ON THE BASICITY FROM EXPONENTS IN LEBESGUE SPACES WITH VARIABLE EXPONENT

BILALOV B.T. 1, HUSEYNOV Z.G 2, §

Abstract. In the paper we consider the systems of exponents \( \{ \exp i(n - \alpha \cdot \text{sign} n) t \}_{n \in \mathbb{Z}}, 1 \cup \{ \exp i(n - \alpha \cdot \text{sign} n) t \}_{n \neq 0} \), cosines \( \{ \cos (n - \alpha) t \}_{n \geq 0} \cup \{ \cos (n - \alpha) t \}_{n \geq 1} \) and sines \( \{ \sin (n - \alpha) t \}_{n \geq 1} \). The basis properties of these systems are completely studied in the spaces \( L_{p(t)} \) with variable exponent \( p(t) \).

Keywords: a system of exponents, basicity, completeness, minimality, Lebesgue space with variable exponent.

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1. Introduction

The paper studies the basicity of the system of exponents

\[
\left\{ e^{i(n - \alpha \cdot \text{sign} n) t} \right\}_{n \in \mathbb{Z}}, 1 \cup \left\{ e^{i(n - \alpha \cdot \text{sign} n) t} \right\}_{n \neq 0}
\]

in Lebesgue spaces of functions with variable exponent \( p(t) \), denoted as \( L_{p(t)} \), where \( \alpha \in \mathbb{C} \) is a complex parameter, \( Z \) is a set of integers. Systems (1),(2) are model systems for studying spectral properties of some differential operators. They are obtained from ordinary system of exponents by linear perturbation. The well-known mathematicians as Paley-Wiener [15], N. Levinson [14] and others were the first who appealed to study basis properties of these systems. Basis properties of systems (1),(2) were completely studied in Lebesgue ordinary spaces \( L_p \) or \( W^k \). Recently, in connection with consideration of some concrete problems of mechanics and mathematical physics,(see [12,16]), interest to studying these or other problems in the spaces \( L_{p(t)} \) or \( W^k_{p(t)} \) increases.

In the present paper we study basicity of systems (1),(2) in \( L_{p(t)} \equiv L_{p(t)} (-\pi, \pi) \) under definite conditions on the function \( p : [-\pi, \pi] \to [1, +\infty) \).

2. Necessary notation and facts

Let \( p : [-\pi, \pi] \to [1, +\infty) \) be some function measurable by Lebesgue. By \( L_0 \) we denote a class of all measurable on \([-\pi, \pi]\) (with respect to Lebesgue measure) functions. Accept the denotation

\[
I_p (f) \overset{def}{=} \int_{-\pi}^{\pi} |f(t)|^{p(t)} \, dt.
\]
Let $\mathcal{L} \equiv \{ f \in L_0 : L_p (f) < +\infty \}$. With respect to ordinary linear operations of addition of functions and multiplication by a number, for $p^+ = \sup_{[-\pi, \pi]} \text{vrai} p (t) < +\infty$, $\mathcal{L}$ turns into a linear space. By the norm

$$
\| f \|_{p_t} \overset{\text{def}}{=} \inf \left\{ \lambda > 0 : L_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}
$$

$\mathcal{L}$ is a Banach space and we denote it by $L_{p_t}$.

Denote

$$
H^{\text{ln}} \overset{\text{def}}{=} \left\{ p : \exists C > 0; \forall t_1, t_2 \in [-\pi, \pi], \left| t_1 - t_2 \right| \leq \frac{1}{2} \implies \left| p (t_1) - p (t_2) \right| \leq \frac{C}{-\ln |t_1 - t_2|} \right\}.
$$

Everywhere $q (t)$ denotes a conjugated to $p (t)$ function: $\frac{1}{p (t)} + \frac{1}{q (t)} \equiv 1$. Accept $p^- = \inf_{[-\pi, \pi]} \text{vrai} p (t) \cdot p^\pi = \max \{ p (\pi) ; p (-\pi) \}$, $p_\pi = \min \{ p (\pi) ; p (-\pi) \}$. It holds Holder’s generalized inequality

$$
\int_{-\pi}^{\pi} |f (t) g (t)| dt \leq C (p^- ; p^+) \| f \|_{p_t} \cdot \| g \|_{q_t},
$$

where $C (p^- ; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

The following property follows directly from definition.

**Property A.** If $\| f (t) \| \leq \| g (t) \|$ a.e. on $(-\pi, \pi)$, then $\| f \|_{p_t} \leq \| g \|_{p_t}$.

We easily prove the following

**Statement 1.** Let $p \in H^{\text{ln}}, p (t) > 0$, $\forall t \in [-\pi, \pi]$ and $\{ \alpha_i \}_{i=1}^m \subset R$ ($R$ is a real axis).

The function $\omega (t) = \prod_{i=1}^m |t - t_i|^{|\alpha_i|}$ belongs to the space $L_{p_t}$, if $\alpha_i > -\frac{1}{p (t_i)}$, $\forall i = 1, m$; where $\{ t_i \}_{i=1}^m \subset [-\pi, \pi]$, $t_i \neq t_j$ for $i \neq j$.

In sequel, we’ll need the following facts.

**Property B [16].** If $p (t) : 1 < p^- \leq p^+ < +\infty$, the class $C_0^\infty (-\pi, \pi)$ (finite and infinitely differentiable) is everywhere dense in $L_{p_t}$.

By $S$ we denote a singular integral:

$$
S (f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f (\tau)}{\tau - t} d\tau, \quad t \in \Gamma,
$$

where $\Gamma \subset C$ is some piecewise-Holder curve on $C$ ($C$ is a complex plane).

Let $\rho : [-\pi, \pi] \to [1, +\infty)$ be some weight function. Determine a weight class $L_{p_t, \rho_t} : L_{p_t, \rho_t} \overset{\text{def}}{=} \{ f : \rho \cdot f \in L_{p_t} \}$ with the norm: $\| f \|_{p_t, \rho_t} \overset{\text{def}}{=} \| \rho f \|_{p_t}$.

The following statement was proved in the paper [11].

**Statement [11].** Let $p \in H^{\text{ln}}, 1 < p^- \text{ and } p (t) = \prod_{k=1}^m |t - \tau_k|^{\alpha_k}$, where $\{ \tau_k \}_{k=1}^m \subset [-\pi, \pi]$, $\tau_i \neq \tau_j$ for $i \neq j$. Then a singular operator $S$ boundedly acts from $L_{p_t, \rho_t}$ to $L_{p_t, \rho_t}$ if

$$
-\frac{1}{p (\tau_k)} < \alpha_k < \frac{1}{q (\tau_k)}, \quad k = 1, m
$$

are fulfilled.

The following classes of analytic functions play an important part while establishing basicity.
3. Hardy classes with variable exponent

These classes were considered in the papers [10,3]. Let \( U \equiv \{ z : |z| < 1 \} \) be a unique ball on a complex plane and \( \Gamma = \partial U \) be a unit circle. For a function \( u(z) \) harmonic in \( U \) we accept
\[
\|u\|_{p_t} \equiv \sup_{0 < r < 1} \|u(re^{it})\|_{p_t},
\]
where \( p : [-\pi, \pi] \to [1, +\infty) \) is some measurable function. Denote
\[
h_{p_t} \equiv \{ u : \Delta u = 0 \text{ in } U \text{ and } \|u\|_{p_t} < +\infty \}.
\]
The continuous imbeddings \( h_+ \hookrightarrow h_{p_t} \hookrightarrow h_- \) are true. The following theorem is valid.

**Theorem [3].** Let \( 1 < p^- \leq p^+ < +\infty \). If
\[
u \in h_{p_t}, \text{ then } \exists f \in L_{p_t} : u(re^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta - t) f(t) \, dt,
\]
where \( p_r(\alpha) = \frac{1 - r^2}{1 + r^2 - 2r \cos \alpha} \) is a Poisson kernel. Vice-versa, if \( f \in L_{p_t}, p \in H^{l_1} \), then (3) belongs to \( h_{p_t} \).

The Hardy class \( H^+_{p_t} \) is introduced in the similar way
\[
H^+_{p_t} \equiv \{ f : f \text{ is analytic in } U \text{ and } \|f\|_{H^+_{p_t}} < +\infty \},
\]
where \( \|f\|_{H^+_{p_t}} = \sup_{0 < r < 1} \|f(re^{it})\|_{p_t} \).

It is easy to see that \( f \in H^+_{p_t} \iff \text{Ref; } Im f \in h_{p_t}, \) where \( \text{Re } z;\text{Im } z \) are real and imaginary parts of \( z \), respectively.

Using the previous theorem we can easily prove the following refined variant of theorem 5 of the paper [10].

**Theorem [10].** Let \( p \in H^{l_1} \) and \( p^- > 1 \). Then
\[
F \in H^+_{p_t} \iff \exists f \in L_{p_t} : F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it} f(t) \, dt\left( \frac{e^{it} - z}{e^{it} + z} \right).
\]

Analogy of Smirnov’s known theorem is also valid.

**Theorem [9].** Let \( p_i(t) : 0 < p^-_i \leq p^+_i < +\infty, \ i = 1, 2; p_1(t) \leq p_2(t), \) a.e. on \([ -\pi, \pi ]\) be measurable function, \( F \in H^+_{p_1}; p_2 \in H^{l_1} \) and \( p^-_2 > 1 \). Then, if \( F^+ \in L_{p_2} \implies F \in H^+_{p_2} \).

Let’s define the class \( mH^-_{p_t} \) of analytic outside of a unit circle functions of order \( \leq m \) at infinity. Let \( f(z) \) be a function analytic in \( C \setminus \overline{U} \ (\overline{U} = U \cup \Gamma) \), having a finite order \( \leq m \) at infinity, i.e. \( f(z) = f_1(z) + f_2(z), \) where \( f_1(z) \) is a polynomial of power \( \leq m, f_2(z) \) is a tame part of expansion of \( f(z) \) in Lorentz series in the vicinity of a point at infinity. If the function \( \varphi(z) \equiv f_2\left( \frac{1}{z} \right) \) (‘\( \ast \)’ is a complex conjugation) belongs to the class \( H^+_{p_t} \), we’ll say that the function \( f(z) \) belongs to the class \( mH^-_{p_t} \).

4. Riemann’s problem in the classes \( H^\pm_{p_t} \)

Let a complex valued function \( G(t) \) on \([ -\pi, \pi ]\) satisfy the conditions:
1) \( |G|_{l_1} \in L_\infty; \)
2) the argument \( \theta(t) \equiv \arg G(t) \) has an expansion of the form \( \theta(t) = \theta_0(t) + \theta_1(t), \) where \( \theta_0(t) \in C \ [-\pi, \pi ]; \) \( \theta_1(t) \) is a bounded variation function on \([ -\pi, \pi ]; \)
3) \( \theta_1(t) \) has a finite number of discontinuity points \( \{ s_k \}_1^r : -\pi < s_1 < ... < s_r < \pi \) on \([ -\pi, \pi ]; \)

4) \( \left\{ \frac{h_k}{2\pi} + \frac{1}{q(s_k)} \right\}_{k=0}^r \cap Z = \{0\} \), where \( h_k = \theta (s_k + 0) - \theta (s_k - 0) \), \( k = 1, r \); \( h_0 = \theta (\pi) - \theta (\pi) \).

It is required to find a piecewise-analytic function \( F^\pm (z) \) on \( C \) with a section \( \Gamma \), satisfying the conditions:

a) \( F^+ \in H^+_p : \ 0 < p^- \leq p^+ < +\infty \);

b) \( F^- \in mH^-_p \);

c) non-tangential boundary values of \( F^\pm (e^{it}) \) on a unit circle \( \Gamma \) a.e. satisfy the relation:

\[
F^+ (e^{it}) + G (t) F^- (e^{it}) = g (t), \ a.e. \ t \in (-\pi, \pi),
\]

where \( g \in L_p \) is an arbitrary function.

When summability indices are constant, the theory of these problems were sufficiently well studied (see. [6]). Let’s consider Riemann’s following homogeneous problem:

\[
\begin{align*}
F^+ (\tau) + G (\tau) F^- (\tau) &= 0, \ \tau \in \Gamma; \\
F^+ &\in H^+_p; \ F^- \in mH^-_p. \\
\end{align*}
\]

Let’s introduce into consideration the following analytic functions \( X^\pm_1 (z) \) interior (the sign ” + ”) and exterior (the sign ” − ”) to a unit circle.

\[
X^\pm_1 (z) \equiv \exp \left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln |G (e^{it})| \frac{e^{it} + z}{e - z} dt \right\},
\]

\[
X^\pm_2 (z) \equiv \exp \left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \theta (t) \frac{e^{it} + z}{e - z} dt \right\}.
\]

Let

\[
Z_i (z) = \begin{cases} 
X^+_i (z), & |z| < 1, \\
[X^-_i (z)]^{-1}, & |z| < 1.
\end{cases}
\]

Denote \( Z^\pm (z) \equiv Z^+_1 (z) \cdot Z^+_2 (z) \). Define \( \{n_i\}^r_{i=1} \subset Z \) from the inequalities

\[
-\frac{1}{q(s_k)} < \frac{h_k}{2\pi} + n_k - n_{k-1} < \frac{1}{p(s_k)}, \ k = 1, r; \\
n_0 = 0.
\]

Let \( \omega_r = \frac{h_0}{2\pi} + n_r \). Earlier we proved

**Theorem [9].** Let \( p \in H^\infty \), \( 1 < p^- \); the conditions 1)-4) be fulfilled. Then, if it holds \(-\frac{1}{q} < \omega_r < \frac{1}{p} \), the general solution of homogeneous problem (4) in the classes \( (H^+_p, mH^-_p) \) is of the form \( F (z) \equiv Z (z) \cdot P_m (z) \), where \( P_m (z) \) is an arbitrary polynomial of power \( \leq m \).

**Corollary.** Let all the requirements of the previous theorem be fulfilled. Then, provided \( F^- (\infty) = 0 \) the Riemann’s homogeneous problem (4) in the classes \( (H^+_p, mH^-_p) \) has only a trivial solution, i.e. zero solution.

Now, let’s consider Riemann’s homogeneous problem

\[
\begin{align*}
F^+ (\tau) + G (\tau) F^- (\tau) &= g (\tau), \ \tau \in \Gamma; \\
F^+ &\in H^+_p; \ F^- \in mH^-_p. \\
\end{align*}
\]

where \( g (\tau) \in L_p \) is an arbitrary function. Obviously, the problem (5) has a unique solution (if it is solvable) iff the appropriate problem (4) has only a trivial solution. In the general case the solution of problem (5) is of the form \( F (z) = F_0 (z) + Z (z) \cdot P_m (z) \), where \( F_0 (z) \) is one of the particular solutions of problem (5), \( P_m (z) \) is a polynomial of power \( \leq m \).
5. MAIN RESULTS

As \( G(\tau) \) we take the concrete function \( G(e^{it}) = e^{2it}, \) \( t \in [-\pi, \pi] \). Suppose \( \alpha \in \mathbb{R} \). The complex case is similarly investigated.

At first we assume that \( g(e^{it}) \) is a Holder function \([-\pi, \pi]\). Solve the problem (5) by the method developed in the monograph [8]. We get the particular solution \( F_0(z) \) of the form:

\[
F_0^+(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha \theta} g(e^{i\theta}) d\theta}{(1 + e^{i\theta})^{2\alpha} (1 - z \cdot e^{-i\theta})} (1 + z)^{2\alpha},
\]

\[
F_0^-(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha \theta} g(e^{i\theta}) d\theta}{(1 + e^{i\theta})^{2\alpha} (1 - z \cdot e^{-i\theta})} z^{-2\alpha} (1 + z)^{2\alpha}.
\]

The fact that \( F_0(z) \) satisfies the relation (5) follows directly from the Sokhotsky-Plancherel formula. Denote

\[
h_n^+(t) = \frac{1 + e^{iit})^{-2\alpha}}{2\pi} e^{i\alpha t} \cdot \sum_{k=0}^{n} C_{2\alpha}^{n-k} e^{ikt}, \quad n = 0, \infty;
\]

\[
h_m(t) = -\frac{1 + e^{iit})^{-2\alpha}}{2\pi} e^{i\alpha t} \cdot \sum_{k=0}^{m} C_{2\alpha}^{m-k} e^{-ikt}, \quad m = 1, \infty,
\]

where \( C_{n}^{\beta} = \frac{\beta (\beta - 1) \cdots (\beta - n + 1)}{n!} \) are binomial coefficients. Expanding the functions \( F_0^+(z) \) and \( F_0^-(z) \) respectively, in the vicinities of zero and a point at infinity in power of \( z \), we get

\[
F_0^+(z) = \sum_{n=0}^{\infty} a_n^+ z^n, \quad F_0^-(z) = \sum_{n=1}^{\infty} a_n^- z^{-n},
\]

where

\[
a_n^+ = \frac{1}{\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) h_n^+(\theta) d\theta, \quad n \geq 0; \quad a_n^- = \frac{1}{\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) h_n^-(\theta) d\theta, \quad n \geq 1.
\]

Let \( |2\alpha| < 1 \). It is easy to see that \( F_0^+ \in H_1^+; \quad F_0^- \in -1 H_1^- \). The relations [6]

\[
\int_{-\pi}^{\pi} |F_0^+(e^{it}) - F_0^+(re^{it})| dt \to 0, \quad r \to 1 - 0;
\]

\[
\int_{-\pi}^{\pi} |F_0^-(e^{it}) - F_0^-(re^{it})| dt \to 0, \quad r \to 1 + 0,
\]

yield

\[
a_n^+ = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_0^+(e^{it}) e^{-int} dt, \quad \forall n \geq 0; \quad a_n^- = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_0^-(e^{it}) e^{int} dt, \quad \forall m \geq 1.
\]

Using the representation of the Cauchy type integral with power character perultiarity in the vicinity of a discontinuity point of first order density (see [8], p.74), it is easy to show that if the conditions \( 0 < 2\alpha < 1 \) and \( g_1(1) = g(-1) = 0 \) hold, the functions \( F_0^+(\tau) \) are continuous on a unit circle. Therefore, the Fourier series of these functions by the system of exponents \( \{e^{int}\} \) converge to them on \([-\pi, \pi]\) uniformly, since they satisfy some Holderian conditions on \( \Gamma \). As the result we get

\[
F_0^+(e^{it}) = \sum_{n=0}^{\infty} a_n^+ e^{int}; \quad F_0^-(e^{it}) = \sum_{n=1}^{\infty} a_n^- e^{-int},
\]
uniformly on \([-\pi, \pi]\). Considering these relations in (5) we get (where \(g(\tau) = f(\tau) \cdot e^{i\alpha \tau}, \tau = e^{it}\)); \(f(e^{it})\) is a Holder function on \([-\pi, \pi]\)

\[
f(e^{it}) = \sum_{n=0}^{\infty} a_n^+ e^{i(n-\alpha)t} + \sum_{n=1}^{\infty} a_n^- e^{-i(n-\alpha)t},
\]

uniformly on \([-\pi, \pi]\). It is proved in [4] that for \(|\alpha| < \frac{1}{2}\) the following relations

\[
\begin{align*}
\int_{-\pi}^{\pi} e^{i(n-\alpha)t} h^+_m(t) dt &= \delta_{nm}, \quad \forall n, m \geq 0; \\
\int_{-\pi}^{\pi} e^{i(n-\alpha)t} h^-_m(t) dt &= 0, \forall n \geq 0; \forall m \geq 1; \\
\int_{-\pi}^{\pi} e^{-i(n-\alpha)t} h^+_m(t) dt &= 0, \forall n \geq 1; \forall m \geq 0; \\
\int_{-\pi}^{\pi} e^{-i(n-\alpha)t} h^-_m(t) dt &= \delta_{nm}, \forall n, m \geq 1.
\end{align*}
\]

are fulfilled.

It directly follows from the Property A that, if \(p(t) \in H^{in}\) and \(p^- > 1\), then the system (1) belongs to \(L_{p^r}\). In this case the space \(L_{q^r}\) is a space conjugated to \(L_{p^r}\) (see [16]). Consequently, it follows from statement 1 and representations for \(h^+_m(t)\) that for \(\alpha < \frac{1}{2q^r}\) the system \(\{h^+_m; h^-_m\}\) belongs to \(L_{q^r}\). Then, from relations (6) we get that while fulfilling the conditions formulated above, the system (1) and \(\{h^+_m; h^-_m\}\) are conjugated and so (1) is minimal in \(L_{p^r}\). Having paid attention to the Property B we get that for \(\frac{1}{2} > \alpha \geq 0\) the system (1) is complete in \(L_{p^r}\). Thus, if the inequality \(0 \leq \alpha < \frac{1}{2q^r}\) is fulfilled, then (1) is complete and minimal in \(L_{p^r}\).

Denote

\[
I(z) = \int_{-\pi}^{\pi} \frac{e^{i\alpha \theta} g_0(\theta) d\theta}{(1 + e^{i\theta})^{2\alpha}(1 - ze^{-i\theta})}, \quad g_0(\theta) = g_0(e^{i\theta}).
\]

Then we can represent \(F_0^\pm(z)\) in the form

\[
\begin{align*}
F_0^+(z) &= \frac{1}{2\pi} I(z) (1 + z)^{2\alpha}, \quad |z| < 1; \\
F_0^-(z) &= \frac{1}{2\pi} I(z) (1 + z^{-1})^{2\alpha}, \quad |z| > 1.
\end{align*}
\]

From the same reasonings we get that for finite functions \(g_0(\theta)\) on \([-\pi, \pi]\), the Fourier series for boundary values \(I^\pm(e^{i\theta})\) converge to them uniformly on \([-\pi, \pi]\). Therewith, if \(2\alpha > -\frac{1}{p^r}\), the functions \((1 + e^{i\theta})^{2\alpha}\) and \((1 - e^{-i\theta})^{2\alpha}\) belong to the space \(L_{p^r}\) and by the results of the paper [5], the Fourier series of these functions converge to them in \(L_{p^r}\). Again, it follows from the Property B that for \(-\frac{1}{2p^r} < \alpha < \frac{1}{2}\) the system (1) is complete in \(L_{p^r}\). Combining the obtained results we arrive at the following conclusion.

**Statement 2.** Let \(p(t) \in H^{in}\), \(p^- > 1\), and the inequality

\[
-\frac{1}{2p^r} < \alpha < \frac{1}{2q^r},
\]

be fulfilled. Then the system (1) is complete and minimal in \(L_{p^r}\).

Now we study the basicity. Let (8) be fulfilled. Then the system (1) is minimal in \(L_{p^r}\) and let \(\{h^+_m(t); h^-_m(t)\}_{n \geq 0; m \geq 1}\) be an appropriate conjugated system. Hence \(\forall f \in L_{p^r}\) and consider the partial sum \(S_m\):

\[
S_m[f] = \sum_{n=0}^{m} a_n^+ e^{i(n-\alpha)t} + \sum_{n=1}^{m} a_n^- e^{-i(n-\alpha)t},
\]
where
\[ a^+_n = \int_{-\pi}^{\pi} f(t) h_n^+(t) dt, \quad n \geq 0; \quad a^-_k = \int_{-\pi}^{\pi} f(t) h_n^-(\theta) dt, \quad k \geq 1. \]

Let's consider the problem (5), where as the right hand side of \( g(\tau) \) we take the function \( g(e^{i\theta}) = e^{i\alpha t} f(\theta) \), furthermore, require \( F^- (\infty) = 0 \). Then, as it follows from Corollary 1, the problem (5) has a unique solution \( F_0^\pm (\tau) \) in the classes \( (H^+_{p1}, -1 H^-_{p1}) \) and thus \( F_0^\pm (e^{i\theta}) \in L_{p1} \.

Show that
\[ \sup_{m, \| f \|_{p1} = 1} \| S_m [f] \|_{p1} < +\infty. \]

As we have already seen
\[ a^+_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_0^+ (e^{i\theta}) e^{-int} d\theta, \quad \forall n \geq 0; \quad a^-_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_0^- (e^{i\theta}) e^{ikt} d\theta, \quad \forall k \geq 1. \]

We have
\[ \| S_m [f] \|_{p1} \leq \left\| e^{-i\alpha t} \sum_{n=0}^{m} a^+_n e^{int} \right\|_{p1} + \left\| e^{i\alpha t} \sum_{n=1}^{m} a^-_n e^{-int} \right\|_{p1}. \]

Since the classic system of exponents \( \{ e^{int} \} \) forms a basis in \( L_{p1} \) (see [5]), then considering the Property A hence we get
\[ \| S_m [f] \|_{p1} \leq M_1 \| F_0^+ (e^{i\theta}) \|_{p1} + M_2 \| F_0^- (e^{i\theta}) \|_{p1}, \]
where \( M_i, \ i = 1, 2 \) are some constants. Applying the Sokhotsky-Planel formula to the expressions \( F_0^+ (\tau) \) and \( F_0^- (\tau) \) we get
\[ F_0^+ (e^{i\theta}) = i e^{i\alpha \theta} f (\theta) + S^+ (f), \quad F_0^- (e^{i\theta}) = i e^{-i\alpha \theta} f (\theta) + S^- (f), \]
where \( S^\pm (f) \) are appropriate singular type integrals

\[ S^+ (f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha \theta} f (\theta) d\theta}{(1 + e^{i\theta})^{2\alpha} (1 - e^{i(s - \theta)})} \cdot (1 + e^{is})^{2\alpha}, \]
\[ S^- (f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\alpha \theta} f (\theta) d\theta}{(1 + e^{i\theta})^{2\alpha} (1 - e^{i(s - \theta)})} \cdot (1 + e^{-is})^{2\alpha}. \]

Then, having paid attention to the Statement [11] we get that the integral operators \( S^+ (f) \) and \( S^- (f) \) act boundedly from \( L_{p1} \) to \( L_{p1} \), i.e.
\[ \| S^\pm (f) \|_{p1} \leq M \| f \|_{p1}, \quad \forall f \in L_{p1}. \]

As the result we have
\[ \| S_m [f] \|_{p1} \leq M_1 \left( M_3 \| f \|_{p1} + \| S^+ (f) \|_{p1} \right) + \]
\[ + M_2 \left( M_4 \| f \|_{p1} + \| S^- (f) \|_{p1} \right) \leq M_5 \| f \|_{p1}, \quad \forall f \in L_{p1}, \]
where \( M_i, i = 3, 5 \) are some constants.

As the result, it follows from the basicity criterium that the system (1) forms a basis in \( L_{p1} \), i.e. the following theorem is valid.

**Theorem 1.** Let \( p(t) \in H^{in}, \ p^- > 1, \) and the inequality \(-\frac{1}{2p^\pi} < \alpha < \frac{1}{2q^\pi}\) be fulfilled. Then the system of exponents (1) forms a basis in \( L_{p1} \).
Separately we consider the case $-\frac{1}{2p}\leq \alpha \leq -\frac{1}{2p}$. In this case, it follows from relations (6) and from expressions for $h_n^\pm (t)$ that the system (1) is minimal in $L_{p_r}$, since it has a biorthogonal system. Represent the system (1) in the form:

$$\left\{ e^{i[(n+1)-(\alpha+1)]t} e^{-i(m-\alpha)t} \right\}_{n \geq 0; m \geq 1}. \tag{9}$$

Multiplication of each term of the system (9) by the function $e^{\frac{i}{2}t}$ doesn’t influence on its completeness in $L_{p_r}$. As the result we get the system $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 1; m \geq 1}$, where $I_{n;m}^{\alpha} (t) \equiv (e^{i(n-\alpha)t} e^{-i(m-\alpha)t})$, $\alpha = \frac{1}{2}$. It is easy to notice that $\frac{1}{p} + \frac{1}{q} = 1$; $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, the inequality $\frac{1}{2q} \leq \alpha \leq \frac{1}{2}$ is fulfilled for $\alpha$. Then by the previous results we get that the system $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 0; m \geq 1}$ is complete in $L_{p_r}$. It follows from the expressions for $\left\{ h_n^\pm (t) \right\}$ and from Statement 1 that in this case the system $\left\{ h_n^\pm (t) \right\}$ doesn’t belong to the space $L_{q_r}$. Since the system $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 1; m \geq 1}$ is complete in $L_{p_r}$ then from the uniqueness of biorthogonal system to the complete system we get that $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 0; m \geq 1}$ is not minimal in $L_{p_r}$ and as a result of that the system $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 0; m \geq 1}$ and so the system (1) is complete and minimal in $L_{p_r}$. The fact that the system (1) doesn’t form a basis in $L_{p_r}$ is proved similar to the paper [13].

We arrive at the following conclusion: if $-\frac{1}{2p} \leq \alpha \leq -\frac{1}{2p}$, the system (1) is complete and minimal in $L_{p_r}$, and now, let $\alpha < -\frac{1}{2p}$, for example $-\frac{1}{2p} - \frac{1}{2} \leq \alpha < -\frac{1}{2p}$. In this case, it holds $-\frac{1}{2p} \leq \alpha < \frac{1}{2q}$ and so the system $\left\{ I_{n;m}^{\alpha} (t) \right\}_{n \geq 0; m \geq 1}$ is complete, and minimal in $L_{p_r}$. As the result the system (1) is not complete, but minimal in $L_{p_r}$. In the similar way we show that for $\alpha \geq \frac{1}{2q}$ the system is complete, but not minimal in $L_{p_r}$.

Combining all the obtained results, we have the following theorem.

**Theorem 2.** Let $p(t) \in H^{in}$, $p^- > 1$. The system (1) is complete in $L_{p_r}$ iff $\alpha \geq -\frac{1}{2p}$; it is minimal in $L_{p_r}$ only for $\alpha < \frac{1}{2q}$.

Let the inequality $\alpha < \frac{1}{2q}$ hold. By theorem 2, in this case the system (1) is minimal in $L_{p_r}$. It directly follows from analytical expressions for the conjugated system $\left\{ h_n^\pm (t) \right\}$ that

$$h_0^+ (t) = \frac{1}{2\pi} \cdot \frac{e^{i\alpha t}}{(1 + e^{it})^{2\alpha}}.$$ 

We have

$$\frac{c_0^+}{c_0^+} = \int_{-\pi}^{\pi} h_0^+ (t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{(1 + e^{-it})^{2\alpha} \cdot (e^{it})^{\alpha}} =$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{(e^{\frac{i\pi}{2}} + e^{-\frac{i\pi}{2}})^{2\alpha}} = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{(2\cos \frac{t}{2})^{2\alpha}} \neq 0.$$ 

We consider the system $\left\{ H_n^+ ; H_m^- \right\}_{n \geq 0; m \geq 1}$

$$H_0^+ = \frac{1}{c_0^+} h_0^+; \ H_n^+ = h_n^+ - \frac{c_n^+}{c_0^+} h_0^+; \ H_n^- = h_n^- - \frac{c_n^-}{c_0^-} h_0^-; \tag{10}$$
where \( c_n = \int_{-\pi}^{\pi} h_n(t) \, dt, \forall n \geq 1. \) It is easy to verify that the systems \( \{H_n^+; H_{n+1}^-\}_{n \geq 0} \) and (2) are biorthonormed. Thus, for \( \alpha < \frac{1}{2q^\pi} \) the system (2) is minimal in \( L_{p^\alpha} \). The remaining cases for the values of \( \alpha \) are similarly proved.

Let \(-\frac{1}{2p^\pi} < \alpha < \frac{1}{2q^\pi}\). Take \( \forall f \in L_{p^\alpha} \) and consider

\[
S_m^0[f] = f_0^+ + \sum_{n=1}^{m} \left[ f_n^+ e^{-i\alpha t} e^{int} + f_n^- e^{i\alpha t} e^{-int} \right],
\]

where \( f_n^\pm \) are biorthogonal coefficients of the function \( f \) by the system (2).

Considering expression (10) for \( H_n^\pm \) it is easy to show that \( \|S_m^0(f) - f\|_{p^\alpha} \to 0, \quad m \to \infty \). This proves the basicity of the system (2) in the considered case. Thus, it is proved.

**Theorem 3.** Let \( p(t) \in H^{lin}, \quad p^- > 1 \). The system (2) forms a basis in \( L_{p^\alpha} \) iff \(-\frac{1}{2p^\pi} < \alpha < \frac{1}{2q^\pi}\). Moreover, it is complete in \( L_{p^\alpha} \) only for \( \alpha \geq -\frac{1}{2p^\pi} \); it is minimal iff \( \alpha < \frac{1}{2q^\pi} \). For \(-\frac{1}{2p^\pi} \leq \alpha \leq -\frac{1}{2q^\pi} \) it is complete and minimal, but doesn’t form a basis in \( L_{p^\alpha} \).

**References**

**Bilalov Bilal** - is professor, head of department of the Non-harmonic analysis of the Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences (ANAS). He got his Ph.D. and D. degrees in Moscow State University. Bilalov Bilal is a member of editorial board of Transactions of ANAS and Proceedings of IMM of ANAS since 2001, and referee of Math.Review since 2005. He is also a member of Editorial Board of "TWMS Journal of Pure and Applied Mathematics".

The research areas of Bilalov Bilal can be roughly listed as follows: theory of an approximation by polynomials of functions theory of basicity in the Hilbert and Banach spaces, and its application to concrete systems; research of basis properties of eigen-functions and adjoint-functions of some not self-conjugate differential operators.

**Huseynov Zafar** - was born in 1953 in Shusha city, Azerbaijan. He graduated from Faculty of Mechanic and Mathematics of Baku State University in 1975. At present he is a dean of Faculty of Mathematics in Sumgait State University.