Proceedings of IAM, V.5, N.2, 2016, pp.156-174

THE FINITENESS OF THE NUMBER OF EIGENVALUES OF AN HAMILTONIAN IN FOCK SPACE

Tulkin H. Rasulov[1]

[1]Bukhara State University, 11 M. Ikbol str., Bukhara, 200100, Uzbekistan
e-mail: rth@mail.ru

Abstract. We consider a model operator A associated with the Hamiltonian of a system describing three particles in interactions, without conservation of the number of particles on a d-dimensional lattice. We precisely describe the location and structure of the essential spectrum of A by the spectra of three families of the generalized Friedrichs models. We obtain a symmetric version of the Weinberg equation for eigenvectors of A and find the sufficient conditions for the finiteness of the discrete spectrum of A.

Keywords: Hamiltonian, Fock space, annihilation and creation operators, generalized Friedrichs model, essential and discrete spectra.

AMS Subject Classification: 35P20, 47N50, 81Q10.

1. Introduction

The main goal of the present paper is to find the conditions which guarantee the finiteness of the number of discrete eigenvalues located in the below of the bottom of the three-particle branch of the essential spectrum of the model operator (Hamiltonian) A in Fock space. The model operator A is associated with the lattice system describing three particles in interactions, without conservation of the number of particles. Such systems are naturally occur in the theory of solid-state physics [10], quantum field theory [5] and statistical physics [8,9]. Often, the number of particles can be arbitrary large as in cases involving photons (see e.g. [4]), in other cases, such as scattering of spin waves on defects, scattering massive particles and chemical reactions, there are only participants at any given time, though their number can be change. Recall that the study of systems describing n particles in interaction, without conservation of the number of particles can be reduced to the investigation of the spectral properties of self-adjoint operators acting in the n-particle cut subspace of the Fock space [5,9,10,16]. The spectral properties of such Hamiltonians in Fock space are studied in [6,9,22] for continuous case and [3,11,13,14,20] for discrete case.

The problem of the finiteness of the number of eigenvalues of the systems with a fixed number of particles has been studied by many articles. The first mathematical result on the finiteness of the discrete spectrum of Schroedinger operators for general interactions was obtained by Uchiyama in [17]. Under natural assumptions on the potential the essential spectrum of the continuous Schroedinger operator A_c coincides with the half-axis $[\kappa;\infty)$, $\kappa \le 0$. In [18,21] it was shown that for $\kappa < 0$ and a sufficiently rapid decrease of the interactions the discrete spectrum of A_c is actually finite. In the case $\kappa = 0$ the finiteness of the discrete spectrum of A_c with certain decreasing interactions was established in [19]. The authors of [1] used the Faddeev and Weinberg type equations and an expansion of the Fredholm determinant to prove the finiteness of the number of eigenvalues of the three-particle discrete Schroedinger operators A_l with pair contact interactions when the corresponding two-particle discrete Schroedinger operators have no virtual levels. The Birman-Schwinger principle was used in [7] to prove that the discrete spectrum of the operator A_l describing systems of three particles (two bosons and the third particle of a different nature) is finite. In all mentioned above papers devoted to the finiteness of lattice Hamiltonians, the case of dimension three, that is, $d = 3$, have been considered.

We note that in the continuous case, considered in [9,22], the "two-particle" and "three-particle" branches of the continuous spectrum are given by half-lines and overlap. In lattice case, in contrast to the continuous case, the "two-particle" and "three-particle" branches of the essential spectrum of the model operator A fill finite-length segments and might not overlap. In the present paper we construct a symmetric version of the Weinberg type equation for the eigenvectors of A and it is used to prove the finiteness of the number of discrete eigenvalues located in the below of the bottom of the three-particle branch of the essential spectrum of A. Our method can also be used in the case where the essential spectrum of A has a gap and $d \geq 3$.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the operator matrix A is described as a bounded self-adjoint operator in the direct sum of the zero-, one- and two-particle subspaces of the Fock space and the main results are formulated. In Section 3, we prove some auxiliary lemmas. In Section 4, we obtain a symmetric version of the Weinberg equation for the eigenvectors of A. Section 5 is devoted to the proof of the main results. At the end of this section provided an interesting example satisfying all technical assumptions.

We remark that if the diagonal element A_{22} of A is the multiplication operator, then this operator was studied in [3]. Here the existence of infinitely many eigenvalues (resp. the finiteness of eigenvalues) below the bottom of the essential spectrum of A was proved for the case where the associated Friedrichs model has a threshold energy resonance (resp. a threshold eigenvalue). The location of the essential spectrum of A were described in [11,13,20] and its structure was studied in [14]. The existence of infinitely many negative eigenvalues of A is proved for the case where the associated Friedrichs model have a zero energy resonance and an asymptotics of the form $U_0 |\log|z||$ for the number of eigenvalues of A lying below $z < 0$, is obtained in [12]. The conditions for the infiniteness of the number of eigenvalues located inside (in the gap, in the below of the bottom) of the essential spectrum of A is find.

Throughout the present paper we adopt the following conventions: Let C, R and Z be the set of all complex, real and integer numbers, respectively. We denote by T^d the d-dimensional torus (the first Brillouin zone, i.e., dual group of Z^d), the cube $(-\pi,\pi]^d$ with appropriately identified sides equipped with its Haar measure. The torus T^d will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space R^d modulo $(2\pi Z)^d$. The spectrum, the essential spectrum, the discrete and point spectrum of a bounded self-adjoint operator will be denoted by $\sigma(\cdot)$, $\sigma_{ess}(\cdot)$, $\sigma_{disc}(\cdot)$ and $\sigma_p(\cdot)$, respectively. For each $\delta>0$, the notation $U_\delta(p_0):=\{p\in T^d : |p-p_0|<\delta\}$ stands for a δ-neighborhood of the point $p_0\in T^d$.

2. The model operator and main results

2.1. The model operator. Let $d\geq 3$ and $L_2((T^d)^n)$ be the Hilbert space of square integrable (complex) functions defined on $(T^d)^n$, $n=1,2$. Denote by H the direct sum of spaces $H_0:=C$, $H_1:=L_2(T^d)$ and $H_2:=L_2((T^d)^2)$, that is, $H:=H_0\oplus H_1\oplus H_2$. The spaces H_0, H_1 and H_2 are called zero-, one- and two-particle subspaces of the Fock space $F(L_2(T^d))$ over $L_2(T^d)$, respectively, where

$$F(L_2(T^d)):=C\oplus L_2(T^d)\oplus L_2((T^d)^2)\oplus\cdots\oplus L_2((T^d)^n)\oplus\cdots.$$

Let us consider the model operator A acting in the Hilbert space H as

$$A:=\begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22}\end{pmatrix},$$

with the entries $A_{ij}:H_j\to H_i$, $i\leq j$, $i,j=0,1,2$ defined by

$$A_{00}f_0=af_0,\ A_{01}f_1=\int_{T^d} v(s)f_1(s)ds,\ (A_{11}f_1)(p)=u(p)f_1(p),$$

$$(A_{12}f_2)(p)=\int_{T^d} v_0(s)f_2(p,s)ds,\ A_{22}=A_{22}^0-V_1-V_2,\ (A_{22}^0f_2)(p,q)=w(p,q)f_2(p,q),$$

$$(V_1f_2)(p,q)=v_1(q)\int_{T^d} v_1(s)f_2(p,s)ds,\ (V_2f_2)(p,q)=v_2(p)\int_{T^d} v_2(s)f_2(s,q)ds.$$

Here $f_i\in H_i$, $i=0,1,2$; a is a fixed real number, $u(\cdot)$, $v(\cdot)$, $v_i(\cdot)$, $i=0,1,2$ and $w(\cdot,\cdot)$ are real-valued continuous functions on T^d and $(T^d)^2$, respectively. The operator $A_{ij}^*\,(i<j)$ denotes the adjoint to A_{ij} and

$$(A_{01}^* f_0)(p) = v(p) f_0,\ (A_{12}^* f_1)(p,q) = v_0(q) f_1(p),\ f_i \in H_i,\ i = 0,1.$$

Under these assumptions the operator A is bounded and self-adjoint in H.

We remark that the operators A_{01} and A_{12} (resp. A_{01}^* and A_{12}^*) are called annihilation (resp. creation) operators. In the present paper we consider the case where the number of annihilations and creations of the particles of the considering system is equal to 1, that is, $A_{ij} \equiv 0$ for all $|i-j|>1$.

Notice that [3] the operator A is associated to the Hamiltonian of a lattice system describing three particles in interaction, without conservation of the number of particles and the operator A_{22} is associated to a system of three quantum particles on the d -dimensional lattice that interact via nonlocal pair potentials [2].

Set $w_1(p,q) := w(p,q)$ and $w_2(p,q) := w(q,p)$. To formulate the main results of the paper we introduce the operators A_0 and A_α, $\alpha = 1,2$ acting in the Hilbert spaces H and H_2, respectively, as

$$A_0 := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22}^0 \end{pmatrix}, \quad A_\alpha := A_{22}^0 - V_\alpha,\ \alpha = 1,2$$

and the following families of bounded self-adjoint operators acting in $H_0 \oplus H_1$ and H_1 as

$$a(p) := \begin{pmatrix} a_{00}(p) & a_{01} \\ a_{01}^* & a_1^0(p) - v_1 \end{pmatrix}, \quad a_0(p) := \begin{pmatrix} a_{00}(p) & a_{01} \\ a_{01}^* & a_1^0(p) \end{pmatrix}$$

and $a_\alpha(p) := a_\alpha^0(p) - v_\alpha$, $\alpha = 1,2$, respectively, where

$$a_{00}(p) f_0 = u(p) f_0, \quad a_{01} f_1 = \int_{T^d} v_0(s) f_1(s) ds,$$

$$(a_\alpha^0(p) f_1)(q) = w_\alpha(p,q) f_1(q), \quad (v_\alpha f_1)(q) = v_\alpha(q) \int_{T^d} v_\alpha(s) f_1(s) ds,\ \alpha = 1,2.$$

We recall that the operator $a(p)$ is called molecular-resonance model and it is associated with the Hamiltonian of the system consisting of at most two particles on the d -dimensional lattice, interacting via both a nonlocal potential, and creation and annihilation operators.

In [14] it was shown that for any $p \in T^d$ the operator $a(p)$ has at most three eigenvalues.

Set

$$m := \min_{p,q \in T^d} w(p,q), \quad M := \max_{p,q \in T^d} w(p,q).$$

The following theorem describes the location of the essential spectrum of the operator A by the spectrum of the families $a(p)$ and $a_2(p)$ of the generalized Friedrichs models [20].

Theorem 2.1. For the essential spectrum of A the following equality holds

$$\sigma_{ess}(A)=\sigma\cup[m,M], \quad \sigma:=\bigcup_{p\in T^d}\{\sigma_{disc}(a(p))\cup\sigma_{disc}(a_2(p))\}. \qquad (1)$$

Moreover, the set $\sigma_{ess}(A)$ is a union of at most five bounded closed intervals.

The subsets σ and $[m,M]$ are called two-particle and three-particle branches of the essential spectrum of A, respectively.

2.2. Main assumptions. Let $T_0\in(0,2\pi)^d$ be a fixed element.

Assumption 2.2. *For $\alpha,\beta=0,1,\ \alpha\neq\beta$ the function $v_\alpha(\cdot)$ is T_0-periodic and $v_\beta(\cdot)$ satisfies the condition*

$$\int_{T^d} v_\beta(s)g(s)ds=0, \qquad (2)$$

for any T_0 -periodic function $g\in L_2(T^d)$.

Assumption 2.3. (i) The function $w(\cdot,\cdot)$ is T_0-periodic on each variable p and q, that is, $w(p+T_0,q)=w(p,q+T_0)=w(p,q)$ for all $p,q\in T^d$;

(ii) The function $w(\cdot,\cdot)$ has the non-degenerate minimum at the points $(p_i,q_i)\in(T^d)^2$, $i=1,\ldots,n$, $1\le n<\infty$. All third order partial derivatives of the functions $u(\cdot)$ and $w(\cdot,\cdot)$ are continuous on T^d and $(T^d)^2$, respectively.

Under the Assumption 2.2 and the part (i) of Assumption 2.3 the discrete spectrum of $a(p)$ coincides (see Lemma 3.1 below) with the union of discrete spectra of the operators $a_0(p)$ and $a_1(p)$. It follows from the definition of the operator $a_\alpha(p)$, $\alpha=0,1$ that its structure is simpler than that of $a(p)$. Using the Weyl theorem one can easily show that

$$\sigma_{ess}(a(p))=\sigma_{ess}(a_0(p))=\sigma_{ess}(a_1(p))=[m_1(p),M_1(p)],$$
$$\sigma_{ess}(a_2(p))=[m_2(p),M_2(p)],$$

where the numbers $m_\alpha(p)$ and $M_\alpha(p)$ are defined by

$$m_\alpha(p):=\min_{q\in T^d} w_\alpha(p,q), \quad M_\alpha(p):=\max_{q\in T^d} w_\alpha(p,q),\ \alpha=1,2.$$

For any fixed $p\in T^d$, we define the analytic functions in the domains $C\setminus[m_1(p),M_1(p)]$ and $C\setminus[m_2(p),M_2(p)]$ by

$$\Delta_0(p;z):=u(p)-z-\int_{T^d}\frac{v_0^2(s)ds}{w_1(p,s)-z},\ \Delta_1(p;z):=1-\int_{T^d}\frac{v_1^2(s)ds}{w_1(p,s)-z}$$

and

$$\Delta_2(p;z):=1-\int_{T^d}\frac{v_2^2(s)ds}{w_2(p,s)-z},$$

respectively. For $\alpha=0,1,2$ the function $\Delta_\alpha(p;\cdot)$ is called the Fredholm determinant associated with the operator $a_\alpha(p)$.

From now on we always assume that $\alpha = 0,1,2$. Since the function $w(\cdot,\cdot)$ has the non-degenerate minimum at the points $(p_i, q_i) \in (T^d)^2$, $i = 1,\dots,n$ and the function $v_\alpha(\cdot)$ is a continuous on T^d, for any $p \in T^d$ the integral

$$\int_{T^d} \frac{v_\alpha^2(s)ds}{w(p,s) - m}$$

is positive and finite. Then the Lebesgue dominated convergence theorem yields $\Delta_\alpha(p_i; m) = \lim_{p \to p_i} \Delta_\alpha(p; m)$, $i = 1,\dots,n$, and hence the function $\Delta_\alpha(\cdot; m)$ is a continuous on T^d.

Note that using the fact [2,3]

$$\sigma_{ess}(A_\alpha) = \sigma_\alpha \cup [m, M], \quad \sigma_\alpha := \bigcup_{p \in T^d} \sigma_{disc}(a_\alpha(p))$$

together with Assumption 2.2 and part (i) of Assumption 2.3 the equality (2.1) can be written as

$$\sigma_{ess}(A) = \sigma_{ess}(A_0) \cup \sigma_{ess}(A_1) \cup \sigma_{ess}(A_2). \tag{3}$$

It was shown in [2,3] that if $\min_{p \in T^d} \Delta_\alpha(p; m) < 0$, then $\sigma_\alpha \cap (-\infty, m] \neq \varnothing$. Assuming $\min_{p \in T^d} \Delta_\alpha(p; m) < 0$, we introduce the following numbers

$$E_{\min}^{(\alpha)} := \min\{\sigma_\alpha \cap (-\infty, m]\},\ E_{\max}^{(\alpha)} := \max\{\sigma_\alpha \cap (-\infty, m]\}.$$

The following theorem [2,3] describes the structure of the part of the essential spectrum of A_α located in $(-\infty, M]$.

Theorem 2.4. Let part (ii) of Assumption 2.3 be fulfilled.

(i) If $\min_{p \in T^d} \Delta_\alpha(p; m) \geq 0$, then

$$(-\infty, M] \cap \sigma_{ess}(A_\alpha) = [m, M];$$

(ii) If $\min_{p \in T^d} \Delta_\alpha(p; m) < 0$ and $\max_{p \in T^d} \Delta_\alpha(p; m) \geq 0$, then

$$(-\infty, M] \cap \sigma_{ess}(A_\alpha) = [E_{\min}^{(\alpha)}, M],\ E_{\min}^{(\alpha)} < m;$$

(iii) If $\max_{p \in T^d} \Delta_\alpha(p; m) < 0$, then

$$(-\infty, M] \cap \sigma_{ess}(A_\alpha) = [E_{\min}^{(\alpha)}, E_{\max}^{(\alpha)}] \cup [m, M],\ E_{\max}^{(\alpha)} < m.$$

We notice that if Assumptions 2.2 and 2.3 hold, then Theorem 2.4 together with the equality (3) describes the structure of the part of the essential spectrum of A located in $(-\infty, M]$.

If $\min_{p \in T^d} \Delta_\alpha(p; m) < 0$, then from $E_{\min}^{(\alpha)}, E_{\max}^{(\alpha)} \in \sigma_\alpha$ it follows that there exist positive integers n_α, k_α and points $\{p_{\alpha i}\}_{i=1}^{n_\alpha}, \{q_{\alpha j}\}_{j=1}^{k_\alpha} \subset T^d$ such that

$$\{p \in T^d : \Delta_\alpha(p; E_{\min}^{(\alpha)}) = 0\} = \{p_{\alpha 1}, \dots, p_{\alpha n_\alpha}\};$$

$$\{p \in T^d : \Delta_\alpha(p; E_{\max}^{(\alpha)}) = 0\} = \{q_{\alpha 1}, \dots, q_{\alpha k_\alpha}\}.$$

Assumption 2.5. There exist positive numbers C, δ and $\beta_{\alpha j} \in (0,2]$, $j = 1, \dots, n_\alpha$ such that

$$|\Delta_\alpha(p; E_{\min}^{(\alpha)})| \geq C|p - p_{\alpha j}|^{\beta_{\alpha j}}, \ p \in U_\rho(p_{\alpha j}), \ j = 1, \dots, n_\alpha,$$

and the inequality $\Delta_\alpha(p; E_{\min}^{(\alpha)}) > 0$ holds for all $p \in T^d \setminus \{p_{\alpha 1}, \dots, p_{\alpha n_\alpha}\}$.

Assumption 2.6. There exist positive numbers K, ρ and $\gamma_{\alpha i} \in (0,2]$, $i = 1, \dots, k_\alpha$ such that

$$|\Delta_\alpha(p; E_{\max}^{(\alpha)})| \geq K|p - q_{\alpha j}|^{\gamma_{\alpha j}}, \ p \in U_\delta(q_{\alpha j}), \ i = 1, \dots, k_\alpha,$$

and the inequality $\Delta_\alpha(p; E_{\max}^{(\alpha)}) < 0$ holds for all $p \in T^d \setminus \{q_{\alpha 1}, \dots, q_{\alpha k_\alpha}\}$.

2.3. Statement of the main results. Here we formulate main results of the paper.

Theorem 2.7. Let part (i) of Assumption 2.3 be fulfilled.

(i) If Assumption 2.2 holds with $\alpha = 0$ and the function $v_2(\cdot)$ satisfies the condition (2), then $\sigma_{disc}(A_0) \subset \sigma_p(A)$.

(ii) If Assumption 2.2 holds with $\alpha = 1$ and in addition, the functions $u(\cdot)$, $v(\cdot)$ are T_0-periodic, then $\sigma_{disc}(A_{22}) \subset \sigma_p(A)$.

Theorem 2.8. Let Assumptions 2.2 and 2.3 be fulfilled. Assume

(α.1) $\min\limits_{p \in T^d} \Delta_\alpha(p; m) > 0$;

(α.2) $\min\limits_{p \in T^d} \Delta_\alpha(p; m) < 0$, $\max\limits_{p \in T^d} \Delta_\alpha(p; m) \geq 0$ and Assumption 2.5 holds;

(α.3) $\max\limits_{p \in T^d} \Delta_\alpha(p; m) < 0$ and Assumptions 2.5, 2.6 hold.

If for some $i, j, k \in \{1,2,3\}$ the conditions (1.i), (2.j) and (3.k) hold, then the operator matrix A has a finite number of discrete eigenvalues lying on the left of m.

Remark 2.9. The class of functions $u(\cdot)$, $v_\beta(\cdot)$, $\beta = 1,2$ and $w(\cdot,\cdot)$ satisfying the conditions in Theorem 2.8 is nonempty (see Lemma 5.1).

3. Some auxiliary statements

The following lemma describes the relation between the eigenvalues of the operators $a(p)$ and $a_\beta(p)$, $\beta = 0,1$.

Lemma 3.1. Let Assumption 2.2 and part (i) of Assumption 2.3 be fulfilled. For any fixed $p \in T^d$ the number $z(p) \in C \setminus [m_1(p), M_1(p)]$ is an eigenvalue of $a(p)$ if and only if $z(p)$ is an eigenvalue of at least one of the operators $a_0(p)$ and $a_1(p)$.

Proof. Let $p \in T^d$ be fixed. Suppose $(f_0, f_1) \in H_0 \oplus H_1$ is an eigenvector of the operator $a(p)$ associated with the eigenvalue $z(p) \in C \setminus [m_1(p), M_1(p)]$. Then f_0 and f_1 satisfy the following system of equations

$$(u(p) - z(p)) f_0 + \int_{T^d} v_0(s) f_1(s) ds = 0;$$

$$v_0(q) f_0 + (w_1(p,q) - z(p)) f_1(q) - v_1(q) \int_{T^d} v_1(s) f_1(s) ds = 0. \qquad (4)$$

Since for any $q \in T^d$ the relation $w_1(p,q) - z(p) \neq 0$ holds, from the second equation of the system (4) for f_1 we have

$$f_1(q) = \frac{C_{f_1} v_1(q) - v_0(q) f_0}{w_1(p,q) - z(p)}, \qquad (5)$$

where

$$C_{f_1} = \int_{T^d} v_1(s) f_1(s) ds. \qquad (6)$$

Substituting the expression (5) for f_1 into the first equation of the system (4) and the equality (6), we conclude that the system of equations (4) has a nontrivial solution if and only if the system of equations

$$\Delta_0(p; z(p)) f_0 + \int_{T^d} \frac{v_0(s) v_1(s) ds}{w_1(p,s) - z(p)} C_{f_1} = 0;$$

$$\int_{T^d} \frac{v_0(s) v_1(s) ds}{w_1(p,s) - z(p)} f_0 + \Delta_1(p; z(p)) C_{f_1} = 0$$

has a nontrivial solution $(f_0, C_{f_1}) \in C^2$, i.e. if the condition

$$\Delta_0(p; z(p)) \Delta_1(p; z(p)) - \left(\int_{T^d} \frac{v_0(s) v_1(s) ds}{w_1(p,s) - z(p)} \right)^2 = 0$$

is satisfied.

By part (i) of Assumption 2.3 for any fixed $p \in T^d$ the function $(w_1(p,\cdot) - z(p))^{-1} \in L_2(T^d)$ is T_0-periodic. Applying Assumption 2.2 we obtain

$$\int_{T^d} \frac{v_0(s) v_1(s) ds}{w_1(p,s) - z(p)} = 0.$$

If we set $v_1(q) \equiv 0$ in the operator $a(p)$, then $a(p) = a_0(p)$; in this case the number $z(p) \in C \setminus [m_1(p), M_1(p)]$ is an eigenvalue of $a_0(p)$ if and only if $\Delta_0(p; z(p)) = 0$. Similarly one can show that the number $z(p) \in C \setminus [m_1(p), M_1(p)]$ is an eigenvalue of $a_1(p)$ if and only if $\Delta_1(p; z(p)) = 0$. The lemma is proved.

Lemma 3.2. Let $\min\limits_{p\in T^d}\Delta_\alpha(p;m)>0$. Then there exists a positive number C_1 such that the inequality $\Delta_\alpha(p;z)\ge C_1$ holds for all $p\in T^d$ and $z\le m$.

Proof. Since for any $p\in T^d$ the function $\Delta_\alpha(p;\cdot)$ is monotonically decreasing in $(-\infty;m]$, we have

$$\Delta_\alpha(p;z)\ge\Delta_\alpha(p;m)\ge\min_{p\in T^d}\Delta_\alpha(p;m)>0$$

for all $p\in T^d$ and $z\le m$. Now setting $C_1:=\min\limits_{p\in T^d}\Delta_\alpha(p;m)$ we complete the proof of lemma.

Lemma 3.3. If Assumption 2.5 resp. 2.6 holds, then for any $\delta>0$ there exist the positive numbers $C_1(\delta)$ and $C_2(\delta)$ such that

(i) $\Delta_\alpha(p;E_{\min}^{(\alpha)})\ge C_1(\delta)$ *for all* $p\in T^d\setminus\bigcup\limits_{i=1}^{n_\alpha}U_\delta(p_{\alpha i})$;

resp.

(ii) $|\Delta_\alpha(p;E_{\max}^{(\alpha)})|\ge C_2(\delta)$ *for all* $p\in T^d\setminus\bigcup\limits_{j=1}^{k_\alpha}U_\delta(q_{\alpha j})$.

Proof. Let Assumption 2.5 be fulfilled. Then the inequality $\Delta_\alpha(p;E_{\min}^{(\alpha)})>0$ holds for any $p\in T^d\setminus\{p_{\alpha 1},\ldots,p_{\alpha n_\alpha}\}$. Since for any $\delta>0$ the set $T^d\setminus\bigcup\limits_{i=1}^{n_\alpha}U_\delta(p_{\alpha i})$ is compact and $\Delta_\alpha(\cdot;E_{\min}^{(\alpha)})$ is the positive continuous function on this set, there exists the number $C_1(\delta)>0$ such that the assertion (i) of lemma holds. Proof of assertion (ii) is similar.

Lemma 3.4. Let part (ii) of Assumption 2.3 be fulfilled. Then there exist positive numbers C_1,C_2,C_3 and δ such that the following inequalities hold

(i) $C_1(|p-p_i|^2+|q-q_i|^2)\le w(p,q)-m\le C_2(|p-p_i|^2+|q-q_i|^2)$ for all $(p,q)\in U_\delta(p_i)\times U_\delta(q_i)$;

(ii) $w(p,q)-m\ge C_3$ *for all* $(p,q)\notin\bigcup\limits_{i=1}^{n}(U_\delta(p_i)\times U_\delta(q_i))$.

Proof. By part (ii) of Assumption 2.3 the all third order partial derivatives of $w(\cdot,\cdot)$ are continuous on $(T^d)^2$ and it has the non-degenerate minimum at the points $(p_i,q_i)\in(T^d)^2$, $i=1,\ldots,n$. Then by the Hadamard lemma [23] there exists a δ-neighborhood of the point $(p_i,q_i)\in(T^d)^2$ such that the following decomposition holds

$$w(p,q)=m+\frac{1}{2}\Big((W_1^{(i)}(p-p_i),p-p_i)+2(W_2^{(i)}(p-p_i),q-q_i)+(W_3^{(i)}(q-q_i),q-q_i)\Big)$$

$$+\sum_{|s|+|l|=3} H_{sl}^{(i)}(p,q)\prod_{k=1}^{d}(p^{(k)}-p_i^{(k)})^{s_k}(q^{(k)}-q_i^{(k)})^{l_k},$$

$$(p,q)\in U_\delta(p_i)\times U_\delta(q_i)$$

where $i=1,...,n$ and

$$W_1^{(i)}:=\left(\frac{\partial^2 w(p_i,q_i)}{\partial p^{(j)}\partial p^{(k)}}\right)_{j,k=1}^{d},\ W_2^{(i)}:=\left(\frac{\partial^2 w(p_i,q_i)}{\partial p^{(j)}\partial q^{(k)}}\right)_{j,k=1}^{d},\ W_3^{(i)}:=\left(\frac{\partial^2 w(p_i,q_i)}{\partial q^{(j)}\partial q^{(k)}}\right)_{j,k=1}^{d},$$

$$s=(s_1,...,s_d), l=(l_1,...,l_d), |s|=s_1+...+s_d, s_j, l_j\in\{0,1,...,d\},\ j=1,...,d,$$

and $H_{sl}^{(i)}(\cdot,\cdot)$ with $|s|+|l|=3$ are continuous functions in $U_\delta(p_i)\times U_\delta(q_i)$. Therefore, there exist positive numbers C_1,C_2,C_3 such that (i) and (ii) hold true.

4. The Weinberg type system of integral equations

In this section we derive an analogue of the Weinberg type system of integral equations for the eigenvectors, corresponding to the eigenvalues of A, lying on the left of m.

Let $\tau_{ess}(A)$ be the lower bound of the essential spectrum of A. It is clear that $\Delta_\alpha(p,z)>0$ for all $p\in T^d$ and $z\in(-\infty,\tau_{ess}(A))$; if $\max_{p\in T^d}\Delta_\alpha(p;m)<0$, then $\Delta_\alpha(p,z)<0$ for all $p\in T^d$ and $z\in(E_{\max}^{(\alpha)},m)$. So $sign(\Delta_\alpha(p,z))$ depends on the location of $z\in(-\infty,m)\setminus\sigma_{ess}(A)$ and does not depend on $p\in T^d$. For $z\in(-\infty,m)\setminus\sigma_{ess}(A)$ we set $\xi_\alpha(z):=sign(\Delta_\alpha(p,z))$.

Let for any $z\in(-\infty,m)\setminus\sigma_{ess}(A)$ the operator $W(z)$ act in the Hilbert space H as a 3×3 operator matrix with entries $W_{ij}(z):H_j\to H_i$, $i,j=0,1,2$ defined by

$$W_{00}(z)g_0=(1+z-a)g_0,\ W_{01}(z)g_1=-\int_{T^d}\frac{v(s)g_1(s)ds}{\sqrt{\xi_0(z)\Delta_0(s,z)}};$$

$$W_{02}(z)\equiv 0,\ (W_{10}(z)g_0)(p)=-\frac{\xi_0(z)v(p)g_0}{\sqrt{\xi_0(z)\Delta_0(p,z)}},\ W_{11}(z)\equiv 0;$$

$$(W_{12}(z)g_2)(p)=-\frac{\xi_0(z)v_2(p)}{\sqrt{\xi_0(z)\Delta_0(p,z)}}\int_{T^d}\int_{T^d}\frac{v_0(t)v_2(s)g_2(s,t)dsdt}{\sqrt{\xi_2(z)\Delta_2(t,z)}(w(p,t)-z)};$$

$$(W_{20}(z)g_0)(p,q)=-\frac{v_0(q)(W_{10}(z)g_0)(p)}{w(p,q)-z};$$

$$(W_{21}(z)g_1)(p,q)=-\frac{\xi_2(z)v_0(q)v_2(p)}{(w(p,q)-z)\sqrt{\xi_2(z)\Delta_2(q,z)}}\int_{T^d}\frac{v_2(s)g_1(s)ds}{\sqrt{\xi_0(z)\Delta_0(s,z)}(w(s,q)-z)};$$

$$(W_{22}(z)g_2)(p,q)=\frac{\xi_1(z)v_1(q)v_2(p)}{(w(p,q)-z)\sqrt{\xi_1(z)\Delta_1(p,z)}}\int_{T^d}\int_{T^d}\frac{v_1(t)v_2(s)g_2(s,t)dsdt}{\sqrt{\xi_2(z)\Delta_2(t,z)}(w(p,t)-z)}$$

$$+\frac{\xi_2(z)v_1(q)v_2(p)}{(w(p,q)-z)\sqrt{\xi_2(z)\Delta_2(q,z)}}\int_{T^d}\int_{T^d}\frac{v_1(t)v_2(s)g_2(s,t)dsdt}{\sqrt{\xi_1(z)\Delta_1(s,z)}(w(s,q)-z)}-\frac{v_0(q)(W_{12}(z)g_2)(p)}{w(p,q)-z},$$

where $g_j \in H_j$, $j=0,1,2$.

We have the following lemma.

Lemma 4.1. Let Assumption 2.2 and part (i) of Assumption 2.3 be fulfilled. If $f \in H$ is an eigenvector corresponding to the eigenvalue $z \in (-\infty, m) \setminus \sigma_{ess}(A)$ of A, then f satisfies the Weinberg equation $W(z)f = f$.

Proof. Let $z \in (-\infty, m) \setminus \sigma_{ess}(A)$ be an eigenvalue of the operator A and $f=(f_0, f_1, f_2) \in H$ be the corresponding eigenvector. Then f_0, f_1 and f_2 satisfy the system of equations

$$(A_{00}-z)f_0 + A_{01}f_1 = 0;$$

$$(A_{10}f_0)(p)+((A_{11}-z)f_1)(p)+(A_{12}f_2)(p)=0; \tag{7}$$

$$(A_{21}f_1)(p,q)+((A_{22}^0-z)f_2)(p,q)-(V_1f_2)(p,q)-(V_2f_2)(p,q)=0.$$

Since $z<m$, from the third equation of the system (4.1) for f_2 we have

$$f_2(p,q)=-\frac{v_1(q)\psi_1(p)+v_2(p)\psi_2(q)}{w(p,q)-z}-\frac{v_0(q)f_1(p)}{w(p,q)-z}, \tag{8}$$

where

$$\psi_1(p)=\int_{T^d} v_1(s)f_2(p,s)ds, \tag{9}$$

$$\psi_2(p)=\int_{T^d} v_1(s)f_2(s,p)ds. \tag{10}$$

Substituting the expression (8) for f_2 into the second equation of the system (7) and the equalities (9), (10) and using Assumptions 2.2, 2.3, we obtain

$$f_0=(1+z-a)f_0-\int_{T^d} v(s)f_1(s)ds;$$

$$\Delta_0(p,z)f_1(p)=-v(p)f_0-v_2(p)\int_{T^d}\frac{v_0(s)\psi_2(s)ds}{w(p,s)-z}; \tag{11}$$

$$\Delta_1(p,z)\psi_1(p)=v_2(p)\int_{T^d}\frac{v_1(s)\psi_2(s)ds}{w(p,s)-z};$$

$$\Delta_2(p,z)\psi_2(p)=-v_0(p)\int_{T^d}\frac{v_2(s)f_1(s)ds}{w(s,p)-z}+v_1(p)\int_{T^d}\frac{v_2(s)\psi_1(s)ds}{w(s,p)-z}.$$

It is clear that the inequality $\xi_\alpha(z)\Delta_\alpha(p,z)>0$ holds for all $z \in (-\infty, m) \setminus \sigma_{ess}(A)$ and $p \in T^d$. Therefore, the system of equations (11) has a nontrivial solution if and only if the following system of equations

$$f_0 = W_{00}(z) f_0 + W_{01}(z) f_1;$$
$$f_1(p) = (W_{10}(z) f_0)(p) - \frac{\xi_0(z) v_2(p)}{\sqrt{\xi_0(z)\Delta_0(p,z)}} \int_{T^d} \frac{v_0(s)\psi_2(s)ds}{\sqrt{\xi_2(z)\Delta_2(s,z)}(w(p,s)-z)};$$
$$\psi_1(p) = \frac{\xi_1(z) v_2(p)}{\sqrt{\xi_1(z)\Delta_1(p,z)}} \int_{T^d} \frac{v_1(s)\psi_2(s)ds}{\sqrt{\xi_2(z)\Delta_2(s,z)}(w(p,s)-z)};$$
$$\psi_2(p) = -\frac{\xi_2(z) v_0(p)}{\sqrt{\xi_2(z)\Delta_2(p,z)}} \int_{T^d} \frac{v_2(s) f_1(s)ds}{\sqrt{\xi_0(z)\Delta_0(s,z)}(w(s,p)-z)}$$
$$+ \frac{\xi_2(z) v_1(p)}{\sqrt{\xi_2(z)\Delta_2(p,z)}} \int_{T^d} \frac{v_2(s)\psi_1(s)ds}{\sqrt{\xi_1(z)\Delta_1(s,z)}(w(s,p)-z)}$$

has a nontrivial solution.

Substituting the last expressions for f_1 and ψ_β, $\beta = 1,2$ into the formula (8) and using the equalities (9), (10), we obtain the Weinberg equation $W(z) f = f$.

Set

$$E_{\min} := \min_\alpha E_{\min}^{(\alpha)}, \; E_{\max} := \min_\alpha E_{\max}^{(\alpha)}, \; \sum := \overline{[\tau_{ess}(A) - 1, m] \setminus \sigma_{ess}(A)}.$$

Lemma 4.2. Let assumptions in Theorem 2.8 be fulfilled. Then the operator $W(z)$ is compact for $z \in \sum$ and the operator-valued function $W(z)$ is continuous in the uniform operator topology for $z \in \sum$.

Proof. We will prove the statement of the lemma for the case $\max_{p \in T^d} \Delta_\alpha(p;m) < 0$ with

$$E_{\min}^{(1)} = E_{\min}^{(2)} = E_{\min}^{(3)}, \; E_{\max}^{(1)} = E_{\max}^{(2)} = E_{\max}^{(3)}.$$

Other cases can be proven in a similar.

In this case we have

$$\sum = [E_{\min} - 1, E_{\min}] \cup [E_{\max}, m], \; E_{\max} < m.$$

Let Assumptions 2.5 and 2.6 be fulfilled. For $z \in (-\infty, m) \setminus \sigma_{ess}(A)$ denote by $W(p,q,s,t;z)$ the kernel of the operator $W_{22}(z)$.

We have the following inequalities

$$w(p,q) - z \geq m - E_{\min} > 0 \text{ for all } p,q \in T^d, \; z \leq E_{\min};$$
$$w(p,q) - z \geq (m - E_{\max})/2 > 0 \text{ for all } p,q \in T^d, \; z \in [E_{\max}; (m + E_{\max})/2].$$

Then by Assumptions 2.5, 2.6 and Lemma 3.3 the function $|W(\cdot,\cdot,\cdot;z)|$ can be estimated by

$$C_1 \left[\left(1 + \sum_{i=1}^{n_2} \frac{\chi_\delta(t - p_{2i})}{|t - p_{2i}|^{\beta_{2i}/2}} \right) \left(1 + \sum_{i=1}^{n_0} \frac{\chi_\delta(p - p_{0i})}{|p - p_{0i}|^{\beta_{0i}/2}} + \sum_{i=1}^{n_1} \frac{\chi_\delta(p - p_{1i})}{|p - p_{1i}|^{\beta_{1i}/2}} \right) \right.$$

$$+\left(1+\sum_{i=1}^{n_2}\frac{\chi_\delta(q-p_{2i})}{|q-p_{2i}|^{\beta_{2i}/2}}\right)\left(1+\sum_{i=1}^{n_1}\frac{\chi_\delta(s-p_{1i})}{|s-p_{1i}|^{\beta_{1i}/2}}\right)\Bigg]$$

for $z\le E_{\min}$ and by

$$C_2\Bigg[\left(1+\sum_{i=1}^{k_2}\frac{\chi_\rho(t-q_{2i})}{|t-p_{2i}|^{\gamma_{2i}/2}}\right)\left(1+\sum_{i=1}^{k_0}\frac{\chi_\rho(p-q_{0i})}{|p-q_{0i}|^{\gamma_{0i}/2}}+\sum_{i=1}^{k_1}\frac{\chi_\rho(p-q_{1i})}{|p-q_{1i}|^{\gamma_{1i}/2}}\right)$$

$$+\left(1+\sum_{i=1}^{k_2}\frac{\chi_\rho(q-q_{2i})}{|q-q_{2i}|^{\gamma_{2i}/2}}\right)\left(1+\sum_{i=1}^{k_1}\frac{\chi_\rho(s-q_{1i})}{|s-q_{1i}|^{\gamma_{1i}/2}}\right)\Bigg]$$

for $z\in[E_{\max},(m+E_{\max})/2]$, where $\chi_\delta(\cdot)$ is the characteristic function of $U_\delta(0)$.

Since $\xi_\alpha(z)=-1$ for any $z\in(E_{\max},m)$ and $\max\limits_{p\in T^d}\Delta_\alpha(p;m)<0$, we have the inequality $\max\limits_{p\in T^d}(\xi_\alpha(z)\Delta_\alpha(p;m))>0$ for any $z\in(E_{\max},m)$. Therefore, Lemmas 3.2 and 3.4 imply that the function $|W(\cdot,\cdot,\cdot,\cdot;z)|$ can be estimated by

$$C_3\sum_{i=1}^{n}\left(1+\frac{\chi_\delta(p-p_i)\chi_\delta(q-q_i)}{|p-p_i|^2+|q-q_i|^2}\right)\left(1+\frac{\chi_\delta(p-p_i)\chi_\delta(t-q_i)}{|p-p_i|^2+|t-q_i|^2}+\frac{\chi_\delta(s-p_i)\chi_\delta(q-q_i)}{|s-p_i|^2+|q-q_i|^2}\right)$$

for $z\in[(m+E_{\max})/2,m]$.

Using elementary inequality

$$\frac{1}{(p^{(1)})^2+\ldots+(p^{(d)})^2}\le\frac{1}{(p^{(1)})^2+(p^{(2)})^2+(p^{(3)})^2},\ (p^{(1)})^2+(p^{(2)})^2+(p^{(3)})^2\ne 0,$$

one can see that the latter three majorant functions are square integrable on $(T^d)^4$ and hence the operator $W_{22}(z)$ is Hilbert Schmidt for any $z\in(-\infty,E_{\min}]\cup[E_{\max},m]$.

A similar argument shows that the operators $W_{12}(z)$ and $W_{21}(z)$ are also Hilbert Schmidt for any $z\in\sum$.

For any $z\in(-\infty,m)\setminus\sigma_{ess}(A)$ the kernel function of $W_{ij}(z)$, $i,j=1,2$ is continuous on its domain. Therefore the continuity of the operator-valued functions $W_{ij}(z)$, $i,j=1,2$ in the uniform operator topology for $z\in\sum$ follows from Lebesgue's dominated convergence theorem.

Since for all $z\in\sum$ the operators $W_{00}(z)$, $W_{01}(z)$, $W_{10}(z)$ and $W_{20}(z)$ are of rank 1 and continuous in the uniform operator topology for $z\in\sum$ one concludes that $W(z)$ is compact for $z\in\sum$ and the operator-valued function $W(z)$ is continuous in the uniform operator topology for $z\in\sum$.

5. Proof of the main results

In this section we prove Theorems 2.7 and 2.8.

Proof of Theorem 2.7. First we prove part (ii). Let $\alpha=1$. If $z_1 \in C \setminus \sigma_{ess}(A_{22})$ is an eigenvalue of the operator A_{22} and $f_2 \in H_2$ is the corresponding eigenfunction, then f_2 satisfies the equation

$$(w(p,q)-z_1) f_2(p,q) - v_1(q) \int_{T^d} v_1(s) f_2(p,s)ds - v_2(p) \int_{T^d} v_2(s) f_2(s,q)ds = 0\,. \qquad (12)$$

Since $z_1 \notin [m,M]$, from the equation (5.1) for f_2 we have

$$f_2(p,q) = \frac{v_1(q)\psi_1(p) + v_2(p)\psi_2(q)}{w(p,q)-z_1}, \qquad (13)$$

where the function $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are defined by (9) and (10), respectively.

Substituting the expression (13) for f_2 into the equalities (9) and (10), we obtain

$$\Delta_1(p,z_1)\psi_1(p) = v_2(p) \int_{T^d} \frac{v_1(s)\psi_2(s)ds}{w(p,s)-z_1};$$

$$\Delta_2(p,z_1)\psi_2(p) = v_1(p) \int_{T^d} \frac{v_2(s)\psi_1(s)ds}{w(s,p)-z_1}.$$

Since $z_1 \notin \sigma_{ess}(A_{22})$ the inequality $\Delta_\alpha(p,z_1) \neq 0$, $\alpha=1,2$ holds for all $p \in T^d$. From the last two equations we have

$$\psi_1(p) = \frac{v_2(p)}{\Delta_1(p,z_1)} \int_{T^d} \frac{v_1(s)\psi_2(s)ds}{w(p,s)-z_1};$$

$$\psi_2(p) = \frac{v_1(p)}{\Delta_2(p,z_1)} \int_{T^d} \frac{v_2(s)\psi_1(s)ds}{w(s,p)-z_1}.$$

For $\alpha=1,2$ the functions $v_\alpha(\cdot)$ and $w_\alpha(\cdot,q)$, $q \in T^d$ are T_0-periodic and hence the function $\psi_\alpha(\cdot)$ is also T_0-periodic. Therefore, for any fixed $p \in T^d$ the function $f_2(p,\cdot)$ defined by (13), is T_0-periodic. Hence this function satisfies the condition (2.2), that is, $A_{12} f_2 = 0$. So the number $z_1 \in \sigma_{disc}(A_{22})$ is an eigenvalue of A and associated eigenvector has form $f=(0,0,f_2) \in H$. Therefore, $\sigma_{disc}(A_{22}) \subset \sigma_p(A)$.

Now we prove part (i). Let $\alpha=0$. If $z_0 \in \sigma_{disc}(A_0)$ and $g=(g_0,g_1,g_2) \in H$ is the eigenfunction associated with the discrete eigenvalue z_0 of A_0, then similar analysis shows that $V_\beta g_2 = 0$, $\beta=1,2$, which guarantee that the number $z_2 \in \sigma_{disc}(A_2)$ is an eigenvalue of A with the same eigenvector $g \in H$, that is, $\sigma_{disc}(A_0) \subset \sigma_p(A)$. Theorem 2.7 is proved.

Proof of Theorem 2.8. We prove the finiteness of the number of discrete eigenvalues located on the left of m for the case, when $\max_{p \in T^d} \Delta_\alpha(p;m) < 0$. Other cases can be proven similarly. Suppose that the operator A has an infinite number of discrete eigenvalues $(E_k)_{k\in N} \subset (E_{\max}, m)$. Then three cases are possible:

(i) $\lim_{k\to\infty} E_k = m$;

(ii) $\lim_{k\to\infty} E_k = E_{\max}$;

(iii) there exist $(E_k^{'})_{k\in N}, (E_k^{''})_{k\in N} \subset (E_k)_{k\in N}$ such that $\lim_{k\to\infty} E_k^{'} = m$ and $\lim_{k\to\infty} E_k^{''} = E_{\max}$.

Let us consider the case (iii). For each $k \in N$ we denote by $\varphi_k^{'} \in H$ and $\varphi_k^{''} \in H$ the orthonormalized eigenvectors corresponding to the eigenvalues $E_k^{'}$ and $E_k^{''}$, respectively. Then it follows from Lemma 4.1 that $\varphi_k^{'} = W(E_k^{'})\varphi_k^{'}$ and $\varphi_k^{''} = W(E_k^{''})\varphi_k^{''}$ for any $k \in N$. By virtue of Lemma 4.2 the operators $W(E_{\max})$, $W(m)$ are compact and $\| W(z) - W(E_{\max}) \| \to 0$ and $\| W(z) - W(m) \| \to 0$ as $z \to E_{\max} + 0$ and $z \to m - 0$, respectively. Therefore,

$$1 = \| \varphi_k^{'} \| = \| W(E_k^{'})\varphi_k^{'} \| \le \| (W(E_k^{'}) - W(E_{\max}))\varphi_k^{'} \| + \| W(E_{\max})\varphi_k^{'} \| \to 0;$$

$$1 = \| \varphi_k^{''} \| = \| W(E_k^{''})\varphi_k^{''} \| \le \| (W(E_k^{''}) - W(m))\varphi_k^{''} \| + \| W(m)\varphi_k^{''} \| \to 0$$

as $k \to \infty$. This contradiction implies that the points $z = E_{\max}$ and $z = m$ can not be limit points of the set of discrete eigenvalues of A belonging to the interval $(E_{\max}, m)$. Similar arguments show that other edges of Σ are also cannot be accumulation point for the set of discrete eigenvalues of A smaller than m.

The following example shows that the class of functions $u(\cdot)$, $v_\alpha(\cdot)$, $\alpha = 1,2$ and $w(\cdot,\cdot)$ satisfying the conditions of Theorem 2.8 is nonempty.

Lemma 5.1. Let

$$\hat{v}_0(p) := \sum_{i=1}^{d} c_{0i} \cos(2p^{(i)}),\ \hat{v}_1(p) := \sum_{i=1}^{d} c_{1i} \cos(p^{(i)}),\ v_\beta(p) := \sqrt{\mu_\alpha}\,\hat{v}_\beta(p),\ \beta = 0,1,$$

$$u(p) \equiv 1,\ w(p,q) = \varepsilon(p) + \varepsilon(q),\ \varepsilon(p) = \sum_{i=1}^{d} (1 - \cos(2p^{(i)})),$$

where $\mu_\beta > 0$; $c_{\beta i}$, $\beta = 0,1$, $i = 1,\ldots,d$ are arbitrary real numbers.

Set

$$\mu_\alpha^{(0)} := \left(\int_{T^d} \frac{\hat{v}_\alpha^2(s)ds}{\varepsilon(s)} \right)^{-1},\ \mu_\alpha^{(1)} := \left(\int_{T^d} \frac{\hat{v}_\alpha^2(s)ds}{6 + \varepsilon(s)} \right)^{-1}.$$

Then the functions $u(\cdot)$, $v_\beta(\cdot)$, $\beta = 1,2$ and $w(\cdot,\cdot)$ are satisfy Assumptions 2.2, 2.3, 2.5, 2.6. Moreover,

(i) if $0 < \mu_\alpha < \mu_\alpha^{(0)}$, then $\max_{p\in T^d} \Delta_\alpha(p;m) > 0$*;*

(ii) if $\mu_\alpha^{(0)} < \mu_\alpha \le \mu_\alpha^{(1)}$, then $\min\limits_{p\in T^d} \Delta_\alpha(p;m) < 0$ and $\max\limits_{p\in T^d} \Delta_\alpha(p;m) \ge 0$;

(iii) if $\mu_\alpha > \mu_\alpha^{(1)}$, then $\max\limits_{p\in T^d} \Delta_\alpha(p;m) < 0$.

Proof. First we recall that for $T_0 := (\pi,...,\pi) \in T^d$ the function $w(\cdot,\cdot)$ is T_0-periodic on each variable p and q. Let the function $g \in L_2(T^d)$ as in Assumption 2.2. Then we have

$$\int\limits_{T^d} v_1(s)g(s)ds = \int\limits_{T^d} v_1(s+T_0)g(s+T_0)ds = -\int\limits_{T^d} v_1(s)g(s)ds,$$

which yields the equality (2.2), that is, Assumption 2.2 holds with $\alpha = 0$ and $\beta = 1$.

We introduce the following subset of T^d:

$$\Lambda := \{p = (p^{(1)},...,p^{(d)}) \in T^d : p^{(k)} \in \{0,\pi\},\ k = 1,...,d\}.$$

From the definition of $w(\cdot,\cdot)$ it follows that this function has zero non-degenerate minimum at the points of $\Lambda \times \Lambda$ and it satisfy all conditions of Assumption 2.3.

The assertions (i)-(iii) directly follows from the definition of the numbers $\mu_\alpha^{(0)}$ and $\mu_\alpha^{(0)}$.

Let $\mu_\alpha^{(0)} < \mu_\alpha \le \mu_\alpha^{(1)}$. We prove that the function $\Delta_\alpha(\cdot, E_{\min}^{(\alpha)})$ has the non-degenerate minimum at the points of Λ. Simple calculations show that for any fixed $p' \in \Lambda$ the inequality $\Delta_\alpha(p, E_{\min}^{(\alpha)}) > \Delta_\alpha(p', E_{\min}^{(\alpha)})$ holds for all $p \in T^d \setminus \Lambda$.

Since $E_{\min}^{(\alpha)} \in (-\infty, 0)$, it is clear that the function $\Delta_\alpha(\cdot, E_{\min}^{(\alpha)})$ is twice continuously differentiable in T^d. Moreover, from the equalities

$$\frac{\partial^2 \Delta_\alpha(p, E_{\min}^{(\alpha)})}{\partial p^{(k)} \partial p^{(k)}} = 4\mu_\alpha \cos(2p^{(k)}) \int\limits_{T^d} \frac{\hat{v}_\alpha^2(s)ds}{(\varepsilon(p)+\varepsilon(s)-E_{\min}^{(\alpha)})^2}$$

$$-8\mu_\alpha (\sin(2p^{(k)}))^2 \int\limits_{T^d} \frac{\hat{v}_\alpha^2(s)ds}{(\varepsilon(p)+\varepsilon(s)-E_{\min}^{(\alpha)})^3},\ k = 1,...,d;$$

$$\frac{\partial^2 \Delta_\alpha(p, E_{\min}^{(\alpha)})}{\partial p^{(k)} \partial p^{(l)}} = -8\mu_\alpha \sin(2p^{(k)})\sin(2p^{(l)}) \int\limits_{T^d} \frac{\hat{v}_\alpha^2(s)ds}{(\varepsilon(p)+\varepsilon(s)-E_{\min}^{(\alpha)})^3}, k \ne l,\ k,l = 1,...,d$$

we get

$$\frac{\partial^2 \Delta_\alpha(p', E_{\min}^{(\alpha)})}{\partial p^{(k)} \partial p^{(k)}} > 0,\quad \frac{\partial^2 \Delta_\alpha(p', E_{\min}^{(\alpha)})}{\partial p^{(k)} \partial p^{(l)}} = 0,\ k \ne l,\ k,l = 1,...,d$$

for any $p' \in \Lambda$.

Using these facts, one may verify that the matrix of the second order partial derivatives of the function $\Delta_\alpha(\cdot, E_{\min}^{(\alpha)})$ at the points of Λ are positive definite. Thus the function $\Delta_\alpha(\cdot, E_{\min}^{(\alpha)})$ has the non-degenerate minimum at the points of Λ.

Since the number of points of Λ is equal to 2^d for convenience we numerate the points of Λ as $p_1,...,p_{2^d}$. Then the equality $\Delta_\alpha(p_i, E_{\min}^{(\alpha)}) = 0$, $i = 1,...,2^d$ implies that there exist the numbers $\delta > 0$ and $C > 0$ such that

$$|\Delta_\alpha(p, E_{\min}^{(\alpha)})| \geq C|p - p_i|^2, \ p \in U_\delta(p_i), \ i = 1,...,2^d,$$

that is, Assumption 2.5 holds with $n_\alpha = 2^d$, $p_{\alpha i} = p_i$ and $\beta_{\alpha i} = 2$ for $i = 1,...,2^d$.

In the case $\mu_\alpha > \mu_\alpha^{(1)}$ one can similarly show that there exist the numbers $\rho > 0$ and $K > 0$ such that

$$|\Delta_\alpha(p, E_{\max}^{(\alpha)})| \geq K|p - p_i|^2, \ p \in U_\delta(p_i), \ i = 1,...,2^d,$$

that is, Assumption 2.6 holds with $k_\alpha = 2^d$, $q_{\alpha j} = p_j$ and $\gamma_{\alpha j} = 2$ for $j = 1,...,2^d$.

Acknowledgements. This work was supported by the TOSCA Erasmus Mundus grant. The author wishes to thank the University of L'Aquila for the invitation and hospitality. The author is indebted to the anonymous referees for a number of constructive comments.

References

1. Abdullaev Zh., Lakaev S., Finiteness of the discrete spectrum of the three-particle Schroedinger equation on a lattice, Theor. Math. Phys., Vol.111, No.1, 1997, pp.467–479.
2. Albeverio S., Lakaev S., Djumanova R., The essential and discrete spectrum of a model operator associated to a system of three identical quantum particles, Rep. Math. Phys., Vol.63, No.3, 2009, pp.359–380.
3. Albeverio S., Lakaev S., Rasulov T., On the spectrum of an Hamiltonian in Fock space. Discrete spectrum asymptotics, J. Stat. Phys., Vol.127, No.2, 2007, pp.191–220.
4. Bach V., Froehlich J., Sigal I., Mathematical theory of non-relavistic matter and radiation, Lett. Math. Phys., Vol.34, 1995, pp.183–201.
5. Friedrichs K., Perturbation of spectra in Hilbert space, Amer. Math. Soc. Providence, Rhole Island, 1965, p.178.
6. Huebner M., Spohn H., Spectral properties of spin boson Hamiltonian, Annl. Inst. Poincare, Vol.62, No.3, 1995, pp.289–323.
7. Lakaev S., Muminov M., Essential and discrete spectra of the three-particle Schroedinger operator on a lattices, Theor. Math. Phys., Vol.135, No.3, 2003, pp.849–871.
8. Malishev V., Minlos R., Linear infinite-particle operators, Translations of Mathematical Monographs. 143, AMS, Providence, RI, 1995, p.297.
9. Minlos R., Spohn H., The three-body problem in radioactive decay: the case of one atom and at most two photons, Topics in Statistical and Theoretical Physics. Amer. Math. Soc. Transl., Ser. 2, Vol.177, AMS, Providence, RI, 1996, pp.159–193.

10. Mogilner A., Hamiltonians in solid state physics as multiparticle discrete Schroedinger operators: problems and results, Advances in Sov. Math., Vol.5, 1991, pp.139–194.
11. Muminov M., Rasulov T., The Faddeev equation and essential spectrum of a Hamiltonian in Fock Space, Methods Funct. Anal. Topology, Vol.17, No.1, 2011, pp.47–57.
12. Muminov M., Rasulov T., Embedded eigenvalues of an Hamiltonian in bosonic Fock space, Comm. Math. Anal., Vol.17, No.1, 2015, pp.1–22.
13. Rasulov T., The Faddeev equation and the location of the essential spectrum of a model multi-particle operator, Russian Math. (Iz. VUZ), Vol.52, No.12, 2008, pp.50–59.
14. Rasulov T., Study of the essential spectrum of a matrix operator, Theor. Math. Phys., Vol.164, No.1, 2010, pp.883–895.
15. Reed M., Simon B., Methods of modern mathematical physics. IV: Analysis of Operators, Academic Press, New York, 1979, p.396.
16. Sigal I., Soffer A., Zielinski L., On the spectral properties of Hamiltonians without conservation of the particle number, J. Math. Phys., Vol.42, No.4, 2002, pp.1844–1855.
17. Uchiyama J., Finiteness of the number of discrete eigenvalues of the Schroedinger operator for a three-particle system, Publ. Res. Inst. Math. Sci. Kyoto Univ. S. Y., Vol.5, 1969, pp.51–63.
18. Yafaev D., The discrete spectrum of the three-particle Schroedinger operator, Dokl. Akad. Nauk SSSR, Vol.206, No.1, 1972, pp.68–70.
19. Yafaev D., The finiteness of the discrete spectrum of the three-particle Schroedinger operator, Theor. Math. Phys., Vol.25, No.2, 1975, pp.185–195.
20. Yodgorov G., Muminov M., Spectrum of a Model Operator in the Perturbation Theory of the Essential Spectrum, Theor. Math. Phys., Vol.144, No.3, 2005, pp.1344–1352.
21. Zhislin G., The finiteness of the discrete spectrum of the energy operators of many particle quantum systems, Dokl. Akad. Nauk SSSR, Vol.207, No.1, 1972, pp.25–28.
22. Zhukov Yu., Minlos R., Spectrum and scattering in a spin-boson model with not more than three photons, Theor. Math. Phys., Vol.103, No.1, 1995, pp.398–411.
23. Zorich V., Mathematical Analysis, I. Springer-Verlag, Heildelberg, 2004, p.574.

Fok fəzasında Hamiltonianın məxsusi ədədləri sayının məhdudluğu

Tulkin Rəsulov

XÜLASƏ

D-ölçülü qəfəsdə hissəciklərin sayının saxlanılmadığı halda 3 hissəcikli qarşılıqlı təsiri təsvir edən, Hamiltonian ilə bağlı olan operator modelinə baxılır. Biz A operatorunun əsas spektrinin dəqiq yerləşməsi və strukturu üçün üç ümumiləşdirilmiş Fridrix modelini veririk. Biz diskret spektrin sonluluğu üçün kafi şərt deyil, A operatorunun məxsusi vektorları üçün Vaynberq tənliyinin simmetrik verisyasını qururuq.

Açar sözlər: Hamiltonian, Fock fəzası, anniqilyasiya və generasiya operator, ümumiləşdirilmiş Fridrix modeli, əsas və diskret spektr.

Ограниченность количества собственных значений Гамильтониа в пространстве Фока

Тулкин Х. Расулов

РЕЗЮМЕ

Рассматривается модел оператора связанного с Гамильтонианом системы трех частиц, описывающих взаимодействие, без сохранения числа частиц на d-мерной решетке. Мы точно описываем расположение и структура существенного спектра A по спектрам трех моделей обобщенных Фридрихсом. Мы получаем симметрическую версию уравнения Вайнберга для собственных векторов A, а не достаточные условия для конечности дискретного спектра.

Ключевые слова: Гамильтониан, пространство Фока, аннигиляция и операторы генерации, обобщенная модель Фридрихса, существенные и дискретные спектры.