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OPTIMAL REAL-TIME CONTROL OF NONDETERMINISTIC MODELS ON IMPERFECT MEASUREMENTS OF INPUT AND OUTPUT SIGNALS

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Abstract. In the paper optimal guaranteed control problems for linear nonstationary dynamical systems under set-membership uncertainties are considered. It is supposed that in the course of control process states of control object are unknown and signals of two measurement devices are only available for use. The first of them implements incomplete and inexact measurements of input signals, the second one makes imperfect measurements of control object states (output signals).By preposterior analysis an optimal output (combined) closable loop is defined. Realization of this loop (forming current values of control actions) is carried out by optimal estimators and optimal regulator. According to the separation principle of control and observation processes, optimal estimators generate in real time estimates of uncertainty using signals of measurement devices. By obtained estimates the optimal regulator produces current values of optimal loop in the same mode. Results are illustrated by examples.

Keywords: optimal observation and control, set-membership uncertainty, preposterior analysis, closable loop, real-time control, width of distribution, preposterior initial and current distributions, positional solution, optimal estimator, optimal regulator.

AMS Subject Classification: 49N05, 93C05.

Introduction

Control is a process in which purposeful control actions to a control object are formed at each current moment depending on available by this moment information about the object behavior and disturbances acting on it. Human deals with control all his conscious life. To ensure constantly desirable course of any processes he either undertakes appropriate actions and makes necessary decisions by himself or charges with part of them to create automatic control systems. He realizes his actions and decisions according to existing situation and changes it in the same rate in which the situation changes, that is in real time mode. With occurrence and rapid progress of computer technology a possibility is appeared to consruct control systems functioning on the real time control principle.

There are two approaches to control: programmed and positional. Programmed control is realized by means of open-loop control systems (Fig. 1). Their control actions $u(t)$, $t \geq 0$, are formed by means of programs composed on a priori information before control process stating and is not corrected in the process course. The positional control $u(\tau, y_\tau(\cdot))$ (τ - current moment, $y_\tau(\cdot)$ - available by moment τ current information, $(\tau, y_\tau(\cdot))$ - position of the problem) is realized by means of closed loops or real-time control. To control by closed-loop principle following loops are used: feedforward (Fig. 2), feedback (Fig. 3), combined (Fig. 4) loops. Feedforward

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loop $u(\tau, y_{w\tau}(\cdot))$ transforms information $y_{w\tau}(\cdot)$ about input signals (disturbances) into control actions, feedback loop $u(\tau, y_{x\tau}(\cdot))$ transforms information $y_{x\tau}(\cdot)$ about output signals. Combined loops $u(\tau, y_\tau(\cdot))$, $y_\tau(\cdot) = (y_{w\tau}(\cdot), y_{x\tau}(\cdot))$, are integration of feedforward and feedback loops. By real-time control the mentioned loops are not created beforehand but their values necessary for control are obtained with help of computer techniques during control process.

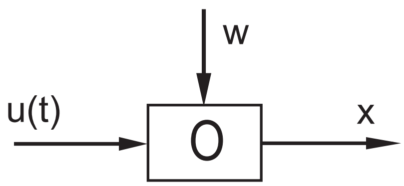


Fig. 1: Open-loop control.

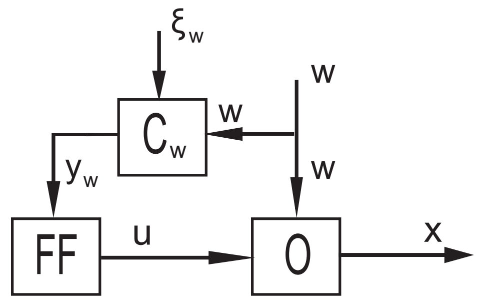


Fig. 2: Feedforward loop.

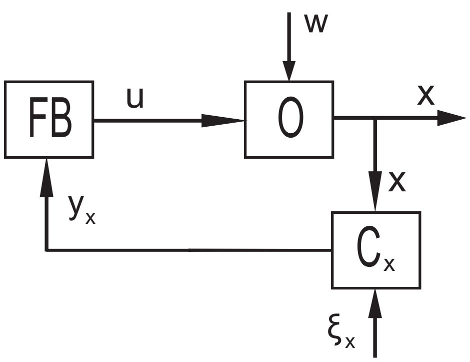


Fig. 3: Feedback loop.

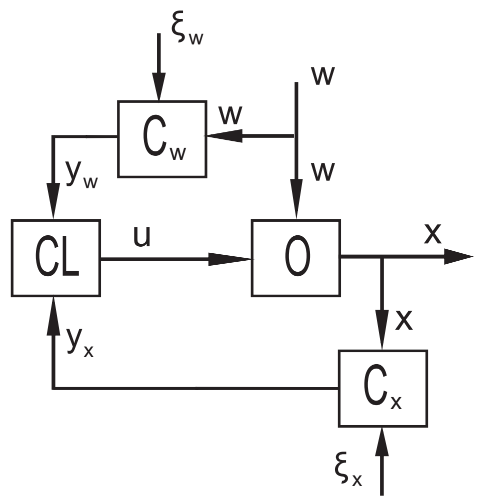


Fig. 4: Combined loop.

In the part of optimal control theory in which direct (geometric) constraints on control actions are considered, the greatest successes are achieved at analysis of optimal programs [6]. But as is known, the construction of positional solutions by classical closed-loop principle encountered serious difficulties [2]. Up to now only individual examples of successful construction of optimal feedback for stationary small order systems are known. In such situation (in the light of rapid computer technology development) attempts of optimal system synthesis according to the real-time control principle are natural and timely.

This work is closely related to researches [1, 3, 4, 5], where some results of authors on optimal real-time control for nonstationary linear nondeterministic models with set-membership uncertainties and inexact measurements of input and output signals are given.

Structure of the paper.

In Section 1 the problem statement is given. A linear nonstationary nondeterministic model of control system is optimized in the class of discrete bounded control actions. It is required to steer the system with guarantee to a terminal set at a fixed time moment and to obtain the maximal guaranteed value of a cost function. In Section 2 an auxiliary optimal observation problem is introduced to construct maximal widthes (or the estimates) of a posteriori distributions of terminal states for the subsystem observing to given directions. On the estimates obtained preposteriori distribution of terminal states is defined. Similarly preposterior analysis that considers current information about the observed subsystem is carried out and notion of a positional solution to the optimal observation problem is introduced. In conclusion the description of work of an optimal estimator which calculates sufficient estimates of the preposterior distribution in real time is described. The results of an initial and current preposterior analysis are illustrated by examples (Section 3). In Section 4 an optimal initial (current) closable program is constructed on the basis of results of preposterior observation. An optimal output closable loop is defined for the problem of optimal control. The Section contains method of constructing quasirealization of optimal loop by an optimal regulator in real-time mode. The article comes to the end (Section 5) with an example.

1. Problem statement

Let $T=[t_*,t^*]$ be a time interval; $T_u=\{t_*,t_*+Let,t^*-h\}$, $h=(t^*-t_*)/N$ (N is a positive integer); $T_w=\{\theta_i^w\in T_u, i=\overline{1,N_w}\}$, $t_*=\theta_0^w<\theta_1^w<...<\theta_{N_w}^w$, is a set of measurement instants of input signals; $T_x=\{\theta_i^x\in T_u, i=\overline{1,N_x}\}$, $t_*=\theta_0^x<\theta_1^x<...<\theta_{N_x}^x$, is a set of measurement instants of output signals; $A(t)\in\mathbb{R}^{n_x\times n_x}$, $A_w(t)\in\mathbb{R}^{n_z\times n_z}$, $B(t)\in\mathbb{R}^{n_x\times r}$, $M(t)\in\mathbb{R}^{n_x\times n_w}$, $M_w(t)\in\mathbb{R}^{n_z\times n_w}$, $t\in T$, are piecewise continuous functions; $C_w(t)\in\mathbb{R}^{q_w\times n_z}$, $C_x(t)\in\mathbb{R}^{q_x\times n_x}$, $t\in T$, are continuous functions; $h_i\in\mathbb{R}^{n_x}$, $h_i'h_i=1$, $i\in I=\overline{1,m}$; $m>n_x$; $H\in\mathbb{R}^{m\times n_x}$ is a matrix with rows h_i, $i\in I$; $c\in\mathbb{R}^{n_x}$; $g_*,g^*\in\mathbb{R}^m$; $u_*,u^*\in\mathbb{R}^r$; $x_0\in\mathbb{R}^{n_x}$; $z_0\in\mathbb{R}^{n_z}$; $U=\{u\in\mathbb{R}^r:\ u_*\le u\le u^*\}$, $\Xi_w=\{\xi\in\mathbb{R}^{q_w}:\xi_{*w}\le\xi\le\xi_w^*\}$, $\Xi_x=\{\xi\in\mathbb{R}^{q_x}:\xi_{*x}\le\xi\le\xi_x^*\}$ are bounded sets; $X^*=\{x\in\mathbb{R}^{n_x}:\ g_{*i}\le h_i'x\le g_i^*,\ i\in I\}$ is a bounded body; $u(\underline{t}:\bar t)=(u(t),\ \underline{t}\le t<\bar t)$.

A function $u(\cdot)=u(t_*:t^*)$ is called *discrete* (with a quantization period h) if

$$u(t)\equiv u(s),\ t\in[s,s+h[,\ s\in T_u.$$

In the class of discrete control actions $u(\cdot)$ we consider the optimal control problem:

$$J(u)=c'x(t^*)\to max; \tag{1}$$

$$\begin{gathered}\dot x=A(t)x+B(t)u+M(t)w(t),\ x(t_*)=x_0;\\ x(t^*)\in X^*;\ u(t)\in U;\end{gathered} \tag{2}$$

$$\begin{gathered}y_w(\theta^w)=C_w(\theta^w)z(\theta^w)+\xi_w(\theta^w),\ \xi_w(\theta^w)\in\Xi_w,\ \theta^w\in T_w;\\ \dot z=A_w(t)z+M_w(t)w(t),\ z(t_*)=z_0,\ t\in T;\end{gathered} \tag{3}$$

$$y_x(\theta_i^x)=\int_{\theta_{i-1}^x}^{\theta_i^x}C_x(v)x(v)dv+\xi_x(\theta_i^x),\ \xi_x(\theta_i^x)\in\Xi_x,\ i=\overline{1,N_x}. \tag{4}$$

Here $x=x(t)\in\mathbb{R}^{n_x}$ is a state of mathematical model of control object (2) at time instant t; $u=u(t)\in\mathbb{R}^r$ is a value of control action; $z=z(t)\in\mathbb{R}^{n_z}$ is a state of mathematical model of a measuring device for input signals (3); $y_w(\theta^w)$, $\theta^w\in T_w$; $y_x(\theta^x)$, $\theta^x\in T_x$, are signals of devices measuring disturbance and state (3), (4); $\xi_w(\theta^w)\in\mathbb{R}^{q_w}$, $\theta^w\in T_w$; $\xi_x(\theta^x)\in\mathbb{R}^{q_x}$, $\theta^x\in T_x$, are unknown errors of measurements (3), (4); $w(t)\in\mathbb{R}^{n_x}$, $t\in T$, is unknown disturbance.

Concerning disturbances $w(t)$, $t\in T$, we assume that it is a finite-parametric function

$$w(t)=L(t)w,\ t\in T, \tag{5}$$

with a given piecewise continuous function $L(t)\in\mathbb{R}^{n_w\times l}$, $t\in T$, and an unknown vector $w\in\mathbb{R}^l$ from a bounded set $W=\{w\in\mathbb{R}^l:\omega_*\le w\le\omega^*\}$.

In general, problem (1)-(5) is to generate in real time bounded discrete control actions $u(t)\in U$, $t\in T$, by inexact and incomplete measurements of measuring devices signals (3), (4). This actions have to steer system (2) to the terminal set X^* at time t^* with guarantee and provide the maximum guaranteed value of the cost function $J(u)$.

Preliminarily we solve an auxiliary problem of optimal observation.

2. Optimal preposterior observation of dynamic systems

2.1. **Initial preposterior distribution.** Let us single out *an observation subsystem* from (1)-(4):

$$\dot{x} = A(t)x + M(t)w(t),\ t \in T;\ x(t_*) = x_0; \tag{6}$$

$$\begin{gathered} y_w(\theta^w) = C_w(\theta^w)z(\theta^w) + \xi_w(\theta^w),\ \xi_w(\theta^w) \in \Xi_w,\ \theta^w \in T_w; \\ \dot{z} = A_w(t)z + M_w(t)w(t),\ z(t_*) = z_0,\ t \in T; \end{gathered} \tag{7}$$

$$y_x(\theta_i^x) = \int_{\theta_{i-1}^x}^{\theta_i^x} C_x(v)x(v)dv + \xi_x(\theta_i^x),\ \xi_x(\theta_i^x) \in \Xi_x,\ i = \overline{1, N_x}. \tag{8}$$

On account of uncertainty of the vector w, the terminal state $x(t^*|w)$ of observation subsystem (6) that plays an important role in control problems, can be established only to within the set

$$X^o_{t^*} = \{x \in \mathbb{R}^{n_x} : x = F(t^*, t_*)x_0 + \int_{t_*}^{t^*} F(t^*, t)M(t)L(t)dtw,\ w \in W\}. \tag{9}$$

Here $F(t,\tau) = F(t)F^{-1}(\tau)$; $F(t) \in \mathbb{R}^{n_x \times n_x}$, $t \in T$; $\dot{F} = A(t)F$, $F(t_*) = E$. Set (9) is called *a priori distribution of the terminal state of the observation subsystem.*

The set $X^o_{t^*}$ is determined by the a priori information about mathematical model (6) of the observation object without regard for a priori information about measuring devices (7), (8). Now we take into consideration a priori information about the whole observation subsystem (6)-(8). To do so, preposterior analysis are supposed to be used before the observation process starting.

We inroduce *the set of (virtual) closing instants* $T_{cl} = T_{clw} \cup T_{clx}$, where $T_{clw} \subseteq T_w$, $T_{clx} \subseteq T_x$. If $T_{clw} = T_w$ and $T_{clx} = T_x$, then we deal with the *full preposterior analysis*, otherwise with *partial* one.

By taking arbitrary $\tilde{w} \in W$, $\tilde{\xi}_w(\theta^w) \in \Xi_w$, where $\theta^w \in T_{clw}$; $\tilde{\xi}_x(\theta^x) \in \Xi_x$, $\theta^x \in T_{clx}$, we simulate a virtual transition process $\tilde{x}(t)$, $t \in T$, in the observation subsystem and by the virtual signals $\tilde{y}(\cdot) = \{\tilde{y}_w(\theta^w),\ \theta^w \in T_{clw};\ \tilde{y}_x(\theta^x),\ \theta^x \in T_{clx}\}$ determine *a posteriori distribution* [4] of the terminal state $X^o_{t^*}(\tilde{y}(\cdot))$ as a set of those and only those terminal states of the subsystem $x \in X^o_{t^*}$ such that together with some vector $w \in W$ and measurement errors $\xi_w(\theta^w) \in \Xi_w$, $\theta^w \in T_{clw}$; $\xi_x(\theta^x) \in \Xi_x$, $\theta^x \in T_{clx}$, they can generate $\tilde{y}(\cdot)$.

As the numerical characterisic (*estimate*) of the set $X^o_{t^*}(\tilde{y}(\cdot))$ *along a given direction* q we take the value (*the set width*):

$$d(\tilde{y}(\cdot)|q) = \max_{x \in X^o_{t^*}(\tilde{y}(\cdot))} q'x - \min_{x \in X^o_{t^*}(\tilde{y}(\cdot))} q'x = \max_{\bar{x}, \underline{x} \in X^o_{t^*}(\tilde{y}(\cdot))} q'(\bar{x} - \underline{x}).$$

Definition 1. Let $\tilde{Y}$ be the set of all possible virtual signals $\tilde{y}(\cdot)$. The number

$$d^0(q) = \max d(\tilde{y}(\cdot)|q),\ \tilde{y}(\cdot) \in \tilde{Y}, \tag{10}$$

is called the *maximal width or initial preposterior estimate* of sets $X^o_{t^*}(\tilde{y}(\cdot))$, $\tilde{y}(\cdot) \in \tilde{Y}$, along the direction q.

Definition 2. Calculation of estimate (10) is called the *optimal initial preposterior observation problem (for direction q)*, its result - *initial prepoterior solution (for direction q)*.

Let Q be a finite totality of identity n_x-vectors (directions), in which each set from n_x vectors are linear-independent.

Definition 3. The set

$$\mathcal{X}^o_{t^*} = \{x \in \mathbb{R}^{n_x} :\ -d^0(q)/2 \leq q'x \leq d^0(q)/2,\ q \in Q\}$$

is said to be an *initial preposterior distribution (on totality Q) at instant t^**. For $|Q| \leq n_x$ each of the sets $X^o_{t^*}(\tilde{y}(\cdot))$, $\tilde{y}(\cdot) \in \tilde{Y}$, can be placed in $\mathcal{X}^o_{t^*}$.

Let us represent problem (10) in analytical form. According to (6)-(8) we have got

$$\begin{array}{l} y_w(\theta^w) = C_w(\theta^w)F_w(\theta^w,t_*)z_0 + \int_{t_*}^{\theta^w} C_w(\theta^w)F_w(\theta^w,t)M_w(t)L(t)dtw + \xi_w(\theta^w), \\ \theta^w \in T_{clw}; \\ y_x(\theta^x_i) = \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v,t_*)x_0dv + \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)\int_{t_*}^{v} F(v,t)M(t)L(t)dtdvw + \xi_x(\theta^x_i), \\ \theta^x_i \in T_{clx}; \end{array} \tag{11}$$

($F_w(t,\tau) = F_w(t)F_w^{-1}(\tau)$; $F_w(t) \in \mathbb{R}^{n_z \times n_z}$, $t \in T: \dot{F}_w = A_w(t)F_w$, $F_w(t_*) = E$).

It follows from th definition of a posteriori distribution of the terminal state that the set $X^o_{t^*}(\tilde{y}(\cdot))$ consists of all $x \in \mathbb{R}^{n_x}$ such that they satisfy:

$$\begin{cases} x = F(t^*,t_*)x_0 + \int_{t_*}^{t^*} F(t^*,t)M(t)L(t)dtw, \\ \xi_{*w} \leq D_w(\theta^w)w - C_w(\theta^w)F_w(\theta^w,t_*)z_0 + \tilde{y}_w(\theta^w) \leq \xi^*_w, \ \theta^w \in T_{clw}; \\ \xi_{*x} \leq D_x(\theta^x_i)w - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v,t_*)x_0dv + \tilde{y}_x(\theta^x_i) \leq \xi^*_x, \ \theta^x_i \in T_{clx}; \\ \omega_* \leq w \leq \omega^*, \end{cases}$$

where

$$\begin{array}{l} D_w(\theta^w) = -\int_{t_*}^{\theta^w} C_w(\theta^w)F_w(\theta^w,t)M_w(t)L(t)dt; \\ D_x(\theta^x_i) = -\int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)\int_{t_*}^{v} F(v,t)M(t)L(t)dtdv. \end{array} \tag{12}$$

Denote

$$q^{x\prime} = q' \int_{t_*}^{t^*} F(t^*,t)M(t)L(t)dt. \tag{13}$$

Using relations (11), we conclude that optimal initial preposterior observation problem (10) is a linear programming problem

$$\begin{cases} d^0(q) = \max\limits_{\bar{w},\underline{w},\tilde{w},\tilde{\xi}_w(\cdot),\tilde{\xi}_x(\cdot)} q^{x\prime}(\bar{w} - \underline{w}), \\ \xi_{*w} \leq D_w(\theta^w)(\bar{w} - \tilde{w}) + \tilde{\xi}_w(\theta^w) \leq \xi^*_w, \\ \xi_{*w} \leq D_w(\theta^w)(\underline{w} - \tilde{w}) + \tilde{\xi}_w(\theta^w) \leq \xi^*_w, \ \theta^w \in T_{clw}; \\ \xi_{*x} \leq D_x(\theta^x)(\bar{w} - \tilde{w}) + \tilde{\xi}_x(\theta^x) \leq \xi^*_x, \\ \xi_{*x} \leq D_x(\theta^x)(\underline{w} - \tilde{w}) + \tilde{\xi}_x(\theta^x) \leq \xi^*_x, \ \theta^x \in T_{clx}; \\ \omega_* \leq \bar{w} \leq \omega^*, \ \omega_* \leq \underline{w} \leq \omega^*, \ \omega_* \leq \tilde{w} \leq \omega^*; \\ \xi_{*w} \leq \tilde{\xi}_w(\theta^w) \leq \xi^*_w, \ \theta^w \in T_{clw}; \ \xi_{*x} \leq \tilde{\xi}_x(\theta^x) \leq \xi^*_x, \ \theta^x \in T_{clx}. \end{cases} \tag{14}$$

Here $\tilde{\xi}_w(\cdot) = \{\tilde{\xi}_w(\theta^w),\ \theta^w \in T_{clw}\}$, $\tilde{\xi}_x(\cdot) = \{\tilde{\xi}_x(\theta^x),\ \theta^x \in T_{clx}\}$. Since preposterior analysis is carried out before the observation process, the time of solving problem (14) does not matter to control.

2.2. **Current preposterior distribution.** Let us carry out preposterior analysis for current instant $\tau \in T_w \cup T_x$ of observation process, assuming that observaion has been accomplished during the interval $T_{+\tau} = [t_*, \tau]$ and by logged signals $y^*_\tau(\cdot) = \{y^*_w(\theta^w),\ \theta^w \in T_w \cap T_{+\tau};\ y^*_x(\theta^x),\ \theta^x \in T_x \cap T_{+\tau}\}$ the *current distribution* $W(\tau, y^*_\tau(\cdot))$ of vector w, corresponding to the *position* $(\tau, y^*_\tau(\cdot))$ has been determined. It consists of those and only those $w \in W$ that together with some measurement errors $\xi_w(\theta^w) \in \Xi_w$, $\theta^w \in T_w \cap T_{+\tau}$; $\xi_x(\theta^x) \in \Xi_x$, $\theta^x \in T_x \cap T_{+\tau}$, are capable to generate $y^*_\tau(\cdot)$.

We choose arbirary $\tilde{w} \in W(\tau, y^*_\tau(\cdot))$, $\tilde{\xi}_w(\theta^w) \in \Xi_w$, $\theta^w \in T_{clw} \cap T^{-\tau}$; $\tilde{\xi}_x(\theta^x) \in \Xi_x$, $\theta^x \in T_{clx} \cap T^{-\tau}$; $T^{-\tau} =]\tau, t*]$, and simulate the virtual transition process $\tilde{x}(t)$, $t \in T^{-\tau}$; $\tilde{x}(\tau) = F(\tau,t_*)x_0 + \int_{t_*}^{\tau} F(\tau,t)M(t)L(t)dt\tilde{w}$, at the observation subsystem. By logged $y^*_\tau(\cdot)$ and virtual $\tilde{y}^\tau(\cdot) = \{\tilde{y}_w(\theta^w),\ \theta^w \in T_{clw} \cap T^{-\tau};\ \tilde{y}_x(\theta^x),\ \theta^x \in T_{clx} \cap T^{-\tau}\}$ measurements we determine the a posteriori distribution of terminal state $X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau, y^*_\tau(\cdot))$ for the position $(\tau, y^*_\tau(\cdot))$. The set

$X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau, y^*_\tau(\cdot))$ is composed from all such terminal states of system (6) which together with some $w \in W$ and measurement errors $\xi_w(\theta^w) \in \Xi_w$, $\theta^w \in (T_w \cap T_{+\tau}) \cup (T_{clw} \cap T^{-\tau})$; $\xi_x(\theta^x) \in \Xi_x$, $\theta^x \in (T_x \cap T_{+\tau}) \cup (T_{clx} \cap T^{-\tau})$, can generate $y^*_\tau(\cdot)$, $\tilde{y}^\tau(\cdot)$.

Definition 4. The *current width of set* $X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau, y^*_\tau(\cdot))$ *in the direction* q is called the number

$$d(\tilde{y}^\tau(\cdot)|q,(\tau, y^*_\tau(\cdot))) = \max_{x\in X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau,y^*_\tau(\cdot))} q'x - \min_{x\in X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau,y^*_\tau(\cdot))} q'x = \max_{\bar{x},\underline{x}\in X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau,y^*_\tau(\cdot))} q'(\bar{x}-\underline{x}).$$

Let $\tilde{Y}(\tau, y^*_\tau(\cdot))$ be a set of all possible virtual signals $\tilde{y}^\tau(\cdot)$ for a position $(\tau, y^*_\tau(\cdot))$.

Definition 5. The number

$$d^0(q|\tau, y^*_\tau(\cdot)) = \max d(\tilde{y}^\tau(\cdot)|q,(\tau, y^*_\tau(\cdot))),\ \ \tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot)), \tag{15}$$

will be called the *current maximal width (current preposterior estimate)* of sets $X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau, y^*_\tau(\cdot))$, $\tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot))$ in the direction q or the *current preposterior solution* to the optimal observation problem (preposteriori solution to the oprimal observation problem for the current position).

Definition 6. The set

$$\mathcal{X}^o_{t^*}(\tau, y^*_\tau(\cdot)) = \{x \in \mathbb{R}^{n_x} : -d^0(q|\tau, y^*_\tau(\cdot))/2 \le q'x \le d^0(q|\tau, y^*_\tau(\cdot))/2,\ q \in Q\},$$

is called the *current preposterior distribution at time instant* t^*. For $|Q| \le n_x$ each of sets $X^o_{t^*}(\tilde{y}^\tau(\cdot)|\tau, y^*_\tau(\cdot))$, $\tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot))$, can be placed in $\mathcal{X}^o_{t^*}(\tau, y^*_\tau(\cdot))$.

By analogy with initial problem (10) it is easy to show analytical form of problem (15) is as follows:

$$\begin{cases} d^0(q|\tau, y^*_\tau(\cdot)) = \max\limits_{\bar{w},\underline{w},\tilde{w},\tilde{\xi}^\tau_w(\cdot),\tilde{\xi}^\tau_x(\cdot)} q^{x\prime}(\bar{w}-\underline{w}),\\ \xi_{*w} \le D_w(\theta^w)\bar{w} - C_w(\theta^w)F_w(\theta^w,t_*)z_0 + y^*_w(\theta^w) \le \xi^*_w,\\ \xi_{*w} \le D_w(\theta^w)\underline{w} - C_w(\theta^w)F_w(\theta^w,t_*)z_0 + y^*_w(\theta^w) \le \xi^*_w,\ \ \theta^w \in T_w \cap T_{+\tau};\\ \xi_{*x} \le D_x(\theta^x_i)\bar{w} - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v,t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x,\\ \xi_{*x} \le D_x(\theta^x_i)\underline{w} - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v,t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x,\ \ \theta^x_i \in T_x \cap T_{+\tau};\\ \xi_{*w} \le D_w(\theta^w)(\bar{w}-\tilde{w}) + \tilde{\xi}_w(\theta^w) \le \xi^*_w,\\ \xi_{*w} \le D_w(\theta^w)(\underline{w}-\tilde{w}) + \tilde{\xi}_w(\theta^w) \le \xi^*_w,\ \ \theta^w \in T_{clw} \cap T^{-\tau};\\ \xi_{*x} \le D_x(\theta^x)(\bar{w}-\tilde{w}) + \tilde{\xi}_x(\theta^x) \le \xi^*_x,\\ \xi_{*x} \le D_x(\theta^x)(\underline{w}-\tilde{w}) + \tilde{\xi}_x(\theta^x) \le \xi^*_x,\ \ \theta^x \in T_{clx} \cap T^{-\tau};\\ \omega_* \le \bar{w} \le \omega^*,\ \omega_* \le \underline{w} \le \omega^*,\ \omega_* \le \tilde{w} \le \omega^*;\\ \xi_{*w} \le \tilde{\xi}_w(\theta^w) \le \xi^*_w,\ \theta^w \in T_{clw} \cup T^{-\tau};\ \ \xi_{*x} \le \tilde{\xi}_x(\theta^x) \le \xi^*_x,\ \theta^x \in T_{clx} \cup T^{-\tau}, \end{cases} \tag{16}$$

where $\tilde{\xi}^\tau_w(\cdot) = \{\tilde{\xi}_w(\theta^w),\ \theta^w \in T_{clw} \cap T^{-\tau}\}$, $\tilde{\xi}^\tau_x(\cdot) = \{\tilde{\xi}_x(\theta^x),\ \theta^x \in T_{clx} \cap T^{-\tau}\}$.

2.3. **Positional solution to the optimal preposterior observation problem.** In order to generate current control actions (with aim of obtaining sufficiently complete information about uncertainty) estimates $d^0(q|\tau, y^*_\tau(\cdot))$ are calculated for several vectors (directions) $q \in Q$ under positional control. A vector

$$d^0(\tau, y^*_\tau(\cdot)) = (d^0(q|\tau, y^*_\tau(\cdot)),\ q \in Q)$$

is called the *vector of sufficient estimates for the position* $(\tau, y^*_\tau(\cdot))$.

Let Y^*_τ be a collection of all possible signals $y^*_\tau(\cdot)$ of measuring devices (7), (8) which can be obtained by τ.

Definition 7. A function

$$d^0(\tau, y^*_\tau(\cdot)),\ \ y^*_\tau(\cdot) \in Y^*_\tau,\ \ \tau \in T_w \cup T_x, \tag{17}$$

is called the *positional solution* to optimal preposterior observation problem. The construction of solutions to (17) represents the *synthesis* of an optimal preposterior obseravtion system.

Knowledge of positional solution (17) makes possible to obtain sufficient estimates for each possible position $(\tau, y^*_\tau(\cdot))$ and generate on the basis of them optimal control actions in the system a) with optimal output closable loop if the partial preposterior analysis is used in the course of observation; b) with optimal output closed loop if the full preposterior analysis is used. At present such method of synthesis of optimal systems is impossible to realize since there is no methods to construct positional solution (17).

2.4. **Optimal preposterior real-time observation.** As can be seen from the foregoing, positional solution (17) is constructed for all possible positions before observation process starting that requires to remember huge amount of information. In contemporary era of rapid development of computer science it is natural to resort to another approach of optimal observation where function (17) is not constructed but its current values required for control are calculated in the course of processes.

To describe this method, first of all we find out how the positional solution is used in particular observation process. We assume that positional solution (17) was constructed. And we consider some particular observation process where unknown w^*, $\xi_w^*(\theta^w)$, $\theta^w \in T_w$; $\xi_x^*(\theta^x)$, $\theta^x \in T_x$ is realized. In subsystem (6)-(8) this collection generates transient process $x^*(t)$, $t \in T$, and known signals $y_w^*(\theta^w)$, $\theta^w \in T_w$; $y_x^*(\theta^x)$, $\theta^x \in T_x$. Knowing positional solution (17), by this signals it is easy to obtain current estimates $d^*(\tau) = d^0(\tau, y_\tau^*(\cdot))$, $\tau \in T_w \cup T_x$. Hence it follows that in a particular observation process positional solution (17) is not used as a whole and only its values along a separate sequence of signals $y_\tau^*(\cdot)$, $\tau \in T_w \cup T_x$ are required.

Definition 8. A function

$$d^*(\tau), \ \tau \in T_w \cup T_x,$$

is called a *realization* of positional solution in a particular observation process. On account of stated above reasons it is imposible to implement such method of observation. Below we describe another principle of optimal observation which we call the *optimal observation in real time mode*. Let us assume that for each instant $\tau \in T_w \cup T_x$ there exists a method to calculate values $d^0(\tau, y_\tau^*(\cdot))$ during time $s^o(\tau)$ not exceeding h.

Definition 9. We call the function

$$d^{**}(t) = \begin{cases} (d^0(q),\, q \in Q),\ t \in [t_*, \bar{t}_* + s^o(\bar{t}_*)]; \\ d^*(\tau),\ t \in [\tau + s^o(\tau), \bar{\tau} + s^o(\bar{\tau})[,\ \ \tau \in T_w \cup T_x; \\ d^*(\underline{t}^*),\ t \in [\underline{t}^* + s^o(\underline{t}^*), t^*], \end{cases}$$

where $\bar{t} = \min\{\tau \in T_w \cup T_x : \tau > t\}$, $\underline{t} = \max\{\tau \in T_w \cup T_x : \tau < t\}$, the *quasirealization* of positional solution, and the device able to construct it the *optimal estimator* (OE).

In other words, quasirealization is a realization of positional solution with regard for expenditure of time on calculation of its current values.

Thus, the problem of synthesis of optimal observation system is reduced to construction of algorithm of operating OE.

In the paper we suggest the following algorithm of work of OE.

Since calculations for each direction $q \in Q$ can be carried out parallel, an algoritm of work of OE will be described only for one OE. Before the observation process starting OE solves problem (14) using the dual method [4] and, thus, calculating the initial preposterior estimate $d^0(q)$ and corresponding optimal support $K_b^0(q, t_*)$.

Let OE worked during the interval $T_{+\tau}$ and constructed the optimal support $K_b^0(q, \tau)$ for the position $(\tau, y_\tau^*(\cdot))$ from the obtained signals $y_\tau^*(\cdot)$ and calculated the current preposterior estimate $d^0(q|\tau, y_\tau^*(\cdot))$. At the nearest next instant $\bar{\tau}$ of measurements the signals a) $y_w^*(\bar{\tau})$ if $\bar{\tau} \in T_w$; b) $y_x^*(\bar{\tau})$ if $\bar{\tau} \in T_x$; c) both signals a), b) if $\bar{\tau} \in T_w \cap T_x$ become known.

During interval $[\,\bar\tau, \bar\tau + s^o(\bar\tau)[$ OE solves problem (16) for the position $(\bar\tau, y^*_{\bar\tau}(\cdot))$. This problem differs from the solved at the previous step for the position $(\tau, y^*_\tau(\cdot))$ that added are the constraints

$$a)\quad \begin{cases} \xi_{*w} \le D_w(\bar\tau)\bar w - C_w(\bar\tau)F_w(\bar\tau, t_*)z_0 + y^*_w(\bar\tau) \le \xi^*_w, \\ \xi_{*w} \le D_w(\bar\tau)\underline{w} - C_w(\bar\tau)F_w(\bar\tau, t_*)z_0 + y^*_w(\bar\tau) \le \xi^*_w; \end{cases}$$

$$b)\quad \begin{cases} \xi_{*x} \le D_x(\theta^x_i)\bar w - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v, t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x, \\ \xi_{*x} \le D_x(\theta^x_i)\underline{w} - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v, t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x,\ \theta^x_i = \bar\tau; \end{cases}$$

$$c)\quad \begin{cases} \xi_{*w} \le D_w(\bar\tau)\bar w - C_w(\bar\tau)F_w(\bar\tau, t_*)z_0 + y^*_w(\bar\tau) \le \xi^*_w, \\ \xi_{*w} \le D_w(\bar\tau)\underline{w} - C_w(\bar\tau)F_w(\bar\tau, t_*)z_0 + y^*_w(\bar\tau) \le \xi^*_w; \\ \xi_{*x} \le D_x(\theta^x_i)\bar w - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v, t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x, \\ \xi_{*x} \le D_x(\theta^x_i)\underline{w} - \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)F(v, t_*)x_0 dv + y^*_x(\theta^x_i) \le \xi^*_x,\ \theta^x_i = \bar\tau, \end{cases}$$

and removed are the constraints

$$\begin{cases} \xi_{*w} \le D_w(\bar\tau)(\bar w - \tilde w) + \tilde\xi_w(\bar\tau) \le \xi^*_w, \\ \xi_{*w} \le D_w(\bar\tau)(\underline{w} - \tilde w) + \tilde\xi_w(\bar\tau) \le \xi^*_w; \end{cases} \text{if } \bar\tau \in T_{clw};$$
$$\begin{cases} \xi_{*x} \le D_x(\bar\tau)(\bar w - \tilde w) + \tilde\xi_x(\bar\tau) \le \xi^*_x, \\ \xi_{*x} \le D_x(\bar\tau)(\underline{w} - \tilde w) + \tilde\xi_x(\bar\tau) \le \xi^*_x; \end{cases} \text{if } \bar\tau \in T_{clx}.$$

OE solves new problem by the dual method, correcting the optimal support $K^0_b(q, \tau)$ of the problem solved at the previous step till constructing the optimal $K^0_b(q, \bar\tau)$. As these problems differ from each other insignificantly, the current support $K^0_b(q, \tau)$ can be corrected rapidly using the dual method.

Remarks: 1. As a set of possible values of vector w, we may consider $W = \{w \in \mathbb{R}^l : \ l_{*w} \le L_w w \le l^*_w, \omega_* \le w \le \omega^*\}$, $L_w \in \mathbb{R}^{m_w \times l}$;
2. The initial state $x_0 \in X_* = \{x \in \mathbb{R}^{n_x} : x = L_0\nu, \nu \in V = \{\nu \in \mathbb{R}^{n_v} : l_{*\nu} \le L_\nu \nu \le l^*_\nu, \nu_* \le \nu \le \nu^*\}\}$, $L_0 \in \mathbb{R}^{n_x \times n_\nu}$, $L_\nu \in \mathbb{R}^{m_x \times n_\nu}$, also can be uncertain.

3. Example 1

Let the mathematical model of observation object be given in the form

$$2\ddot x + 5.4x = w(t);\ x(0) = 0.8,\ \dot x(0) = -1.0,\ T = [0, 12]\,.$$

Suppose we have measuring devices

$$y_w = z + \xi_w(t),\ |\xi_w(t)| \le \xi^*_w;\ \dot z + 1.8z = w(t),\ z(0) = -3.0;$$
$$y_x(\theta^x_i) = \int_{\theta^x_{i-1}}^{\theta^x_i} (x + \dot x)dv + \xi_x(\theta^x_i),\ |\xi_x(\theta^x_i)| \le \xi^*_x,\ i = \overline{1, N_x};$$

and disurbance

$$w(t) = w_1 \sin(t) + w_2 \sin(3t) + w_3 \sin(5t),\ t \in T;$$
$$(w_1, w_2, w_3) \in W = \{w \in \mathbb{R}^3 : |w_i| \le 1.6,\ i = \overline{1, 3}\};$$

$Q = (q_{(i)} = (\cos(\pi i/12), \sin(\pi i/12)),\ i = \{1, 2, .., 24\})$.

Aim of experiments is to construct initial and current preposterior distributions at terminal moment $T = 12$.

In the first series of experiments it was assumed that $\xi^*_w = 0.1$, $\xi^*_x = 0.1$, and distributions were constructed for the following cases (Fig. 5, $x_1 = x$, $x_2 = \dot x$): 1. $X^0_{t^*} = \{x \in \mathbb{R}^{n_x} : x = \int_{t_*}^{t^*} F(t^*, t)M(t)L(t)dtw,\ w \in W\}$; 2. $\mathcal{X}^o_{t^*}$, $T_{clw} = T_{clx} = \{9\}$; 3. $\mathcal{X}^o_{t^*}$, $T_{clw} = T_{clx} = \{6\}$; 4. $\mathcal{X}^o_{t^*}$, $T_{clw} = T_{clx} = \{3, 6, 9\}$; 5. $\mathcal{X}^o_{t^*}$, $T_{clw} = T_{clx} = \{1, 3, 6, 9, 11\}$; 6. $\mathcal{X}^o_{t^*}$, $T_{clw} = T_{clx} = \{1, 2, ..., 11\}$.

In the second series it was supposed that $T_{clw} = T_{clx} = \{3, 6, 9\}$. In the first part of the series it was assumed that $\xi^*_x = 0.1$ and the following cases are considered (Fig. 6a): 1. $X^0_{t^*}$; 2. $\mathcal{X}^o_{t^*}$, $\xi^*_w = 0.6$; 3 $\mathcal{X}^o_{t^*}$, $\xi^*_w = 0.3$; 4. $\mathcal{X}^o_{t^*}$, $\xi^*_w = 0.1$; 5. $\mathcal{X}^o_{t^*}$, $\xi^*_w = 0.05$; 6. $\mathcal{X}^o_{t^*}$, $\xi^*_w = 0.005$. In the second

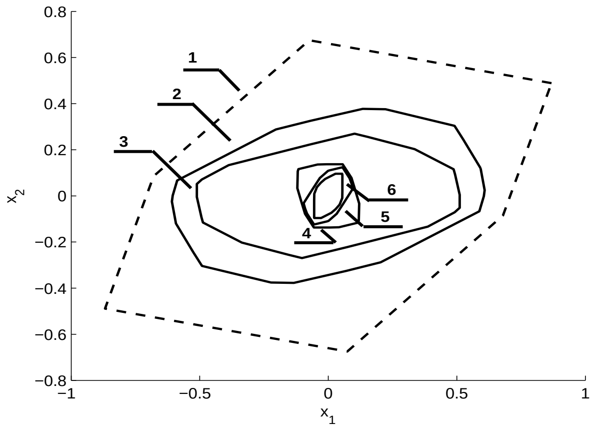
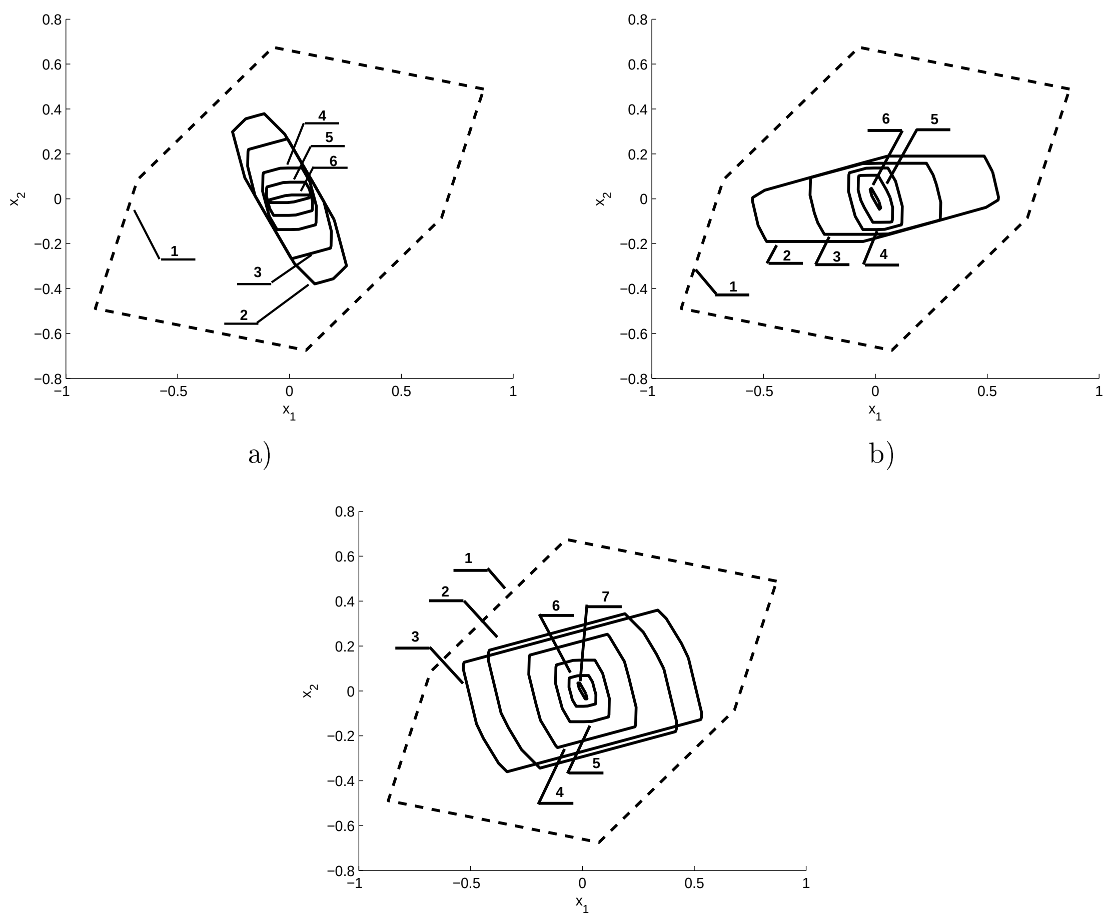


Fig. 5

part it was assumed that $\xi_w^* = 0.1$ and the following cases are considered (Fig. 6b): 1. $X_{t^*}^0$; 2. $\mathcal{X}_{t^*}^o$, $\xi_x^* = 0.6$; 3. $\mathcal{X}_{t^*}^o$, $\xi_x^* = 0.3$; 4. $\mathcal{X}_{t^*}^o$, $\xi_x^* = 0.1$; 5. $\mathcal{X}_{t^*}^o$, $\xi_x^* = 0.05$; 6. $\mathcal{X}_{t^*}^o$, $\xi_x^* = 0.005$. In the third part the following cases are considered (Fig. 6c): 1. $X_{t^*}^0$; 2. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.35$, $\xi_x^* = 0.35$; 3. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.3$, $\xi_x^* = 0.5$; 4. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.2$, $\xi_x^* = 0.2$; 5. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.1$, $\xi_x^* = 0.1$; 6. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.05$, $\xi_x^* = 0.05$; 7. $\mathcal{X}_{t^*}^o$, $\xi_w^* = 0.05$, $\xi_x^* = 0.005$.

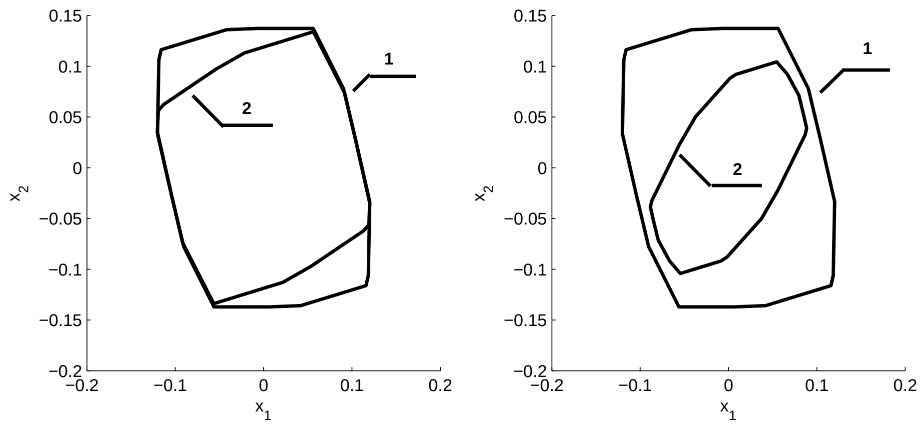


a) b)

c)

Fig. 6

In the third series of experiments current preposterior distributions were constructed for $\xi_w^* = 0.1$, $\xi_x^* = 0.1$. At that, values of simulation elements were: vector $w^* = (0.2, -1.0, -1.2)$, measurement errors $\xi_w^*(t) = \xi_w^* \cos(3t)$, $\xi_x^*(t) = \xi_x^* \sin(5t)$, $t \in T$. In Fig. 7a the sets were obtained by full preposterior analysis for $T_w = T_x = T_{clw} = T_{clx} = \{3, 6, 9\}$: 1. $\mathcal{X}^o_{t^*}$; 2. $\mathcal{X}^o_{t^*}(\tau, y^*_\tau(\cdot))$, $\tau = 3$. In Fig. 7b the sets were obtained by partial preposterior analysis for $T_w = T_x = \{1, 2, ..., 11\}$, $T_{clw} = T_{clx} = \{3, 6, 9\}$: 1. $\mathcal{X}^o_{t^*}$; 2. $\mathcal{X}^o_{t^*}(\tau, y^*_\tau(\cdot))$, $\tau = 2$.



a) b)

Fig. 7

4. Optimal closable output loop and its realization

4.1. **Optimal initial closable program.** To introduce a new type of optimal loops in problem (1)-(5), we continue preposterior analysis and for each closing instant $t_j \in T_{cl}$, $j = \overline{1,p}$, and construct closure sets $\mathcal{X}^p$, $\mathcal{X}^{p-1}, ..., \mathcal{X}^1$.

We begin with the set $\mathcal{X}^p$. Denote: $\mathcal{X}^o_{t+0}$ is an initial preposterior distribution for the observation subsystem at the instant t constructed by virtual signals along the interval $[t_*, t]$; $\mathcal{X}^o_{t-0}$ - by signals along the interval $[t_*, t[$. Let $\mathcal{X}^o_{t_p+0}$, $\mathcal{X}^o_{t^*-0}$ be initial preposterior distributions at the instants t_p и t^* respectively. Let $\mathcal{X}^c_{t_p+0}(z) = z + \mathcal{X}^o_{t_p+0}$. We introduce a set Z^p composed of all such vectors $z \in \mathbb{R}^{n_x}$ for which there exist available control actions $u(t_p : t^*|\mathcal{X}^c_{t_p+0}(z))$ that $X^c_{t^*}(u(t_p : t^*|\mathcal{X}^c_{t_p+0}(z))) = \{F(t^*, t_p)z + \int_{t_p}^{t^*} F(t^*, s)B(s)u(s|\mathcal{X}^c_{t_p+0}(z))ds + \mathcal{X}^o_{t^*-0}\} \subset X^*$. Family of sets

$$\mathcal{X}^p = \{\mathcal{X}^c_{t_p+0}(z),\ z \in Z^p\}$$

is called the closure set of control system at instant t_p.

Let nonemply sets $\mathcal{X}^p$, $\mathcal{X}^{p-1}, ..., \mathcal{X}^{j+1}$ are determined. Using constructed by moment t_j sets $\mathcal{X}^o_{t_j+0}$, $\mathcal{X}^o_{t_{j+1}-0}$ we define

$$Z^j = \{z \in \mathbb{R}^{n_x} :\ \exists u(t|\mathcal{X}^c_{t_j+0}(z)) \in U,\ t \in [t_j : t_{j+1}[,\ X^c_{t_{j+1}}(u(t_j : t_{j+1}|\mathcal{X}^c_{t_j+0}(z))) \subset \mathcal{X}^{j+1}\},$$

where $X^c_{t_{j+1}}(u(t_j : t_{j+1}|\mathcal{X}^c_{t_j+0}(z))) = F(t_{j+1}, t_j)z + \int_{t_j}^{t_{j+1}} F(t_{j+1}, s)B(s)u(s|\mathcal{X}^c_{t_j+0}(z))ds + \mathcal{X}^o_{t_{j+1}-0}$. Denote

$$\mathcal{X}^j = \{\mathcal{X}^c_{t_j+0}(z),\ z \in Z^j\}.$$

Continuing the process we construct $\mathcal{X}^j$, $j = \overline{1,p}$. Let the inclusion $X^c_{t_1}(u(t_* : t_1|x_0)) = \int_{t_*}^{t_1} F(t_1, s)B(s)u(s|x_0)ds + X^o_{t_1} \subset \mathcal{X}^1$ holds. Totality $u(\cdot) = \{u(t_* : t_1|x_0);\ u(t_1 : t_2|\mathcal{X}),\ \mathcal{X} \in \mathcal{X}^1; ...;\ u(t_p : t^*|\mathcal{X}),\ \mathcal{X} \in \mathcal{X}^p\}$ is called an *initial closable program.* It is guaranteed to transfer system (2) to the terminal set for any implementations of uncertainty if measurements are carried out at the instants $t \in T_{cl}$.

We choose $\beta > \min c'x$, $x \in X^*$, replace the set X^* by $X^{*\beta} = X^* \cap \{x \in \mathbb{R}^{n_x} : c'x \geq \beta\}$ and construct the sets $\mathcal{X}^{p\beta}$, $\mathcal{X}^{p-1\beta}, ..., \mathcal{X}^{1\beta}$ while following the above rules. The maximum β^0 for which an initial closable program exists, is equal to the maximum guaranteed value of the cost function of problem (1)-(5).

Definition 10. The totality

$$u^0(\cdot) = \{u^{\beta^0}(t_* : t_1|x_0);\, u^{\beta^0}(t_1 : t_2|\mathcal{X}), \mathcal{X} \in \mathcal{X}^{1\beta^0}; ...;\, u^{\beta^0}(t_p : t^*|\mathcal{X}), \mathcal{X} \in \mathcal{X}^{p\beta^0}\}$$

is called an *optimal initial closable program* (a *program preposterior solution*).

4.2. **Optimal current closable program.** Following classical rule of constructing optimal current closable programs we introduce a *positional preposterior solution* to the problem. Let $\tau \in T_w \cup T_x$ be a current instant and the control process is carried out during the time interval $T_{-\tau} = [t_*, \tau[$, the control actions $u^*_\tau(\cdot) = u^*(t_* : \tau)$ are generated and "pure" from $u^*_\tau(\cdot)$ signals $y^*_\tau(\cdot)$ known by the instant τ are logged (measuring devices signals of the observation object). Denote: $T^\tau_{cl} = T_{cl} \cap T^{-\tau} = \{t_{k(\tau)}, t_{k(\tau)+1}, ..., t_p\}$, $t_{k(\tau)} = \min\{t \in T_{cl} :\ \tau < t\}$; $T^\tau_{cl} = \emptyset$, $\tau \geq t_p$. Having replaced the a priori information $\{t_*, W\}$ by current $\{\tau, W(\tau, y^*_\tau(\cdot)\}$, we perform described above preposterior analysis on the time interval $T^{-\tau}$. As a result, we get closure sets $\mathcal{X}^p(\tau, y^*_\tau(\cdot))$, $\mathcal{X}^{p-1}(\tau, y^*_\tau(\cdot)), ..., \mathcal{X}^{k(\tau)}(\tau, y^*_\tau(\cdot))$ and determine an *optimal current closable program* $u^0(t|\tau, y^*_\tau(\cdot))$, $t \in T^{+\tau} = [\tau, t^*]$, for the position $(\tau, y^*_\tau(\cdot))$. Note that for $\tau \geq t_p$ the optimal current closable program turns into disclosable [3].

4.3. **Positional solution to optimal control problem.** Denote $Y_{\theta(\tau)}(\cdot)$, $\tau \in T_u$, a set of all signals $y_{\theta(\tau)}(\cdot)$ such that for the position $(\theta(\tau), y_{\theta(\tau)}(\cdot))$ a closable program exists; $\theta(\tau) = \max\{\theta^w \in T_w \cap T_{+\tau};\ \theta^x \in T_x \cap T_{+\tau};\ t_*\}$.

Definition 11. A functional

$$u^0(\tau, y_\tau(\cdot)) = u^0(\tau|\theta(\tau), y_{\theta(\tau)}(\cdot)),\, y_{\theta(\tau)}(\cdot) \in Y_{\theta(\tau)}(\cdot),\, \tau \in T_u, \tag{18}$$

is called an *optimal closable (combined, discrete) output loop* (OCOL) (a *positional solution* to the control problem in the class of output closable loops); contraction of (18) to a set of signals of measuring device (3) is an *optimal output closable feedforward loop*; contraction of (18) to a set of signals of measuring device (4) is an *optimal output closable feedback loop*. The construction of OCOL is the *synthesis of the optimal control system in the class of output closable loops*.

Note, if $\tau \geq t_p$ then OCOL becomes an optimal output disclosable loop.

4.4. **Optimal real-time control.** To control dynamical objects by the classical closed-loop principle OCOL has to be constructed before control process starting, that it is not a success even for optimal state feedbacks yet. Therefore, like in the case of optimal observation(Section 2), we adhere to the principle of *optimal real-time control*, by which OCOL is not constructed wholly, but in each particular control process its current values (a *realization of OCOL*) $u^*(\tau) = u^0(\tau, y^*_\tau(\cdot))$, $\tau \in T_u$, are generated by an *optimal regulator* (OR) for the time $s^c(\tau)$, and $s^o(\tau) + s^c(\tau) < h$ (Fig. 8).

Definition 12. The function

$$u^{**}(t) = \begin{cases} u^*(t_*), t \in [t_*, t_* + h + s^o(t_* + h) + s^c(t_* + h)]; \\ u^*(\tau), t \in [\tau + s^o(\tau) + s^c(\tau), \tau + h + s^o(\tau + h) + s^c(\tau + h)[, \tau \in T_u \backslash \{t_*, t^* - h\}; \\ u^*(t^* - h), t \in [t^* - h + s^o(t^* - h) + s^c(t^* - h), t^*], \end{cases}$$

constructed by OE and OR, is called a *quasirealization of OCOL*.

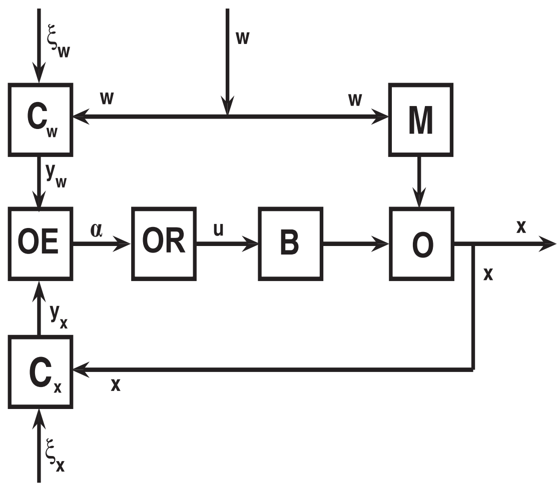


Fig. 8: Optimal real-time control.

We suggest the folllowing algorithm of operating OR. Before the control process starts, OR carries out initial preposterior analysis and constructs closure sets $\mathcal{X}^{p\beta}$, $\mathcal{X}^{p-1\beta}, ..., \mathcal{X}^{1\beta}$, where $\beta = \min c'x,\, x \in X^*$.

At first, we describe a method of constructing $\mathcal{X}^{p\beta}$ (the other closure sets are constructed similarly). Let $\bar{\eta}_{t^*}(q) = \max q'x,\, x \in \mathcal{X}^o_{t^*-0}$, $\underline{\eta}_{t^*}(q) = \min q'x,\, x \in \mathcal{X}^o_{t^*-0}$, be estimates of the initial preposterior distribution in the direction q at terminal instant. Then $\mathcal{X}^{p\beta} = \mathcal{X}^o_{t_p+0} + Z^{p\beta}$, where $Z^{p\beta}$ consists of all $z \in \mathbb{R}^{n_x}$ that the inequalities hold:

$$\begin{cases} g_{*i} - \underline{\eta}_{t^*}(h_i) \le h_i' F(t^*, t_p) z + h_i' \int_{t_p}^{t^*} F(t^*, s) B(s) u(s) ds \le g_i^* - \bar{\eta}_{t^*}(h_i), \ i \in I; \\ \beta - \underline{\eta}_{t^*}(c) \le c' F(t^*, t_p) z + c' \int_{t_p}^{t^*} F(t^*, s) B(s) u(s) ds; \\ u(t) \in U,\, t \in [t_p, t^*]. \end{cases}$$

We note that estimates are calculated along the directions h_i, $i \in I$; c, because the set $X^{*\beta}$, by which $\mathcal{X}^{p\beta}$ is constructed, is determined on this directions.

Let sets $\mathcal{X}^{p\beta}$, $\mathcal{X}^{p-1\beta}, ..., \mathcal{X}^{1\beta}$ have been constructed; $H^{1\beta}$ are directions, by which the outer approximation of set $\mathcal{X}^{1\beta}$ is under construction; $\gamma^{1\beta} = (\gamma^{1\beta}(q),\, q \in H^{1\beta})$, $\gamma^{1\beta}(q) = \max q'x,\, x \in \mathcal{X}^{1\beta}$ is an estimate of the closure set at instant t_1 in direction q; $\gamma^0 = (\gamma^0(q),\, q \in H^{1\beta})$, $\gamma^0(q) = \max q'x,\, x \in X^o_{t_1}$ is an estimate of a priori distribution $X^o_{t_1}$ of the observation subsystem state at the instant t_1 in direction q. The initial closable program $u(t)$, $t \in [t_*, t_1[$ is a solution to problem (19):

$$\begin{cases} \alpha \to \min\limits_{\alpha,\, u(t_*:t_1)}; \\ H^{1\beta} \int_{t_*}^{t_1} F(t_1, s) B(s) u(s) ds - \alpha \le \gamma^{1\beta} - \gamma^0; \\ u(t) \in U,\, t \in [t_*, t_1[. \end{cases} \tag{19}$$

Step by step increasing β and solving problem (19) by the dual method (iterations begin with the empty support), OR generates the optimal initial closable program $u^{\tau 0}(\cdot)$, computes the maximum value $\beta^0(t_*) = \beta^0$ of the cost function, the optimal support $K_b^0(t_*)$ and constructs a set $S_b^*(\bar{t}_*)$ of supporting control action indices which are to "freeze" at the nearest next measurement instant $\bar{t}_*$. As initial supports for solving linear programming problems the empty supports are taken. Since operations are implemented in advance, the time expence is not significant.

To the input of the control object OR sends the control action $u^*(t) = u^0(t)$, $t \in [t_*, \bar{t}_* + s^o(\bar{t}_*) + s^c(\bar{t}_*)[$, where $s^o(\bar{t}_*)$ и $s^c(\bar{t}_*)$ is a time of operating OE and OR correspondingly.

Suppose that OR has operated on the interval $T_{-\tau}$, $\tau \in T_w \cup T_x$, $\tau < t_p$, has computed the optimal closable program $u^{\underline{\tau}0}(\cdot|\underline{\tau}, y_{\underline{\tau}}(\cdot))$, respective to it sets $K_b^0(\underline{\tau})$, $S_b^*(\tau)$ and calculated the maximum value $\beta^0(\underline{\tau})$ of the cost function for the nearest previous measurement instant $\underline{\tau}$. At instant τ OE receives a new signal from measuring devices and for the current position $(\tau, y_\tau^*(\cdot))$ solves the optimal preposterior observation problem. By results of OE work, OR computes the closure sets $\mathcal{X}^{p\beta}(\tau, y_\tau^*(\cdot))$, $\mathcal{X}^{p-1\beta}(\tau, y_\tau^*(\cdot)), ...,$ $\mathcal{X}^{k(\tau)\beta}(\tau, y_\tau^*(\cdot))$ and the optimal current closable program $u^{\tau 0}(\cdot|\tau, y_\tau^*(\cdot))$.

Now we describe a method of constructing $\mathcal{X}^{p\beta}(\tau, y_\tau^*(\cdot))$. Let $\bar{\eta}_{t^*}(q|\tau, y_\tau^*(\cdot)) = \max q'x$, $x \in \mathcal{X}^o_{t^*-0}(\tau, y_\tau^*(\cdot))$; $\underline{\eta}_{t^*}(q|\tau, y_\tau^*(\cdot)) = \min q'x$, $x \in \mathcal{X}^o_{t^*-0}(\tau, y_\tau^*(\cdot))$, be estimates of the current preposterior distribution in direction q at the terminal instant. Then $\mathcal{X}^{p\beta}(\tau, y_\tau^*(\cdot)) = \mathcal{X}^o_{t_p+0}(\tau, y_\tau^*(\cdot)) + Z^{p\beta}(\tau, y_\tau^*(\cdot))$, where $Z^{p\beta}(\tau, y_\tau^*(\cdot))$ consists of all $z \in \mathbb{R}^{n_x}$, on which inequalities hold:

$$\begin{cases} g_{*i} - \underline{\eta}_{t^*}(h_i|\tau, y_\tau^*(\cdot)) \le h_i' F(t^*, t_p) z + h_i' \int_{t_p}^{t^*} F(t^*, s) B(s) u(s) ds \le g_i^* - \bar{\eta}_{t^*}(h_i|\tau, y_\tau^*(\cdot)); \\ \beta - \underline{\eta}_{t^*}(c|\tau, y_\tau^*(\cdot)) \le c' F(t^*, t_p) z + c' \int_{t_p}^{t^*} F(t^*, s) B(s) u(s) ds; \\ u(t) \in U,\ t \in [t_p, t^*]. \end{cases}$$

Let sets $\mathcal{X}^{p\beta}(\tau, y_\tau^*(\cdot))$, $\mathcal{X}^{p-1\beta}(\tau, y_\tau^*(\cdot)), ..., \mathcal{X}^{k(\tau)\beta}(\tau, y_\tau^*(\cdot))$ have been constructed; $H^{k(\tau)\beta}$ are directions, by which the outer approximation of set $\mathcal{X}^{k(\tau)\beta}(\tau, y_\tau^*(\cdot))$ is under construction; $\gamma^{k(\tau)\beta}(\tau, y_\tau^*(\cdot)) = (\gamma^{k(\tau)\beta}(q|\tau, y_\tau^*(\cdot)), q \in H^{k(\tau)\beta})$, $\gamma^{k(\tau)\beta}(q|\tau, y_\tau^*(\cdot)) = \max q'x$, $x \in \mathcal{X}^{k(\tau)\beta}(\tau, y_\tau^*(\cdot))$ is the estimate of the current closure set in direction q at the instant $t_{k(\tau)}$; $\gamma^0(\tau, y_\tau^*(\cdot)) = (\gamma^0(q|\tau, y_\tau^*(\cdot)),\ q \in H^{k(\tau)\beta})$, $\gamma^0(q|\tau, y_\tau^*(\cdot)) = \max q'x$, $x \in X^o_{t_{k(\tau)}}(\tau, y_\tau^*(\cdot))$ is the estimate of current distribution $X^o_{t_{k(\tau)}}(\tau, y_\tau^*(\cdot))$ of the observation subsystem state in direction q at the instant $t_{k(\tau)}$. The current closable program $u(t|\tau, y_\tau^*(\cdot))$, $t \in [\tau, t_{k(\tau)}[$ is a solution to problem (20):

$$\begin{cases} \alpha \to \min\limits_{\alpha,\, u(\tau : t_{k(\tau)})}; \\ H^{k(\tau)\beta} \int_\tau^{t_{k(\tau)}} F(t_{k(\tau)}, s) B(s) u(s) ds - \alpha \le \\ \qquad \le \gamma^{k(\tau)\beta}(\tau, y_\tau^*(\cdot)) - \gamma^0(\tau, y_\tau^*(\cdot)) - H^{k(\tau)\beta} \int_{t_*}^{\tau} F(t_{k(\tau)}, s) B(s) u^*(s) ds; \\ u(t) \in U,\ t \in [\tau, t_{k(\tau)}[. \end{cases} \tag{20}$$

Algorithm of constructing the optimal current closable program $u^{\tau 0}(\cdot|\tau, y_\tau^*(\cdot))$ begins with value $\beta = \beta^0(\underline{\tau})$ of the cost function and initial support $K_b^0(\underline{\tau})$. Solving problem (20) for $\beta = \beta^0(\tau)$, OR computes set $S_b^*(\tau)$ and sends the control action $u^*(t) = u^0(t|\tau, y_\tau^*(\cdot))$, $t \in [\tau + s^o(\tau) + s^c(\tau), \bar{\tau} + s^o(\bar{\tau}) + s^c(\bar{\tau})[$ to the input of the control object.

5. Example 2

Consider the following example:

$$\begin{gathered} x(t^*) + \dot{x}(t^*) \to \max; \\ \ddot{x} + 2.7x = 0.5u + 0.5w(t);\ x(0) = -1.0,\ \dot{x}(0) = -1.7,\ T = [0, 12]; \\ (x(12), \dot{x}(12)) \in X^* = \{x \in \mathbb{R}^2 : |x_1| \le 0.5,\ |x_2| \le 0.5\}; \\ |u(t)| \le 1.0,\ t \in T; \end{gathered}$$

$$\begin{gathered} y_w = z + \xi_w(t),\ |\xi_w(t)| \le 0.1,\ t \in T;\ \dot{z} + 1.8z = w(t),\ z(0) = -3.0; \\ y_x(t) = \int_{t-3}^{t} (x(t) + \dot{x}(t)) ds + \xi_x(t),\ |\xi_x(t)| \le 0.1,\ t \in T; \end{gathered}$$

$$w(t) = w_1 + w_2 \sin(t) + w_3 \sin(3t),\ t \in T;$$
$$(w_1, w_2, w_3) \in W = \{w \in R^3 : |w_1| \le 2.4,\ |w_2| \le 0.8,\ |w_3| \le 0.8\};$$
$$w^* = (1.0, -0.1, -0.5),$$
$$\xi^*_w(t) = 0.1\cos(t),\ \xi^*_x(t) = 0.1\sin(t),\ t \in T;$$
$$Q = (q_{(i)} = (\cos(\pi i/12), \sin(\pi i/12)),\ i = \{1, 2, .., 24\});$$
$$h = 1;\ T_w = T_x = \{3, 6, 9\};\ T_{cl} = T_{clw} = T_{clx} = \{6\}.$$

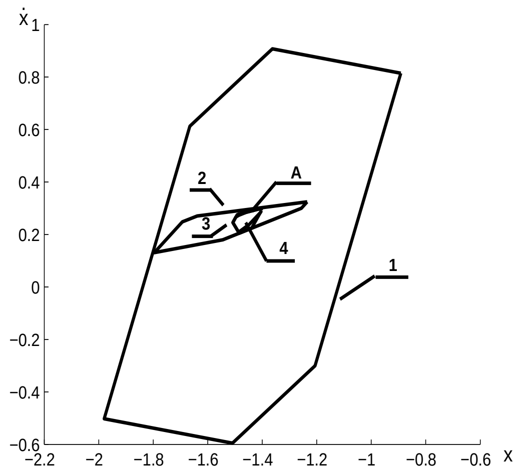


Fig. 9

In Fig. 9 a priori and current distributions of terminal states of the observation system are represented: 1.$X^o_{t^*}$; 2. $X^o_{t^*}(\tau, y^*_\tau(\cdot))$, $\tau = 3$; 3. $X^o_{t^*}(\tau, y^*_\tau(\cdot))$, $\tau = 6$; 4. $X^o_{t^*}(\tau, y^*_\tau(\cdot))$, $\tau = 9$; A is a terminal state of the observation subsystem for $w(t) = 0$, $t \in T$.

Fig. 10 introduces realizations of optimal output closable (solid line) and optimal disclosable [3] (dashed line) loops in a particular control process. In it guaranteed value of the cost function with use of the optimal output disclosable loop is $J(u^*(\cdot)) = 0.7028$, and with use of the optimal output closable loop is $J(\tilde{u}^*(\cdot)) = 0.7920$. It appears on the realized trajectory $\tilde{x}(t)$, $t \in T$, that corresponds to the optimal output closable (combined) loop, value of the cost function is equal to 0.9250; on the realized trajectory $x(t)$, $t \in T$, that corresponds to the optimal output disclosable two-phase (combined) loop with parameter $\varepsilon = 0.001$, value of the cost function is equal to 0.8358.

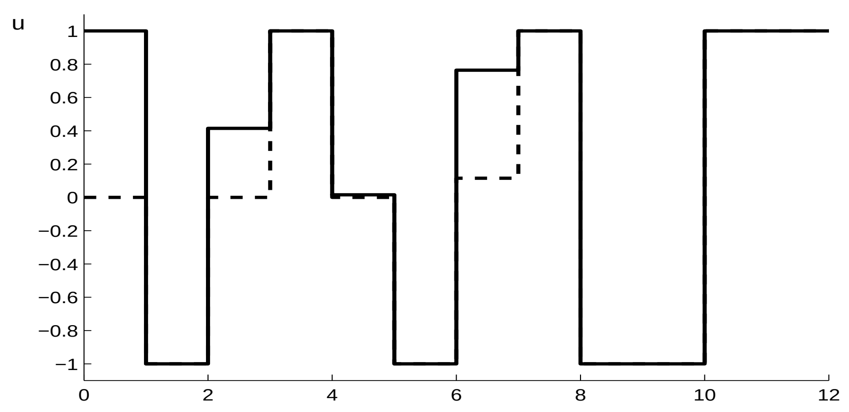
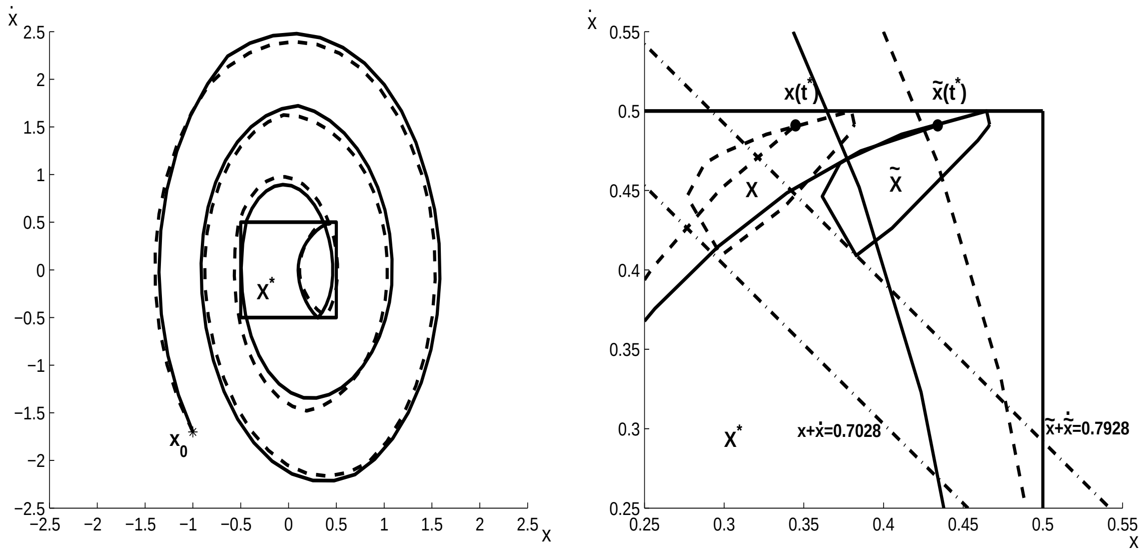


Fig. 10

The phase trajectories that correspond that two types of loops are depicted in Fig. 11a; Fig. 11b contains on enlarge scale fragments of the phase trajectories at the final control stage; $\tilde{X}$, X are the posteriori distributions of terminal states of the control system.



a) b)

Fig. 11

Conclusions

In the paper optimal preposterior observation and optimal control problems for dynamic systems under uncertainty with use of a priori and current information about the control object behaviour and uncertainty are considered. The method of implementing the optimal output closable loop by the optimal estimators and the optimal regulator is suggested. The algorithm of operating the optimal estimators and the optimal regulator that implements positional solutions to the problem in real-time mode is described. Obtained results can find application in solving other (not extreme) control problems (in particular, stabilization problems for dynamic systems under uncertainty).

References

[1] Balashevich N.V., Gabasov R., Kirillova F.M.,(2004), The Construction of Optimal Feedback From Mathematical Models with Uncertainty,Computational Mathematics and Mathematical Physics, 44 (2), pp. 247-267.

[2] Bellman R., Dreyfus S., (1962), Applied dynamic programming, Princeton University Press, Princeton, New Jersey.

[3] Gabasov R., Dmitruk N.M., Kirillova F.M.,(2004), Optimal Control of Multidimensional Systems by Inaccurate Measurements of Their Output Signals, Proceedings of the Steklov Institute of Math. Suppl. 2, Moscow, Russia, pp. 957-980.

[4] Gabasov R., Kirillova F.M. ,(1990), Adjoint problems of control, observation and identification, Doklady Academy of Sci., Belarus, 34(9), pp. 777-780.

[5] Gabasov R., Kirillova F.M., Kostina E.A., (1996), Closable State Feedbacks for Optimization of Undefined Control Systems, Dokl., Akad. Nauk., 347 (2),pp. 180-183.

[6] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mischenko E.F., (1962), Mathematical Theory of Optimal Processes, New York, Interscience Publishers Inc.

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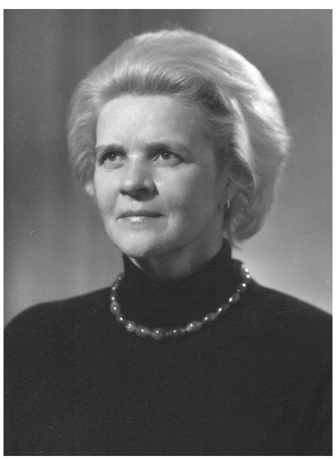

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