space for E and the Cameron-Martin space for H or $E = I\!R^{\mathcal{N}}$, $H = l^2$.

Definition 5. *A Radon measure ρ on E is called α-s.l.c.*

1. *if $\lim_n \pi_n(x) = x$ ρ-almost everywhere,*
2. *if $\pi_n(\rho)$ is α-s.l.c. on the Euclidean space spanned by $\{e_1, \ldots, e_n\}$, for any $n \geq 1$.*

We have the following result which is the immediate consequence of the finite dimensional case (cf. Theorem 4):

Theorem 7. *If ρ is an α-s.l.c. measure on E, then it satisfies the logarithmic Sobolev inequality:*

$$\rho(f^2 \log f^2) \leq \frac{2}{\alpha} \rho(|\nabla f|_H^2)$$

for any smooth, cylindrical function f with $\rho(f^2) = 1$.

5. The case of abstract Wiener space

While working in this frame one encounters often the difficulty of defining a proper Jacobian due to the lack of regularity of the corresponding transformation. This happens especially in the case of the measure transportation theory. Consequently it is reasonable to push the ways to extend as much as possible the notion of Jacobian of a transformation with unsufficient regularity.

5.1. **Sub-jacobians for monotone transformations.** Let (W, H, μ) be an abstract Wiener space, we say that a map $U = I_W + u$, where $u : W \to H$ is a measurable map is monotone or a monotone shift, if $h \to (h + u(w + h), h)_H \geq 0$ μ-almost surely.

Lemma 6. *Assume that $U = I_W + u$ is a monotone shift with $u \in I\!D_{p,1}(H)$, $p > 1$. Then*

$$E[f \circ U \, \Lambda(U)] \leq E[f] \,,$$

for any positive $f \in C_b(W)$, where

$$\Lambda(U) = \det{}_2(I + \nabla u) \exp\left(-\delta u - \frac{1}{2}|u|_H^2\right)$$

and $\det_2(I_H + \nabla u)$ denotes the modified Carleman-Fredholm determinant.

Proof. The necessary background about the subject and the proof follows from [15], Theorem 6.3.1.

Remark. If A is a nuclear operator on a Hilbert space, then $\det_2(I_H + A)$ is defined as

$$\det{}_2(I_H + A) = \det(I_H + A) \exp - \text{ trace } (A) \,,$$

and this function has an analytic extension to the space of Hilbert-Schmidt operators, consequently, the log-concavity of the ordinary determinant implies the log-concavity of the map $A \to \det_2(I_H + A)$.

Lemma 7. *Assume that $U = I_W + u$ is a monotone shift and $u \in \mathbb{D}_{p,0}(H) = L^p(\mu, H)$ for some $p > 1$. Then there exists some $\Lambda(u) \geq 0$ a.s., $E[\Lambda(u)] \leq 1$ and*

$$E[f \circ U\, \Lambda(u)] \leq E[f]$$

for any positive and measurable f. In particular, $U\mu$ is absolutely continuous w.r.to μ on the set on which $\Lambda(u) > 0$.

Proof. Let $(P_t, t \geq 0)$ be the Ornstein-Uhlenbeck semigroup, let $U_n = I_W + u_n$, with $u_n = P_{1/n} u$. Then apply Lemma 6 to U_n. Define

$$\Lambda(U) = \liminf_n \Lambda(U_n)\,.$$

$\Lambda(U)$ does exist and, from the Fatou lemma, it is integrable and satisfies the claim.

Corollary 7. *The map $U \to \Lambda(U)$ is almost surely log-concave on the set of monotone shifts.*

Proof. If $U_i = I_W + u_i$, with $u_i \in L^p(\mu, H)$, $i = 1, 2$ are two monotone shifts, define $u_i^n = P_{1/n} u_i$, $U_i^n = I_W + u_i^n$, $i = 1, 2$. Then, by the log-concavity of the Carleman-Fredholm determinant, for $s, t \in [0, 1]$ with $s + t = 1$, we have

$$\Lambda(sU_1^n + tU_2^n) \geq \Lambda(U_1^n)^s\, \Lambda(U_2^n)^t$$

a.s. If we take the $\liminf$ of both sides as $n \to \infty$, the inequality is preserved.

Proposition 5. *Assume that $U_n = I_W + u_n$, $n \geq 1$, is a sequence of shifts, where $u_n \in \mathbb{D}_{p,1}(H)$, for some $p > 1$. Assume that $U_n\mu \ll \mu$ for all $n \geq 1$ and denote*

$$L_n = \frac{dU_n\mu}{d\mu}\,.$$

Assume further that $L_n \circ U_n\, \Lambda(U_n) = 1$ a.s. for all $n \geq 1$, also that $L_n \to L$ in $L^1(\mu)$ and finally that $U_n \to U = I_W + u$ in $L^0(\mu, W)$. Then $L_n \circ U_n \to L \circ U$, $\Lambda(U_n) \to \Lambda(U)$ in $L^0(\mu)$ and we have

$$L \circ U\, \Lambda(U) = 1$$

a.s. Besides, under the additional hypothesis:

$$\sup_n E[\log^+ L_n] < \infty\,,$$

the sequence $(\Lambda(U_n), n \geq 1)$ is uniformly integrable, hence it converges to $\Lambda(U)$ also in $L^1(\mu)$ and we have

$$E[f \circ U\, \Lambda(U)] = E[f]\,,$$

for any $f \in C_b(W)$.

Proof. Since $L_n \to L$ in $L^1(\mu)$ and $U_n \to U$ in probability, it follows from the Lusin theorem that $L_n \circ U_n \to L \circ U$ in probability, hence $(\Lambda(U_n),\, n \geq 1)$ converges in probability and hence

$$\lim \Lambda(U_n) = \liminf_n \Lambda(U_n) = \Lambda(U)\,.$$

The rest is obvious, since the last hypothesis implies precisely the uniform integrability of the sequence $(\Lambda(U_n),\, n \geq 1)$.

Remark. Proposition 5 is astonishing in the sense that we do not make any assumption about the convergence of the H-valued parts of the shifts at all. Assume now that $u_n \to u$ in $L^0(\mu, H)$, then, although δu either ∇u do not exist, the sequence

$$(\det_2(I + \nabla u_n) e^{-\delta u_n},\, n \geq 1)$$

converges in probability to a non-trivial limit that we denote by $J(U)$, hence $\Lambda(U)$ can be represented as

$$\Lambda(U) = J(U) \exp -\frac{1}{2}|u|^2 .$$

5.2. **The transport case and applications.** Assume that $L \in L \log L(\mu)$, $E[L] = 1$. Let $T = I_W + \nabla\phi$, with ϕ in $\mathbb{D}_{2,1}$, be the transport map which maps $d\mu$ to $Ld\mu$ whose properties are proved in [6]. Define $L_n = E[P_{1/n}L|V_n]$, where V_n is the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$, $(e_i, i \geq) \subset W^\star$ being an orthonormal basis of H. Let $T_n = I_W + \nabla\phi_n$, $\phi_n \in \mathbb{D}_{2,1}$ be the transport map which maps $d\mu$ to $L_n d\mu$. Recall that ϕ_n is 1-convex, since it is V_n-measurable, $\nabla^2\phi$ and $\mathcal{L}\phi$ are lower bounded distributions, hence they are measures. We have

$$E[f \circ T_n] = E[f\, L_n]$$

for any $f \in C_b(W)$. Besides, since $L_n > 0$ a.s., we have

$$L_n \circ T_n\, \Lambda(T_n) = 1$$

a.s., where

$$\Lambda(T_n) = \det\nolimits_2(I + \nabla_a^2\phi_n) \exp\left(-\mathcal{L}_a\phi_n - \frac{1}{2}|\nabla\phi_n|^2\right) ,$$

where $\nabla_a^2\phi_n$ and $\mathcal{L}_a\phi_n$ denote respectively the absolutely continuous parts of the measures $\nabla^2\phi_n$ and $\mathcal{L}\phi_n$ and $\det_2$ denotes, as usual, the modified Carleman-Fredholm determinant (c.f.[15] for further information). It follows from [6] that $\phi_n \to \phi$ in $\mathbb{D}_{2,1}$, $(L_n, n \geq 1)$ being uniformly integrable, $L_n \circ T_n \to L \circ T$ in probability, hence $\Lambda(T_n) \to \Lambda(T)$ also in probability, where $\Lambda(T)$ can be represented as

$$\Lambda(T) = J(T) \exp -\frac{1}{2}|\nabla\phi|^2 .$$

In [6], we have shown that the sequence $(\mathcal{L}_a\phi_n, n \geq 1)$ is a submartingale with respect to the increasing sequence of sigma algebras $(V_n, n \geq 1)$ and the inequality $\mathcal{L}_a\phi_n \leq -\log\Lambda_n$ implies

$$(\mathcal{L}_a\phi_n)^+ \leq (-\log\Lambda(T_n))^+ .$$

Consequently

$$\begin{aligned} E[(\mathcal{L}_a\phi_n)^+] &\leq E[(-\log\Lambda(T_n))^+] = E[(\log L_n \circ T_n)^+] = \\ &= E[L_n \log^+ L_n] \leq 2e^{-1} + E[L_n \log L_n] \end{aligned}$$

and Jensen inequality implies that

$$\sup_n E[(\mathcal{L}_a\phi_n)^+] < \infty ,$$

which is a sufficient condition for the almost everywhere convergence of the submartingale $(\mathcal{L}_a\phi_n, n \geq 1)$ whose limit we denote as $\mathcal{L}(\phi) \in L^1(\mu)$. Note that, as a consequence of this observation, combined with the convergence of $(\Lambda(T_n), n \geq 1)$, we deduce also the convergence of $(\det_2(I_H + \nabla_a^2\phi_n), n \geq 1)$ in probability. Moreover, we have also

$$E[f \circ T\, \Lambda(T)] \leq E[f] ,$$

for any measurable, positive f and moreover, this inequality becomes an equality if $\log L$ is integrable.

Proposition 6. *Let L_1, L_2 be as above, denote by T_1, T_2 the corresponding transport maps and define $M = aT_1 + bT_2$, where $a + b = 1, a \geq 0$. Define $M_n = aT_1^n + bT_2^n$ as in Corollary 7 and define finally $\Lambda(M) = \liminf_n \Lambda(M_n)$. We then have $M\mu$ is absolutely continuous w.r.to μ and*

$$\Lambda(M) \geq \Lambda(T_1)^a\, \Lambda(T_2)^b$$

a.s.

Proof. We have

$$\begin{aligned}\Lambda(M) &= \liminf \Lambda(M_n) \geq \\ &\geq \liminf \Lambda(T_1^n)^a \Lambda(T_2^n)^b \geq \\ &\geq \liminf \Lambda(T_1^n)^a \liminf \Lambda(T_2^n)^b \geq \\ &= \Lambda(T_1)^a \Lambda(T_2)^b\end{aligned}$$

a.s. Hence $\Lambda(M) > 0$ a.s. This result combined with the following consequence of the Fatou lemma

$$\int f \circ M \Lambda(M) d\mu \leq \int f d\mu \,,$$

for any $0 \leq f \in C_b(W)$, implies the absolute continuity $M\mu \ll \mu$.

Theorem 8. *Assume that a, b and c are measurable, positive functions on W such that, for given $s, t \in [0,1]$ with $s+t=1$ and for any $h, k \in H$, we have*

$$a(w+sh+tk) \exp\left[-\frac{1}{2}|sh+tk|_H^2\right] \geq \left(b(w+h)\exp -\frac{1}{2}|h|_H^2\right)^s \left(c(w+k)\exp -\frac{1}{2}|k|_H^2\right)^t$$

almost surely. Let also q be any H-logconcave density and denote by ν the measure $d\nu = q d\mu$. Then we have

$$\int a\, d\nu \geq \left(\int b\, d\nu\right)^s \left(\int c\, d\nu\right)^t .$$

Proof. First we shall prove the case $q = 1$, then the general case can be reduced to this particular case by replacing a, b and c by $a\,q$, $b\,q$ and by $c\,q$ respectively. Moreover, by replacing a, b, c by $a \wedge n$, $b \wedge n$ and $c \wedge n$ we may suppose that they are bounded. Finally, by multiplying them with adequate constants, we can also suppose that their integrals w.r.to μ are all equal to unity. Let $T_1 = I + \nabla\phi_1$ and $T_2 = I + \nabla\phi_2$ be the transport maps such that $T_1\mu = b \cdot \mu$, $T_2\mu = c \cdot \mu$, where $l \cdot \mu$ denotes the measure with density l. It follows from above explanations, $\Lambda(T_1)$ and $\Lambda(T_2)$ are well-defined and

$$b \circ T_1\, \Lambda(T_1) = c \circ T_2\, \Lambda(T_2) = 1$$

a.s. Let $M = sT_1 + tT_2$, then $M\mu \ll \mu$ and as explained above

$$\Lambda(M) \geq \Lambda(T_1)^s\, \Lambda(T_2)^t$$

a.s. Hence

$$\begin{aligned}1 &= (b \circ T_1)^s (c \circ T_2)^t \Lambda(T_1)^s \Lambda(T_2)^t \\ &\leq a \circ (sT_1 + sT_2) \Lambda(T_1)^s \Lambda(T_2)^t \\ &\leq a \circ M\, \Lambda(M) \,.\end{aligned}$$

Therefore

$$1 \leq \int a \circ M\ \Lambda(M) d\mu \leq \int a\ d\mu$$

and this accomplishes the proof.

Remark. Here is another proof of the theorem: let $(\pi_n, n \geq 1)$ be a sequence of orthogonal projections, constructed from the elements of $W^\star$, of the Cameron-Martin space H increasing to identity such that $\lim \pi_n w = w$ μ-a.s. Let $w_n = \pi_n w$ and $w_n^\perp = w - w_n$. For a measurable function f on W, denote the partial map $w_n \to f(w_n + w_n^\perp) \exp -\frac{1}{2}|w_n|^2$ by $f_{w_n^\perp}$. The hypothesis above is equivalent to (cf.[5])

$$a_{w_n^\perp}(sx+ty) \geq b_{w_n^\perp}(x)^s\, c_{w_n^\perp}(y)^t \,.$$

Since $\mu_n = \pi_n\mu$ is log-concave, we have

$$E[a|\pi_n^\perp] \geq E[b|\pi_n^\perp]^s E[c|\pi_n^\perp]^t$$

and the (second) proof follows from the (reverse) martingale convergence theorem.

The following concept has been studied already in [5]:

Definition 6. *A measurable, $\mathbb{R}_+$-valued function on W is called* 1*-log concave if, for any $s+t=1$, $s\geq 0$,*

$$f(w+sh+th')\exp\left(-\frac{1}{2}|sh+th'|_H^2\right)\geq$$
$$\geq\left(f(w+h)\exp-\frac{1}{2}|h|_H^2\right)^s\left(f(w+h')\exp-\frac{1}{2}|h'|_H^2\right)^t$$

almost surely for any $h,h'\in H$.

Corollary 8. *Assume that (W_1,H_1,μ_1) and (W_2,H_2,μ_2) be two abstract Wiener spaces. Assume that $f:W_1\times W_2\to\mathbb{R}_+$ is an* 1*-log concave on the abstract Wiener space $(W_1\times W_2,H_1\times H_2,\mu_1\times\mu_2)$. Then*

$$\hat{f}(x)=\int_{W_2}f(x,y)\,\mu_2(dy)$$

is 1*-log concave on (W_1,H_1,μ_1).*

Proof. Let $h,h'\in H_1$, $s+t=1$. For $(x,y)\in W_1\times W_2$, define

$$a_{x+sh+th'}(y)=f((x,y)+s(h,0)+t(h',0))\exp\left(-\frac{1}{2}\,|sh+th'|_{H_1}^2\right).$$

For any $k,k'\in H_2$, we have

$$a_{x+sh+th'}(y+sk+tk')\exp\left(-\frac{1}{2}\,|sk+tk'|_{H_2}^2\right)\geq$$
$$\geq\quad a_{x+h}(y+k)^s\exp\left(-\tfrac{s}{2}|k|_{H_2}^2\right)\,a_{x+h'}(y+k')^t\left(\exp-\tfrac{t}{2}|k'|_{H_2}^2\right).$$

Applying Theorem 8 to $a_{x+sh+th'}$, a_{x+h} and $a_{x+h'}$, we get

$$\hat{f}(x+sh+th')\exp\left(-\frac{1}{2}\,|sh+th'|_{H_1}^2\right)=\int_{W_2}a_{x+sh+th'}(y)\mu_2(dy)\geq$$
$$\geq\left(\int_{W_2}a_{x+h}(y)\mu_2(dy)\right)^s\left(\int_{W_2}a_{x+h'}(y)\mu_2(dy)\right)^t=$$
$$=\hat{f}(x+h)^s\hat{f}(x+h')^t\exp-\frac{s}{2}|h|_{H_1}^2\exp-\frac{t}{2}|h'|_{H_1}^2\,.$$

Corollary 9. *Assume B,C are two measurable subsets of W, then, for any measure ν as in Theorem 8, we have*

$$\nu(sB+tC)\geq\nu(B)^s\nu(C)^t\,,$$

where $s+t=1$, $s,\,t\geq 0$.

Proof. Let a, b and c be the indicator functions of the sets $sB+tC$, B and C respectively.
Before further infinite dimensional considerations let us give a lemma which is a version of Theorem 2.1 of [1]

Lemma 8. *Assume that f,f_0,f_1 are positive, bounded, measurable functions on $\mathbb{R}^n$ such that for any Borel sets C_0,C_1, some $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$, we have*

$$\int_C f(x)dx\geq\left(\int_{C_0}f_0(x)dx\right)^\alpha\left(\int_{C_1}f_1(x)dx\right)^\beta,\tag{3}$$

where $C=\alpha C_0+\beta C_1$. Then, for any $x_0,x_1\in\mathbb{R}^n$, we have

$$f(z+\alpha x_0+\beta x_1)\geq f_0(z+x_0)^\alpha\,f_1(z+x_1)^\beta$$

dz-almost surely. In particular, if the above identity holds whenever we replace the Lebesgue integral with the Gaussian integral, then f, f_0 and f_1 satisfy the following identity

$$
\begin{aligned}
& f(z+\alpha x_0+\beta x_1)\exp\frac{1}{2}\,|\alpha x_0+\beta x_1|^2 \geq \\
\geq & \left(f_0(z+x_0)\exp -\frac{1}{2}|x_0|^2\right)^{\alpha}\left(f_1(z+x_1)\exp -\frac{1}{2}|x_1|^2\right)^{\beta}
\end{aligned}
$$

dz-almost surely.

Proof. We may suppose that the functions are of bounded support. Let $C_0 = z + x_0 + \epsilon I_n$, $C_1 = z + x_1 + \epsilon I_n$, where $I_n = [-1/2, 1/2]^n$. Then the l.h.s. of the inequality (3) can be written as

$$\frac{1}{\epsilon^n}\int 1_{\epsilon I_n}(x-(z+\alpha x_0+\beta x_1))f(x)dx,$$

which converges in measure to $f(z+\alpha x_0+\beta x_1)$ as $\epsilon \to 0$. For the terms at the r.h.s. we have similar convergence results in measure. For the Gaussian case, it suffice to replace the functions f, f_0, f_1 with $f\,q$, $f_0\,q$ and $f_1\,q$ respectively, where q denotes the Gaussian density.

Let $(e_n, n \geq 1) \subset W^*$ be a CONB basis of H and denote by V_n the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$ and completed with μ-negligeable sets. We have

Theorem 9. *Assume that a, b and c are measurable, positive functions on W. For given $s, t \in [0,1]$ with $s + t = 1$ and for any $h, k \in H$, we have*

(1) *For any $h, k \in H$, the following inequality holds μ-a.s.*

$$
\begin{aligned}
a(w+sh+tk)\exp -\frac{1}{2}\,|sh+tk|_H^2 \geq & \left(b(w+h)\exp -\frac{1}{2}\,|h|_H^2\right)^s \times \\
& \times\left(c(w+k)\exp -\frac{1}{2}\,|k|_H^2\right)^t
\end{aligned}
\tag{4}
$$

if and only if

$$
\begin{aligned}
& E[a|V_n](w+s\pi_n h+t\pi_n k)\exp -\frac{1}{2}\,|s\pi_n h+t\pi_n k|_H^2 \geq \\
\geq & \left(E[b|V_n](w+\pi_n h)\exp -\frac{1}{2}\,|\pi_n h|_H^2\right)^s \times \\
\times & \left(E[c|V_n](w+\pi_n k)\exp -\frac{1}{2}\,|\pi_n k|_H^2\right)^t,
\end{aligned}
$$

μ-a.s., where π_n denotes the orthogonal projection of H onto the space spanned by $\{e_1, \ldots, e_n\}$.

(2) *Similarly, the relation (1) is equivalent to*

$$
\begin{aligned}
P_\tau a(w+sh+tk)\exp -\frac{1}{2}\,|sh+tk|_H^2 \quad \geq \quad & \left(P_\tau b(w+h)\exp -\frac{1}{2}\,|h|_H^2\right)^s \times \\
& \times\left(P_\tau c(w+k)\exp -\frac{1}{2}\,|k|_H^2\right)^t
\end{aligned}
$$

for any $\tau \geq 0$, where P_τ denotes the Ornstein-Uhlenbeck semigroup.

Proof. We can suppose that a, b, c are bounded. Denote by a_n, b_n, c_n the conditional expectations of a, b, c respectively w.r.to V_n. Let now B and C be V_n-measurable sets, hence $sB + tC$

is also V_n-measurable (recall that V_n is completed!). It then follows from Theorem 8

$$\begin{aligned}\int_{sB+tC} a_n\, d\mu &= \int_{sB+tC} a\, d\mu \geq \\ &\geq \left(\int_B b\, d\mu\right)^s \left(\int_C c\, d\mu\right)^t = \\ &= \left(\int_B b_n\, d\mu\right)^s \left(\int_C c_n\, d\mu\right)^t .\end{aligned}$$

Since this true for any V_n-measurable set, it follows from Lemma 8 that a_n, b_n and c_n satisfy the inequality claimed in the first part of the theorem (with the Gaussian measure). To prove the second part, we can replace a, b, c by a_n, b_n, c_n since P_τ commutes with the conditional expectation w.r.to V_n. Hence the problem is reduced to the finite dimensional case. Let us denote again by the same notation the Ornstein-Uhlenbeck semigroup on $I\!R^n$. Then we can write

$$\int_{sB+tC} P_\tau a_n(x) d\mu(x) = \int_{sB+tC} \int_{I\!R^n} a_n(y) q_\tau(x,y) dy d\mu(x)\,,$$

where

$$q_\tau(x,y) = (2\pi(1-e^{-2\tau}))^{-n/2} \exp -\frac{|y-e^{-\tau}x|^2}{2(1-e^{-2\tau})}\,,$$

which is a log-concave function in two variables. We can also write

$$(sB+tC)\times I\!R^n = s(B\times I\!R^n) + t(C\times I\!R^n)\,.$$

It then follows from the Prékopa-Leindler inequality in $I\!R^n \times I\!R^n$ that

$$\int_{sB+tC} P_\tau a_n d\mu \geq \left(\int_B P_\tau b_n d\mu\right)^s \left(\int_C P_\tau c_n d\mu\right)^t .$$

It follows then from Lemma 8 that we have

$$\begin{aligned}P_\tau a_n(w+sh+tk)\exp -\frac{1}{2}\,|s\pi_n h + t\pi_n k|_H^2 \quad &\geq \quad \left(P_\tau b_n(w+h)\exp -\frac{1}{2}\,|\pi_n h|_H^2\right)^s \times \\ &\times \left(P_\tau c_n(w+k)\exp -\frac{1}{2}\,|\pi_n k|_H^2\right)^t\end{aligned}$$

almost surely and we can pass to the limit as $n\to\infty$ due to the martingale convergence theorem.

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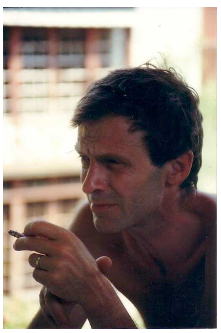

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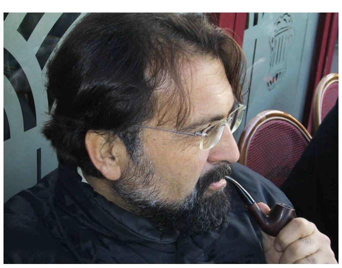

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