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OPTIMAL CONTROL OF THE NEUMANN PROBLEM FOR THE NONLINEAR ELLIPTIC EQUATION WITHOUT THE DIFFERENTIABILITY OF THE CONTROL-STATE MAPPING

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Abstract. The optimal control problem for the nonlinear elliptic equation is considered. The corresponding control-state mapping is not Gataux differentiable. Necessary conditions of optimality are obtained by means of the extended derivatives theory.

Keywords: optimal control, nonlinear elliptic equation, control-state mapping, extended derivative.

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1. Problem statement

Necessary conditions of optimality and the gradient methods require the differentiation of the minimizing functional. The natural approach of the proving of these results is apparently the direct computing of the functional derivative. Then we receive the problem of control-state mapping differentiability. This property could be obtained with using of the Inverse Function Theorem or Implicit Function Theorem. Unfortunately the conditions of these theorems are transgressed for the large class of nonlinear infinite dimensional control systems. Besides, the corresponding inverse or implicit operator can be nondifferentiable in this case, although the direct operator is differentiable.

We consider as example the control systems described by Neumann problem for the nonlinear elliptic equation. If the parameter of nonlinearity and the dimension of the set are small enough, then the control-state mapping is continuously differentiable by the Inverse Function Theorem. Therefore the well-known results of optimal control theory for the nonlinear elliptic equations (see, for example, [1], [4], [6]-[9], [11]-[13], [15], [18]-[20], [22]-[27], [29], [30]) are applicable. However the conditions of the Inverse Function Theorem are disturbed in the general case. Besides the corresponding inverse operator is not Gataux differentiable. Then well-known methods become not applicable. Nevertheless the desired result could be attained by means of the weaker form of the operator derivative (see [24] and [25]).

Let Ω be an opened bounded set from $\mathbb{R}^n$ with a smooth boundary Γ. We consider the nonlinear elliptic equation

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial x_j})+a_0y+|y|^{\rho}y \;=\; f_1+v,\; x\in\Omega \tag{1}$$

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with the boundary condition

$$\frac{\partial y}{\partial \nu} \;=\; f_2,\; x \in \Gamma. \tag{2}$$

Here the known functions a_{ij}, a_0, f_1 and f_2 satisfies to the inclusions $a_{ij} \in\, C^1(\overline{\Omega})$, $a_0 \in\, C(\overline{\Omega})$, $f_1 \in L_2(\Omega)$, $f_2 \in H^{1/2}(\Gamma)$. Besides we shall assume the inequalities

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \;\ge\; \alpha|\xi|^2,\; \forall\xi \in \mathbb{R}^n,\; a_0(x) \ge \alpha,\; x \in \Omega,$$

where $\alpha > 0$. The parameter of nonlinearity ρ is positive. The conormal derivative is determined by

$$\frac{\partial y}{\partial \nu} \;=\; \sum_{i,j=1}^{n} a_{ij}\frac{\partial y}{\partial x_j}\cos(n, x_j),$$

where $cos(n, x_j)$ is the cosine of the exterior normal direction n to the surface Γ. The control v is choose from the nonempty convex closed subset U of the space $V = L_2(\Omega)$.

We consider the spaces $Y_1 = H^1(\Omega)$, $Y_2 = L_q(\Omega)$, $Y = Y_1 \cap Y_2$, where $q = \rho + 2$. The set Y is the reflexive Banach space with norm

$$\|y\|_Y \;=\; \|y\|_1 \,+\, \|y\|_2,$$

where $\|y\|_1$ and $\|y\|_1$ are the norms of the spaces Y_1 and Y_2. We determine the linear continuous operator $A_1 : Y_1 \to Y_1'$ and the nonlinear continuous operator $A_2 : Y_2 \to Y_2'$ by the equalities

$$\langle A_1 y, \lambda\rangle \;=\; \int\limits_{\Omega}(\sum_{i,j=1}^{n} a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial \lambda}{\partial x_i} \,+\, a_0 y\lambda)dx,\; \forall y, \lambda \in Y_1,\; A_2 y = |y|^\rho y,\; \forall y \in Y_2,$$

where $\langle \varphi, \lambda\rangle$ is the value of the linear continuous functional φ in the point λ. Let us determine also the operator $A : Y \to Y'$ by $A = A_1 + A_2$. The point $f \in Y'$ is given by

$$\langle f, \lambda\rangle \;=\; \int\limits_{\Omega}\lambda f_1 dx \,+\, \int\limits_{\Gamma}\lambda f_2 dx,\; \forall\lambda \in Y$$

using the Trace Theorem. Let operator B be the injection of the space V into Y'. Then the Neumann problem (1), (2) can be transform to the equation

$$Ay \;=\; Bv \,+\, f. \tag{3}$$

We have

$$\langle Ay - Az, y - z\rangle \;=\; \int\limits_{\Omega}[\sum_{i,j=1}^{n} a_{ij}\frac{\partial(y-z)}{\partial x_j}\frac{\partial(y-z)}{\partial x_i} \,+\, a_0(y-z)^2]dx +$$

$$+\int\limits_{\Omega}(|y|^\rho y - |z|^\rho z)(y - z)dx \;\ge\; \alpha\|y - z\|_1^2 \,+\, \|y - z\|_2^q,\; \forall y, z \in Y$$

according Green Formula and the properties of the coefficients of our equation. Then we obtain the strong monotony of the operator A. We have also

$$\langle Ay, y\rangle \;=\; \int\limits_{\Omega}[\sum_{i,j=1}^{n} a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial y}{\partial x_i} \,+\, a_0 y^2]dx \,+\, \int\limits_{\Omega}|y|^q dx \;\ge\; \alpha\|y\|_1^2 \,+\, \|y\|_2^q\; \forall y \in Y.$$

Hence

$$\frac{\langle Ay, y\rangle}{\|y\|_Y} \;\ge\; \frac{\alpha\|y\|_1^2 \,+\, \|y\|_2^q}{\|y\|_1 \,+\, \|y\|_2}\;\; \forall y \in Y.$$

We suppose now the convergence $\|y\|_Y \to \infty$. Then the norm of y for one of the composite of the space Y at least tends to infinity. Using the previous inequality we get $\langle Ay, y\rangle \|y\|_Y^{-1} \to \infty$. So the operator A is coercitive. Then we prove that the equation (3) and the boundary problem (1), (2) consequently have the unique solution $y = y(v)$ from the space Y for all $v \in V$ according the monotone operators theory (see [16], Chapter 2, Theorem 2.2). Besides the mapping is weak continuous. We determine the functional

$$I(v) = \frac{\chi}{2}\int\limits_{\Omega} v^2 dx + \int\limits_{\Omega} [y(v) - y_d^2] dx,$$

where $\chi > 0$ and y_d is a known function from the space $L_2(\Omega)$. We shall study the following optimal control problem:

(P) *Minimize* $I(v)$ *for all* $v \in U$.

We obtain the weak lower semicontinuity of the minimizing functional using the weak continuity of the control-state mapping. Then the optimal control problem (P) has a solution.

The optimal control problems for the systems described by the nonlinear elliptic equations are well-known enough (see, for example, [1], [4], [6]-[9], [11]-[13], [15], [18]-[20], [22]-[27], [29], [30]). Besides it seems even that harder problems are solved. For example the nonlinearity can be described by general function with larger velocity of the increment [1], [9]. It can include in the general part of the operator [4] or in the boundary conditions [9], [19]. The uniqueness of the state function can be absent [11], [18]. The degenerate equation [6], non local boundary conditions [20], underdetermined and overspecified boundary conditions [11], high order equations [13] are permitted. Problems with state constraints [7], [9], [23], domain optimization problems [26], minimax problems [22], vector optimization problems [27], problems without convexity of the state functional [4], [29] are considered. Problems with nonsmooth terms in the equation [12] are solved too, even for the more difficult parabolic case [5], [21], [28]. It is known not only qualitative but also quantitative methods for the considered problems (see, for example, [8], [15], [19], and [21]). Then it could seem the absence of actuality of the researching of the optimal control problem for the elliptic equation with concrete easy nonlinearity, usual quadratic functional and standard control constraints. However we suppose that known results do not cover the case of nondifferentiable corresponding inverse operator, i.e. control-state mapping. It is important that such property is typical even for the enough easy and natural problems. It is a motive of the choice of the Problem (P) as the object of researching.

2. Weak nonlinear system

The necessary condition for optimality of the Gataux differentiable functional I in the point v_0 on the convex set U is the variational inequality (see [17], Chapter 1, Theorem 1.3)

$$\langle I'(v_0), v - v_0\rangle \geq 0, \ \forall v \in U. \tag{4}$$

The natural means of its using in optimal control problems is the direct finding of the minimizing functional derivative. However a state functional depends usually on the state and the control directly. So the computing of the functional derivative requires the differentiability of the mapping $y(\cdot) : V \to Y$ according the Compodite Function Theorem. This dependence is determined by the equality

$$y(v) = A^{-1}(Bv + f) \tag{5}$$

because of (3). Then it will be sufficiently to substantiate the existence of the derivative of the inverse operator A^{-1} for the proof of the minimizing functional differentiability. The last property is guarantee by the Inverse Function Theorem (see [20], Chapter 2).

Let A be an operator from Banach space Y into Banach space Z. It is determined points y_0 and $z_0 = Ay_0$. We suppose that this operator is continuously differentiable on the neighborhood of the point and the derivative $A'(y_0)$ is reversible. Then the operator A^{-1} is differentiable by the Inverse Function Theorem. Besides it is satisfy to the equality

$$(A^{-1})'(z_0) \;=\; A'(y_0)^{-1}. \tag{6}$$

Hence the decisive phase of the proof of the minimizing functional differentiability is the reversibility of the derivative $A'(y_0)$. So it is necessary to prove, that the linearized equation

$$A'(y_0)y = z \tag{7}$$

has a unique solution $y \in Y$ for all $z \in Z$. Let us satisfy to the following assumption

$$n = 2 \text{ or } \rho \leq 4/(n-2) \text{ for } n \geq 3. \tag{8}$$

Lemma 2.1. *If the condition* (8) *is true for the operator A of the boundary problem* (1), (2), *then the conditions of the Inverse Function Theorem are right.*

Proof. The mapping $y \to |y|^\rho y$ is the Frechet differentiable operator from $L_q(\Omega)$ into $L_{q'}(\Omega)$ in the arbitrary point y_0 by Krasnosel'sky Theorem (see [14], p. 312). Besides its derivative maps the arbitrary point $y \in L_q(\Omega)$ in $(\rho+1)|y_0|^\rho y$. The considered mapping is continuous differentiable by means of the continuity of the last function with respect to y_0. Then the derivative of the operator A is determined by the equality

$$\langle A'(y_0)y, \lambda\rangle \;=\; \langle A_1 y, \lambda\rangle \;+\; (\rho+1)\int\limits_\Omega |y_0|^\rho y\lambda dx,\ \forall y, \lambda \in Y.$$

The corresponding unique solvability of the equation (7) is the corollary of the coercitivity of the derivative $A'(y_0)$ by Theorem 1.1 (see [17], Chapter 2).

We have

$$\langle A'(y_0)y, y\rangle \;=\; \sum_{i,j=1}^n \int\limits_\Omega (a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial y}{\partial x_i} \;+\; a_0 y^2)dx \;+\; (\rho+1)\int\limits_\Omega |y_0|^\rho y^2 dx \;\geq$$

$$\geq\; \alpha\|y\|_1^2 \;+\; (\rho+1)\int\limits_\Omega |y_0|^\rho y^2 dx,\ \forall y \in Y.$$

This condition is not guarantee the coercitivity of the considered operator not for a while yet because of the absence of the $L_q(\Omega)$ norm in the right side of the last inequality. However we get the continuous injection $H^1(\Omega) \subset L_q(\Omega)$ by classical Sobolev Theorem with assumption (8). Then we obtain $Y = Y_1$ and $Z = Y' = Y_1'$ consequently. Besides Y is the Hilbert space. Hence the linear continuous operator $A'(y_0) : Y \to Y'$ satisfies to the inequality

$$\langle A'(y_0)y, y\rangle \;\geq\; \alpha\|y\|_Y^2,\ \forall y \in Y.$$

Thus the equation (7) has a unique solution $y \in Y$ for all $z \in Y'$. The lemma is proved.

Lemma 2.2. *Let the condition* (8) *be true. Then the mapping $y(\cdot) : V \to Y$ for the boundary problem* (1), (2) *has the Frechet derivative in the arbitrary point $v_0 \in V$. Furthermore*

$$\langle \mu, y'(v_0)h\rangle \;=\; \langle B^* p(\mu), h\rangle\ \forall h \in V, \mu \in Y', \tag{9}$$

where the conjugate operator B^* is the injection from Y into V, and $p(\mu)$ is the solution of the conjugate equation

$$A'(y_0)^* p(\mu) = \mu. \tag{10}$$

Proof. We have

$$y'(v_0) \;=\; (A^{-1})'(Bv_0 + f)B \;=\; A'(y_0)^{-1}B$$

by the equality (5) and the differentiability of the inverse operator, where $y_0 = y(v_0)$. By Lemma 2.1 we get

$$\langle \mu, y'(v_0)h \rangle \;=\; \langle \mu, A'(y_0)^{-1}Bh \rangle \;=\; \langle [A'(y_0)^*]^{-1}\mu, Bh \rangle \;\forall h \in V, \mu \in Y'.$$

The equation (10) has a unique solution $p(\mu) \in Y$ for all $\mu \in Y'$ if the condition (8) is true because of the conjugate operator properties. The last formula can be transform to the equality (9). The lemma is proved.

Now it will not be so difficult to determine the derivative of the minimizing functional.

Lemma 2.3. *Let the condition* (8) *is true. Then the functional I for the problem* (P) *has the Gataux derivative in the arbitrary point $v_0 \in V$. Besides $I'(v_0) = \chi v_0 - p_0$, where p_0 is the solution of the boundary problem*

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ji}\frac{\partial p_0}{\partial x_j}\Big) + a_0 p_0 + (\rho+1)|y_0|^\rho p_0 \;=\; y_d - y_0,\; x \in \Omega, \tag{11}$$

$$\frac{\partial p_0}{\partial \nu^*} = 0,\; x \in \Gamma, \tag{12}$$

Here the conjugate conormal derivative is determined by formula

$$\frac{\partial y}{\partial \nu^*} \;=\; \sum_{i,j=1}^{n} a_{ji}\frac{\partial y}{\partial x_j}\cos(n, x_j).$$

We obtain in really the equality

$$\langle I'(v_0), h \rangle \;=\; \int_\Omega [\chi v_0 h + (y_0 - y_d) y'(v_0) h]dx,\; \forall h \in V$$

using Lemma 2.1.

We get the following conclusion after the substitution of the found functional derivative to the formula (4).

Theorem 2.1. *Let the condition* (8) *is true. Then the solution of the Problem* (P) *satisfies to the variational inequality*

$$\int_\Omega (\chi v_0 - p_0)(v - v_0)dx \geq 0,\; \forall v \in U. \tag{13}$$

Thus we have the system, which includes the state equations (1), (2), the conjugate equations (11), (12) and the variational inequality (13). We note that the proved theorem corresponds to the well known results of the optimal control theory for the nonlinear elliptic equations (see [1], [4], [6]-[9], [11]-[13], [15], [18]-[20], [22]-[27], [29], [30]). However the theorem 2.1 uses fundamentally the assumption (8). It guarantees that the solution of the boundary problem is included into the space, which is not depending from the nonlinear term of the equation. It is imply that the exclusion of the nonlinear term (i.e. operator A_2) does not change the corresponding functional space. We shall name such system *weak nonlinear*. Then the proof of the optimality conditions can be realized by standard methods using the differentiation of the control-state mapping, Inverse Function Theorem or Implicit Function Theorem in fact.

However the solution of the problem (P) exists without the condition (8), i.e. in the *strong nonlinear* case too. Thereupon it will be very interesting to analyze our problem in the general case without any additional conditions.

3. Strong nonlinear system

We can not guarantee the coercitivity of the operator derivative in sense of the space Y without the assumption (8). So the necessary properties of the linearized equation (7) get broken. Hence we do not have any possibilities to use the Inverse Function Theorem for the proof of the control-state mapping differentiability. Besides we have the following proposition.

Lemma 3.1. *If the injection $H^1(\Omega) \subset L_q(\Omega)$ gets broken, then the operator A^{-1} is not Gataux differentiable in some point $z_0 \in Y'$.*

Proof. We suppose on the contrary, that the inverse operator has the Gataux derivative D in the arbitrary point $z_0 \in Y'$. If $\sigma \to 0$, then we have the convergence $[A^{-1}(z_0+\sigma h)-A^{-1}z_0]/\sigma \to Dh$ in Y for all $h \in Y'$. We have $Ay_\sigma - Ay_0 = \sigma h$ where y_σ and y_0 are the solutions of the equation $Ay = z$ for $z = z_0 + \sigma h$ and $z = z_0$. So we obtain

$$A'(y_0)Dh = h. \tag{14}$$

Then the linearized equation

$$A'(y_0)y = h \tag{15}$$

has the solution $y = Dh$ from the space Y for all $h \in Y'$.

We choose the point z_0 such as the function y_0 became continuously. Then $\langle A'(y_0)y, \lambda\rangle$ has the sense for all functions $y, \lambda \in H^1(\Omega)$. Hence $A'(y_0)y$ includes into the set $Z = [H^1(\Omega)]'$ for all $y \in Y$. But this set is not equal to Y' without the injection $H^1(\Omega) \subset L_q(\Omega)$. Then the equation (15) can not have any solutions $y \in Y$ for $h \in Y' \setminus Z$ at least. However it contradicts the received earlier result. Hence the hypothesis of the differentiability of the inverse operator in the chosen point is false.

Nevertheless we shell prove, that the dependence of the state function with respect to the control has some property, which could be interpreted as the weak form of the Gataux differentiability.

Definition. (see [24] and [25]). *Let L be an operator from V into Y, and there exist linear topologic spaces V_0, Y_0, V_*, Y_* with continuous embeddings $V_* \subset V_0 \subset V$, $Y \subset Y_0 \subset Y_*$ and the linear continuous operator $D : V_0 \to Y_0$, such as $[L(v_0+\sigma h)-Lv_0]/\sigma \to Dh$ in Y_* for all $h \in V_*$ if $\sigma \to 0$. Then L is named (V_0, Y_0, V_*, Y_*)-extended differentiable in the point $v_0 \in V$.*

Domain and codomain of the standard derivatives (Gataux, Freshet, Hadamard, Sebastiaoe e Silva, Michael, Lamadrid, and other, see, for example, [3]) coincide exactly with domain and codomain of the considered operator. It does not depend from properties of this operator and the point of differentiation. These sets are distinguished in our case. Furthermore these spaces depend from corresponding operator and the point of differentiation. The extended derivative resemble to the Gataux one. But its domain V_0 is larger and the codomain Y_0 is narrower in comparison with given spaces V and Y. Besides, the passing to the limit is realized in the weaker topology of the space Y_* and for the narrower class of the directions h. It is obvious that the $(V, Y; V, Y)$-extended derivative is equal to the standard Gataux derivative. Then the extended derivative is the extension of the classical Gataux one.

We shall prove that the mapping $Y(\cdot) : V \to Y$ for the problem (1), (2) is extended differentiable in the arbitrary point $v_0 \in V$. We determine the set $Y_0 = \{p|\, p \in\ Y_1,\ |y(v_0)|^{\rho+2}p \in L_2(\Omega)\}$. It is Hilbert space with the scalar product

$$(\varphi, \lambda) = \alpha(\varphi, \lambda)_1 + (\rho + 1) \int\limits_{\Omega} |y(v_0)|^{\rho} \varphi\lambda dx,$$

where $(\varphi, \lambda)_1$ is the scalar product of corresponding points of the space Y_1. Its conjugate space is $Y_0' = \{\mu_1 + |y(v_0)|^{\rho}\mu_2|\, \mu_1 \in Y_1',\ \mu_2 \in L_2(\Omega)\}$.

Lemma 3.2. *The mapping $y(\cdot) : V \to Y$ for the boundary problem* (1), (2) *has the* (V, Y_0, V, Y_*)-*extended derivative in the arbitrary point $v_0 \in V$, such as*

$$\langle \mu, y'(v_0)h\rangle = \langle B^* p(\mu), h\rangle,\ \forall h \in V, \mu \in Y_0', \tag{16}$$

where $p(\mu)$ is the solution of the equation (10).

Proof. The difference $y_\sigma - y_0$ satisfies to the equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}\frac{\partial(y_\sigma - y_0)}{\partial x_j}] + a_0(y_\sigma - y_0) + (g_\sigma)^2(y_\sigma - y_0) \ =\ \sigma h,\ x \in \Omega$$

with homogenous boundary conditions for all $h \in V$ and number σ, where $y_0 = y(v_0)$, $y_\sigma = y(v_0 + \sigma h)$, $(g_\sigma)^2 = (\rho + 1)|y_0 + \varepsilon(y_\sigma - y_0)|^{\rho}$, $\varepsilon \in [0, 1]$. We determine the space $Y_\sigma = \{y|\, y \in Y_1,\ g_\sigma y \in L_2(\Omega)\}$. It is equal to Y_0 for $\sigma = 0$. It is the Hilbert space with the scalar product

$$(y, p) = (y, p)_1\ +\ \int\limits_{\Omega} (g_\sigma)^2 ypdx.$$

The corresponding conjugate space is $Y_\sigma' = \{\mu_1 + g_\sigma\mu_2|\, \mu_1 \in Y_1',\ \mu_2 \in L_2(\Omega)\}$. We determine the linear continuous operator $L_\sigma : Y_\sigma \to Y_\sigma'$ by the equality

$$\langle L_\sigma\varphi, \lambda\rangle\ =\ \langle A_1\varphi, \lambda\rangle\ +\ \int\limits_{\Omega}(g_\sigma)^2\varphi\lambda dx,\ \forall\varphi, \lambda \in Y_\sigma.$$

Then we have

$$\langle L_\sigma(y_\sigma - y_0), \lambda\rangle\ = \sigma\, \langle B^*\lambda, h\rangle,\ \forall\lambda \in Y_\sigma. \tag{17}$$

We consider the equation

$$(L_\sigma)^* p\ =\ \mu. \tag{18}$$

It is equal to the equation (10) for $\sigma = 0$. We obtain

$$\langle (L_\sigma)^* p, p\rangle\ =\ \int\limits_{\Omega}[\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j} + a_0p^2]dx\ +\ \int\limits_{\Omega}|g_\sigma p|^2 dx\ \geq$$

$$\geq\ \alpha\|p\|_1^2\ +\ \|g_\sigma p\|^2_{L_2(\Omega)}\ =\ \|p\|^2_{Y_\sigma},\ \forall p \in Y_\sigma.$$

Then the equation (18) has a unique solution $p = p_\sigma(\mu)$ from the set Y_σ for all number σ and function $\mu \in Y_\sigma'$.

Let $\sigma \to 0$. Then we have the convergence $y_\sigma \to y_0$ weakly in Y by the weak continuity of the mapping $Y(\cdot) : V \to Y$. Hence we receive the boundedness of in space $L_q(\Omega)$. So $\{g_\sigma\}$ is bounded in $L_{2q/\rho}(\Omega)$. From the equation (18) we have the inequality

$$\alpha\|p_\sigma(\mu)\|_1^2\ +\ \|g_\sigma p_\sigma(\mu)\|^2_{L_2(\Omega)}\ \leq\ |\langle\mu, p_\sigma(\mu)\rangle|.$$

So we obtain the estimates

$$\sup_{\mu\in M}\|p_\sigma(\mu)\|_1\ \leq 1,\ \sup_{\mu\in M}\|g_\sigma p_\sigma(\mu)\|_{L_2(\Omega)}\ \leq 1,$$

where $M = \{\mu \in Y_1' |\, \|\mu\| = 1\}$. Using the Holder Inequality we get

$$\sup_{\mu\in M} \|(g_\sigma)^2 p_\sigma(\mu)\|_{L_{q'}(\Omega)} \le \|g_\sigma\|_{L_{2q/\rho}(\Omega)} \sup_{\mu\in M} \|g_\sigma p_\sigma(\mu)\|_{L_2(\Omega)}.$$

Hence we obtain the boundedness of $\{p_\sigma(\mu)\}$ in space Y_1 and $\{(g_\sigma)^2 p_\sigma(\mu)\}$ in $L_{q'}(\Omega)$ uniformly with respect to $\mu \in M$.

So we have the convergence $y_\sigma \to y_0$ and $p_\sigma(\mu) \to r(\mu)$ weakly in Y_1 uniformly with respect to $\mu \in M$ after the extraction of subsequences. By Rellich - Kondrashov Theorem we decide that this convergence is realized also strongly in $L_2(\Omega)$ and a.e. on Ω. Then $(g_\sigma)^2 p_\sigma(\mu) \to (\rho+1)|y_0|^\rho r(\mu)$ a.e. uniformly with respect to μ. Hence (see [16], Lemma 1.3, Chapter 1) we obtain the uniform convergence $(g_\sigma)^2 p_\sigma(\mu) \to (\rho+1)|y_0|^\rho r(\mu)$ weakly in $L_{q'}(\Omega)$. Using (18) we have

$$\langle (L_\sigma)^* p_\sigma(\mu), \varphi\rangle = \langle \mu, \varphi\rangle, \ \forall \varphi \in Y.$$

Therefore we get

$$\langle A'(y_0)^* r(\mu), \varphi\rangle = \langle \mu, \varphi\rangle, \ \forall \varphi \in Y$$

after the passing to the limit. Then $r(\mu) = p(\mu)$. It is obvious, that the equality (16) really determines the linear continuous mapping $y'(v_0) : V \to Y_0$. We obtain

$$\langle \mu, y(v_0+\sigma h) - y(v_0)\rangle = \sigma \langle B^* p_\sigma(\mu), h\rangle \tag{19}$$

after the determination $\lambda = p_\sigma(\mu)$ in the equality (17). From (16) and (19) we deduce

$$\|[y(v_0+\sigma h) - y(v_0)]/\sigma - y'(v_0)h\| = \sup_{\mu\in M} |\langle \mu, [y(v_0+\sigma h) - y(v_0)]/\sigma - y'(v_0)h\rangle| =$$

$$= \sup_{\mu\in M} |\langle B^*[p_\sigma(\mu) - p(\mu)], h\rangle|, \ \ \forall \mu \in Y_0', h \in V.$$

Using the passing to limit we conclude, that the operator $y'(v_0)$ is the extended derivative of the investigated mapping in fact. The lemma is proved. It is obvious, that the extended derivative coincides with Gataux one, if the condition (8) is true. So the formula (16) can be transform to (9). Then Lemma 2.2 becomes the corollary from the last proposition. Let us note, that $(V, L_2(\Omega); V, L_2(\Omega))$-extended differentiability of the control-state mapping has as corollary the Gataux differentiability of the minimizing functional. So by Lemma 3.2 we obtain, that our functional is Gataux differentiable without the condition (8) too. Then we get the following proposition.

Theorem 3.1. *The results of the Theorem* 2.1 *are true without the condition* (8) *too.*

Thus the necessary conditions of optimality for the Problem (P) can be proved in strong nonlinear case too, i.e. without any limitations on the dimension of the set and the parameter of the nonlinearity. It is significant that these restrictions are not use for the proof of the existence of the optimal control. We note also that these results could not obtained on the basis of the known methods of optimization for the systems described by nonlinear elliptic equations (see, for example, [1] – [19]), because it depends really on the classical operator derivatives.

However it is known the necessary conditions of optimality for larger class of elliptic equations. For example equations could include arbitrary nonlinear functions with larger velocity of the nonlinearity increasing [1], [9]. Furthermore the similar results are known for the more difficult parabolic equations (see [28], p. 35). Necessary conditions of optimality were obtained even for the abstract equations with monotone operators (see, for example, [30] and [21]). But the proof of the last results requires necessarily the corresponding assumptions, which are similar to the known limitations for the concrete equations. It seems only the formalization of the known results but not its qualitative generalization. The control in [1] and [9] was chosen from the space L_∞. Then the state function becomes smooth enough. Thus the assertions of the Lemma 3.1 are broken, and the control-state mapping becomes Gataux differentiable. So the necessary conditions of optimality can be proved by means of the classical operator derivatives. The state function of the cited paper [28] (see also the earlier paper of Barbu [5]) was chosen from a Hilbert space. Thereby the difficulties of the Lemma 3.1 are leaven out too. The mentioned difficulties were avoided, but it not was overcome in these results. The solution of boundary problems was chosen so regular, that the control-state mapping became in fact differentiable in classical sense. But it requires with necessity the additional assumptions, specifically smoothness of the control, data, coefficients, and boundary. However we prefer to consider the boundary problem with natural functional spaces, which correspond to the very easy a priori estimates. The using of the additional restrictions for the increase of the state regularity seems undesirable in this situation, because it is not necessary for the existence of the optimal control. If the state was include into the natural spaces then it was used without fail the assumptions of the dimension of the set and the parameter of the nonlinearity (see, for example, [11], Chapter 2, Theorem 7.1; [18], Chapter 3, Theorem 2.2). The similar limitations were used in the evolution case too (see, for example, [11], Chapter 2, Theorems 8.1 and 8.2; [18], Chapter 1, Theorem 5.5 and Chapter 2, Theorem 2.1; [21], Chapter 4, p. 184). These results are analogous to our weak nonlinear case with classical differentiability of the control-state mapping. Thus the difference of our outcomes from well known results is the solution of the optimal control problem with really Gataux undifferentiated control-state mapping.

It could be suppose that our difficulties would be able overcome by means of the nonsmooth analysis (subdifferentiation, Clarke derivative, some others, see, for example, [10]) or with using of the smooth approximation of the systems (see, for example, [5], [21], [28]). These methods are used with success if the problem statement includes nonsmooth terms, for example, the absolute value of a function or the maximum of functions. However our non differentiability has the another sense. It is caused of the absence of topological properties, which are used in the determination of the operator derivatives. The nonsmooth analysis and smooth approximation methods seem inapplicable in our situation because it intends for other difficulties. Particularly the using of the Clarke derivatives will be apparently ineffective because of the absence of the effective nonsmooth infinite dimensional analogues of the Inverse Function Theorem and Implicit Function Theorem. The methods from [5], [21], [28] use the approximation of the nonsmooth terms, proving of the necessary conditions of optimality for the approximation problem and the passing to the limits. However the nonlinear term of our equation is Frechet differentiable by Krasnosel'sky Theorem, but the inverse operator is not differentiable. The citing results use the approximation of the known direct operator, but not unknown inverse operator. We do not understand, how could be approximate an unknown operator. It seems that the proposed method allows solving the optimal control problems, which could not be solve by means of the known results.

4. Other problems

We shall take advantage of the described technique for the solution of other optimal control problems. We consider the equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial x_j}) + a_0 y + |y|^\rho y \,=\, f_1,\ x \in \Omega \tag{20}$$

with the boundary condition

$$\frac{\partial y}{\partial \nu} = f_2 + v, \ x \in \Gamma. \tag{21}$$

Here the known functions have the former properties, but the control v is a point of nonempty convex closed set U from $V = L_2(\Gamma)$. We determine the operator A as before. The linear continuous operator $B : V \to V'$ now is characterized by the equality

$$\langle Bv, \lambda \rangle = \int\limits_{\Omega} \lambda v dx, \ \forall \lambda \in Y.$$

Then the boundary problem (20), (21) is transformed to the equation (3) too. Then this problem has a unique solution $y = y(v)$ from the space Y for all $v \in V$. Besides the mapping is weak continuous. We determine the cost functional

$$I(v) = \frac{\chi}{2} \int\limits_{\Gamma} v^2 dx + \int\limits_{\Omega} [y(v) - y_d^2] dx,$$

where $\chi > 0$ and y_d is a known function from the space $L_2(\Omega)$. We shall study the following optimal control problem:

(P_1) *Minimize* $I(v)$ *for all* $v \in U$.

The properties of the minimizing functional of the Problem (P) and (P_1) are the same. Therefore the Problem (P_1) is resolvable. We shell use the variational inequality (4) for the deducing of the necessary conditions of optimality once again. It is necessary to find the functional derivative for it. Toward this end we shall try to find the derivative of the control-state mapping. By analogy with Lemma 2.1 it is not very difficult to prove the differentiability of this dependence in classical sense according with Inverse Function Theorem if the condition (8) is true. But we can not obtain the analogical result in the general case because the operator A^{-1} is not Gataux differentiable by Lemma 3.1.

Then we shell obtain the extended differentiability of the mapping $y(\cdot) : V \to Y$ in the arbitrary point $v_0 \in V$ by analogy with Lemma 3.2. Let $p(\mu)$ be the solution of the equation (10) with function $y_0 = y(v_0)$ which satisfies to (20), (21).

Lemma 4.1. *The mapping* $y(\cdot) : V \to Y$ *for the system* (20), (21) *has the* $(V, Y_0; V, Y_1)$*-extended derivative, characterized by the equality* (16) *in the arbitrary point* $v_0 \in V$. *Here the space* Y_0 *is determined by Lemma 3.2, and the operators* A *and* B *correspond to the Problem* (P_1).

Proof. We have the equalities

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}\frac{\partial (y_\sigma - y_0)}{\partial x_j}] + a_0(y_\sigma - y_0) + (g_\sigma)^2 (y_\sigma - y_0) = 0, \ x \in \Omega,$$

$$\frac{\partial (y_\sigma - y_0)}{\partial \nu} = \sigma h, \ x \in \Gamma$$

with previous symbols for all function $h \in V$ and number σ. Then the equality (17) is true, but B^* is the superposition of the trace operator γ from Ω to the boundary Γ and the injection of the space $H^{1/2}(\Gamma)$ into V. We consider the equation (18) with the state function which is the solution of the boundary problem (20), (21). It has a unique solution $p = p_\sigma(\mu)$ from the set Y_σ for all number σ and point $\mu \in Y'_\sigma$. Besides if $\sigma \to 0$ then $p_\sigma(\mu) \to p(\mu)$ weakly in Y_1 uniformly with respect to $\mu \in M$. So we obtain $\gamma p_\sigma(\mu) \to \gamma p(\mu)$ weakly in $H^{1/2}(\Omega)$ according to the Trace Theorem. Then the equality (16) characterizes the linear continuous operator $y'(v_0) : V \to Y_0$ in fact. We determine $\lambda = p_\sigma(\mu)$ in the equality (17). So we obtain the formula (19) once again. The proof is finished also, as well as in Lemma 3.2.

Now we obtain immediately the differentiability of the minimizing functional.

Lemma 4.2. *The functional I for the problem (P_1) has Gataux derivative $I'(v_0) = \alpha v_0 - \gamma p_0$ in the arbitrary point $v_0 \in V$, where p_0 is the solution of the boundary problem* (11), (12).

We substitute the founded derivative to the variational inequality (2). So we obtain the necessary condition of optimality.

Theorem 4.1. *The solution of the problem (P_1) satisfies to the condition*

$$\int\limits_{\Gamma} (\chi v_0 - p_0)(v - v_0)dx \geq 0, \; \forall v \in U. \tag{22}$$

We consider now the problem with the nonlinear term in the boundary condition but not in the equation. We have the following boundary problem:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial x_j}) + a_0 y = f_1 + v, \; x \in \Omega \tag{23}$$

with the boundary condition

$$\frac{\partial y}{\partial \nu} + |y|^{\rho} y = f_2, \; x \in \Gamma \tag{24}$$

with previous symbols. The known functions and the control have the same properties as in the Problem (P).

We determine the spaces $Y_2 = \{y| \, \gamma y \in L_q(\Gamma)\}, Y = Y_1 \cap Y_2$, where q and Y_1 were determined before. The set Y is the reflexive Banach space with the norm

$$\|y\|_Y = \|y\|_1 + \|\gamma y\|_{L_q(\Gamma)}.$$

The nonlinear continuous operator $A_2 : Y_2 \to Y_2'$ is characterized by $A_2 y = |\gamma y|^{\rho} \gamma y$. We determine the mapping $A : Y \to Y'$ by equality

$$\langle Ay, \lambda \rangle = \langle A_1 y, \lambda \rangle + \int\limits_{\Gamma} \lambda A_2 y dx, \; \forall y, \lambda \in Y.$$

Let operator B and the point f be assigned also, as well as in the Problem (P). Then the boundary problem (23), (24) transforms to the equation (3). Besides the operator A is strong monotone and coercitive once again. So the problem (23), (24) has a unique solution $y = y(v)$ from Y, and the mapping $y(\cdot) : V \to Y$ is weak continuously as before. We determine the minimizing functional also, as well as in the Problem (P). Then we obtain the following optimal control problem:

(P_2) *Minimize $I(v)$ for the system* $(23), (24)$ *for all* $v \in U$.

Using the weak continuity of the control-state mapping we get the weak lower semicontinuity of the minimizing functional. Then we prove the existence of the optimal control. We require the differentiability of the state function for the conditions of optimality. It is connected with the differentiation of inverse operator A^{-1}. We could prove easily its Gataux differentiability according the Inverse Function Theorem (see Lemma 2.2), if the injection $H^{1/2}(\Gamma) \subset L_q(\Gamma)$ is true. But we have the following proposition:

Lemma 4.3. *If the injection $H^{1/2}(\Gamma) \subset L_q(\Gamma)$ gets broken, then the operator A^{-1} for the boundary problem* $(23), (24)$ *is not Gataux differentiable in a point $z_0 \in Y'$.*

Proof. The operator is continuous differentiable, besides the corresponding derivative in the arbitrary point $y_0 \in Y$ is characterized by the equality

$$\langle A'(y_0)y, \lambda\rangle \;=\; \langle A_1 y, \lambda\rangle \;+\; (\rho+1)\int\limits_{\Gamma} |y_0|^{\rho} y\lambda dx,\; \forall y, \lambda \in Y.$$

If this operator has a Gataux derivative D in the point $z_0 \in Y'$ then the equality (14) is true. So the linearized equation (15) has the solution $y = Dh$ from Y for all $h \in Y'$. We choose the point z_0 such as the corresponding function y_0 becomes continuous. Then the term $\langle A'(y_0)y, \lambda\rangle$ has the sense for all functions $y, \lambda \in H^1(\Omega)$. Therefore the value $A'(y_0)y$ includes into the set $Z = [H^1(\Omega)]'$ for all $y \in Y$, but Z is not equal to Y' without the injection $H^{1/2}(\Gamma) \subset L_q(\Gamma)$. Hence the equation (15) can not have any solutions $y \in Y$ for $h \in Y' \setminus Z$ at least. However it contradicts the obtained earlier result. Then the hypothesis of the differentiability of the inverse operator in the chosen point is false. The lemma is proved.

We shell obtain the extended differentiability of the dependence of the control-state mapping in the arbitrary point $v \in V$. We determine the space $Y_0 \;=\; \{p|\, p \in \; Y_1, \gamma[|y(v_0)|^{\rho/2}p] \in L_2(\Gamma)\}$. It is the Hilbert space with the scalar product

$$(\varphi, \lambda) \;=\; \alpha(\varphi, \lambda)_1 \;+\; (\rho+1)\int\limits_{\Gamma} |y(v_0)|^{\rho} \varphi\lambda dx.$$

Lemma 4.4. *The mapping $y(\cdot) : V \to Y$ for the problem* (24), (25) *has the* $(V, Y_0; V, Y_1)$-*extended derivative in the arbitrary point $v_0 \in V$, characterized by the equality* (16), *where the function $p(\mu)$ is the solution of the equation* (10) *with the derivative of the operator A, described by Lemma* 4.3.

Proof. We have the following equalities

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}\frac{\partial (y_\sigma - y_0)}{\partial x_j}] + a_0(y_\sigma - y_0) \;=\; \sigma h,\; x \in \Omega,$$

$$\frac{\partial (y_\sigma - y_0)}{\partial \nu} + (g_\sigma)^2 (y_\sigma - y_0) \;=\; 0,\; x \in \Gamma$$

for all function $h \in V$ and number σ. We determine the space $Y_\sigma \;=\; \{p|\, p \in Y_1,\; \gamma(g_\sigma p) \in L_2(\Gamma)\}$. It is equal to Y_1 for $\sigma = 0$. We have the Hilbert space with the scalar product

$$(\varphi, \lambda) \;=\; \alpha(\varphi, \lambda)_1 \;+\; \int\limits_{\Gamma} (g_\sigma)^2 \varphi\lambda dx.$$

Let $L_\sigma : Y_\sigma \to Y'_\sigma$ be the linear continuous operator determined by

$$\langle L_\sigma \varphi, \lambda\rangle \;=\; \langle A_1\varphi, \lambda\rangle \;+\; \int\limits_{\Gamma} (g_\sigma)^2 \varphi\lambda dx, \forall \varphi,\; \lambda \in Y_\sigma.$$

It satisfies also to the condition (17).

We have the inequality

$$\langle L_\sigma p, p\rangle \;\geq\; \alpha\|p\|_1^2 \;+\; \|\gamma(g_\sigma p)\|^2_{L_2(\Gamma)} \;=\; \|p\|^2_{Y_\sigma},\; \forall \varphi \in Y_\sigma.$$

Then the equation (18) with corresponding operator L_σ has a unique solution $p = p_\sigma(\mu)$ from set Y_σ for all number σ and point $\mu \in Y'_\sigma$.

If $\sigma \to 0$ then we have the convergence $y_\sigma \to y_0$ weakly in Y. Then $\{\gamma y_\sigma\}$ is the bounded set of the space $L_q(\Gamma)$ and $\{\gamma g_\sigma\}$ is bounded in $L_{2q/\rho}(\Gamma)$. Using the described methods we obtain the inequality

$$\alpha\|p_\sigma(\mu)\|_1^2 \;+\; \|\gamma[g_\sigma p_\sigma(\mu)]\|^2_{L_2(\Gamma)} \;\leq\; |\langle \mu, p_\sigma(\mu)\rangle|.$$

Hence we get the estimates

$$\sup_{\mu \in M} \|p_\sigma(\mu)\|_1 \le 1, \ \sup_{\mu \in M} \|\gamma[g_\sigma p_\sigma(\mu)]\|_{L_2(\Gamma)} \le 1.$$

Therefore $\{p_\sigma(\mu)\}$ and $\{\gamma[(g_\sigma^2 p_\sigma(\mu)]\}$ are the bounded (uniformly with respect to $\mu \in M$) sets of the spaces Y_1 and $L_{q'}(\Gamma)$ correspondingly. Then we obtain the weak in the sense of uniformly convergence $y_\sigma \to y_0$ and $p_\sigma(\mu) \to r(\mu)$. So we have $\gamma y_\sigma \to \gamma y_0$ and $\gamma p_\sigma(\mu) \to \gamma r(\mu)$ weakly in $H^{1/2}(\Gamma)$ according to the Trace Theorem. Using the compact injection of $H^{1/2}(\Gamma)$ into $L_2(\Gamma)$ we conclude, that this convergence is true also with respect to the strong topology of $L_2(\Gamma)$ and a.e. in Γ. Hence $\gamma[(g_\sigma)^2 p_\sigma(\mu)] \to (\rho+1)\gamma[|y_0|^\rho r(\mu)]$ a.e. in Γ uniformly with respect to $\mu \in M$. Then the last convergence is true also in sense of weak topology of $L_{q'}(\Gamma)$. We obtain $r(\mu) = p(\mu)$ after the proceeding to the limit in the equality (18). The proof is finished also, as well as in Lemma 3.2.

The differentiability of minimizing functional is proved by analogy with Lemma 3.3.

Lemma 4.5. *The functional I for the Problem (P_2) has Gataux derivative $I'(v_0) = \chi v_0 - p_0$ in the arbitrary point $v_0 \in V$, where p_0 is the solution of the boundary problem*

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ji}\frac{\partial p_0}{\partial x_j}) + a_0 p_0 = y_d - y_0, \ x \in \Omega, \tag{25}$$

$$\frac{\partial p_0}{\partial \nu^*} + (\rho+1)|y_0|^\rho p_0 = 0, \ x \in \Gamma. \tag{26}$$

The necessary conditions of optimality are obtained after the substitution of the derivative of the cost functional to (4).

Theorem 4.2. *The solution of the Problem (P_2) satisfies to the variational inequality*

$$\int_\Omega (\chi v_0 - p_0)(v - v_0)dx \ge 0, \ \forall v \in U, \tag{27}$$

where p_0 is the solution of the boundary problem (25), (26).

We consider again the control system described by the boundary problem (1), (2) with boundary observation. The minimizing functional is determined particularly by the equality

$$I(v) = \frac{\chi}{2}\int_\Omega v^2 dx + \frac{1}{2}\int_\Gamma [y(v) - y_d]^2 dx,$$

where $\chi > 0$ and y_d is a function from $L_2(\Gamma)$. We have the following optimal control problem:

(P_3) *Minimize $I(v)$ for all $v \in U$.*

This problem has a solution obviously. We shell prove the differentiability of its functional.

Lemma 4.6. *The functional I for the Problem (P_3) has Gataux derivative $I'(v_0) = \chi v_0 - p_0$ in the arbitrary point $v_0 \in V$, where p_0 is the solution of the boundary problem*

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ji}\frac{\partial p_0}{\partial x_j}) + a_0 p_0 + (\rho+1)|y_0|^\rho p_0 = 0, \ x \in \Omega, \tag{28}$$

$$\frac{\partial p_0}{\partial \nu^*} = y_d - y_0, \ x \in \Gamma. \tag{29}$$

Proof. Using Lemma 3.2 we obtain $[y(v_0 + \sigma h) - y(v_0)]/\sigma \to y'(v_0)h$ in Y_1 for all $h \in V$ if $\sigma \to 0$. So we have $\gamma[y(v_0 + \sigma h) - y(v_0)]/\sigma \to \gamma[y'(v_0)h]$ in $H^{1/2}(\Gamma)$ according the Trace

Theorem. Hence

$$\langle I'(v_0), h\rangle = \int\limits_{\Omega} \chi v_0 h dx + \int\limits_{\Gamma} (y_0 - y_d) y'(v_0) h dx, \ \forall h \in V.$$

We determine the point $\mu_0 \in Y_1'$ by

$$\langle \mu_0, \lambda\rangle = \int\limits_{\Gamma} (y_0 - y_d)\lambda dx, \ \forall \lambda \in Y_1.$$

Then the equation (10) become equal to the boundary problem (28), (29) i.e. $p(\mu_0) = p_0$. So the functional derivative is determined by the equality

$$\langle I'(v_0), h\rangle = \int\limits_{\Omega} (\chi v_0 - p_0) h dx, \ \forall h \in V,$$

and the statements of the lemma are true. Hence we obtain the following result:

Theorem 4.3. *The solution of the problem* (P_3) *satisfies to the variational inequality* (13)*, where* p_0 *is the solution of the boundary problem* (28), (29).

5. Conclusions

The control, observation and nonlinearity are distributed for the Problem (P). One of these values is boundary, and another are distributed for the Problems (P_1), (P_2) and (P_3). We could consider obviously the optimal control problems if all or two of these properties are boundary. The systems with two controls (distributed and boundary), observations or nonlinearities could be analyzed analogously. The control could be determined only in the part of the set Ω or its boundary. The power-mode nonlinear function can be replaced by an arbitrary continuous coercitive monotone function with limited velocity of increasing. We note that all specific properties of the concrete nonlinearity become apparent directly. On the contrary many results could be obtained for the general nonlinearity with appropriate assumptions. So the using of the concrete nonlinearity helps better to comprehend nonlinear effects. The minimizing functional could be not only quadratic, but the arbitrary weak lower semicontinuous integral which is differentiable with respect to the state and the control separately. It is significant that the corresponding control-state mapping is extended differentiable but not Gataux differentiable in general case. Therefore the standard methods for the optimal control systems described by nonlinear elliptic equations which use the classical operator derivatives are not applicable for these problems.

The extended differentiability theory permits to obtain the more exact properties of the nonlinear differential partial equations. Particularly the classical differentiability theory affirms that the properties of the dependence of the solution with respect to the absolute term of the equation changes with a jump by the increasing of the parameter of nonlinearity. If the condition (8) is true, then this dependence is differentiable according Lemma 2.2. But it becomes not differentiable by the increasing of the parameter ρ by Lemma 3.1, if the injection $H^1(\Omega) \subset L_{\rho+2}(\Omega)$ becomes false. However Lemma 3.2 declares that this dependence is always extended differentiable. But the spaces which are used in the determination of the extended derivative depend from ρ essentially. Particularly these spaces equal to the natural spaces for the small ρ. Then the extended derivative becomes equal to the classical one. But these objects become differing after the increasing of this parameter if the indicate injection gets broken. This distinction is the more strongly, than more than parameter of nonlinearity. Hence the properties of the investigated dependence change continuously, but not with jump according the extended differentiability theory. We note also, that the spaces used in the derivative definition do not depend on individual features of the operator and point of differentiation in the classical differentiability theory. It is characterized only on spaces, in which the operator is determined. In our case those spaces take account of the individual properties of the operator and the point of differentiation.

It is possible to prove a general theorem of the extended differentiability of the inverse operator with weak restrictions for the linearized equation (see [24] and [25]). Then Theorems 3.1, 4.1, 4.2, 4.3 become its corollaries just as Theorem 2.1 is the corollary of the classical inverse function theorem.

We could determine extended analogues of other operator derivatives (Frechet and other), including the operators in linear topological spaces. Correlations between different forms of the extended derivatives will be similar to the properties of the standard derivatives [3].

We could use the extended differentiation along with results of the abstract extremum theory (see, for example, [11], Chapter 2, Subsection 1) for proving the necessary conditions of optimality for problems with more difficult constraints. If the control is included nonlinearly to the system (for example, in the coefficients of the state operator), then we could replace the Inverse Function Theorem with Implicit Function Theorem. It is possible in classical and extended cases. The analogical results could be obtained for other nonlinear infinite dimensional control systems, for example, for the nonlinear parabolic equations.

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