SOME COMMON FIXED POINT THEOREMS OF COMPATIBLE MAPS WITH INTEGRAL TYPE CONTRACTION IN G-METRIC SPACES

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Abstract. In this paper, we prove some common fixed point theorems in G-metric space by using the notion of integral type contraction. An example is also provided to illustrate our results.

Keywords: G-metric space, common fixed point, compatible maps, integral type contractive mapping.

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1. Introduction

In 2003, Z. Mustafa and B. Sims [16] introduced a more appropriate and robust notion of a generalized metric space. In such kind of spaces a nonnegative real number is assigned to every triplet of elements. In [17] they proved some fixed point results for mapping satisfying sufficient conditions on complete G-metric space. After that several other fixed point theorems have been proved in G-metric spaces by many researchers, see [2, 3, 4, 6, 7, 8, 9, 16, 24, 25, 27]. The studies relevant to metric spaces are being extended to G-metric spaces by several other researchers. For instance, we noted that a best approximation result in G-metric spaces established by Nezhad and Mazaheri in [19], the notion of w-distance, which is relevant to minimization problem in metric spaces [13], has been extended by Saadati et al. [22] to G-metric spaces. Also, Shatanawi [26] gave the concept of ordered generalized metric spaces and presented some fixed point results in ordered G-metric spaces. There has been an important interest to study common fixed point for a pair of mappings that satisfying some contractive conditions in metric spaces. Some elegant and interesting results were obtained in this direction by various authors. In 1976, G. Jungck [10] introduced the notion of commutativity and presented some common fixed point theorems. Also G. Jungck [11] introduced the concept of compatible mappings and proved fixed point results. It is noticed that the problems of fixed point of non-compatible mappings are very important and considered in a number of research studies, see [12, 21]. Also weaker version of commutativity has been considered in a large number of works. One such notion is R-weakly commutativity. This is an extension of weakly commuting mappings [20, 23].
2. Preliminaries

The following definitions and results will be needed in this paper.

**Definition 1.** [16] Let \( Y \) be a nonempty set, and let \( G: Y \times Y \times Y \to \mathbb{R}^+ \) be a function satisfying the following axioms:

(W1) \( G(a, b, c) = 0 \) if \( a = b = c \),

(W2) \( 0 < G(a, a, b) \), for all \( a, b \in Y \) with \( a \neq b \),

(W3) \( G(a, a, b) \leq G(a, b, c) \), for all \( a, b, c \in Y \) with \( a \neq c \),

(W4) \( G(a, b, c) = G(a, c, b) = G(b, c, a) = \ldots \) (symmetry in all variables),

(W5) \( G(a, b, c) \leq G(a, s, s) + G(s, b, c) \), \( \forall a, b, c, s \in Y \), (rectangle inequality).

Then the function \( G \) is called a generalized metric, or more specifically a G-metric on \( Y \), and the pair \( (Y, G) \) is called a G-metric space.

**Example 1.** [16] Let \( Y = \{x, y\} \). Define \( G \) on \( Y \times Y \times Y \) by

\[
G(x, x, x) = G(y, y, y) = 0, G(x, x, y) = 1, G(x, y, y) = 2
\]

and extend \( G \) to \( Y \times Y \times Y \) by using the symmetry in the variables. Then it is clear that \( (Y, G) \) is a G-metric space.

**Definition 2.** [16] Let \( (Y, G) \) be a G-metric space and \( (a_n) \) a sequence of points of \( Y \). A point \( a \in Y \) is said to be the limit of the sequence \( (a_n) \), if

\[
\lim_{n \to +\infty} G(a_n, a, a) = 0
\]

and we say that the sequence \( (a_n) \) is G-convergent to \( a \).

**Proposition 1.** [16] Let \( (Y, G) \) be a G-metric space. Then the following are equivalent:

1. \((a_n)\) is G-convergent to \( a \).
2. \( G(a_n, a, a) \to 0 \) as \( n \to +\infty \).
3. \( G(a_n, a, a) \to 0 \) as \( n \to +\infty \).

**Definition 3.** [14] Let \( (Y, G) \) be a G-metric space. A sequence \( (a_n) \) is called G-Cauchy if for every \( \varepsilon > 0 \), there is \( k \in \mathbb{N} \) such that

\[
G(a_n, a_m, a_l) < \varepsilon, \quad \text{for all } n, m, l \geq k;
\]

that is \( G(a_n, a_m, a_l) \to 0 \) as \( n, m, l \to +\infty \).

**Proposition 2.** [16] Let \( (Y, G) \) be a G-metric space. Then the following are equivalent:

1. The sequence \( (a_n) \) is G-Cauchy.
2. For every \( \varepsilon > 0 \), there is \( k \in \mathbb{N} \) such that \( G(a_n, a_m, a_l) < \varepsilon \), for all \( n, m, l \geq k \).

**Definition 4.** [16] A G-metric space \( (Y, G) \) is called G-complete if every G-Cauchy sequence in \( (Y, G) \) is G-convergent in \( (Y, G) \).

**Proposition 3.** [16] Let \( (Y, G) \) be a G-metric space. Then for any \( a, b, c, e \in Y \), it follows that

1. if \( G(a, b, c) = 0 \), then \( a = b = c \);
2. \( G(a, b, c) \leq G(a, a, b) + G(a, a, c) \);
3. \( G(a, b, b) \leq 2G(b, a, a) \);
4. \( G(a, b, c) \leq G(a, e, c) + G(e, b, c) \).
(v) \( G(a, b, c) \leq 2 \left( G(a, b, e) + G(a, e, c) + G(e, b, c) \right) \);
(vi) \( G(a, b, c) \leq G(a, e, e) + G(b, e, e) + G(c, e, e) \).

**Proposition 4.** [16] Let \((Y, G)\) be a G-metric space. Then the function \( G(a, b, c) \) is jointly continuous in all three of its variables.

**Proposition 5.** [11] Let \( f \) and \( g \) be weakly compatible self-mappings on a set \( Y \). If \( f \) and \( g \) have unique fixed point of coincidence \( w = fa = ga \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

**Definition 5.** [11] Let \( f \) and \( g \) be two self-mappings on a metric space \((Y, d)\). The mappings \( f \) and \( g \) are said to be compatible if \( \lim_{n \to \infty} d(fa_n, ga_n) = 0 \), whenever \( \{a_n\} \) is a sequence in \( Y \) such that \( \lim_{n \to \infty} fa_n = \lim_{n \to \infty} ga_n = z \) for some \( z \in Y \).

**Definition 6.** [4] Let \((Y, G)\) be a G-metric space and \( H : Y \to Y \) be a self-mappings on \((Y, G)\). Now \( H \) is said to be a contraction if
\[
G(Ha, Hb, Hc) \leq \alpha G(a, b, c) \quad \text{for all} \quad a, b, c \in Y \quad \text{where} \quad \alpha \in (0, 1). \tag{1}
\]

Clearly every self-mapping \( H : Y \to Y \) satisfying condition (1) is continuous. To generalize the condition (2.1) for a pair of self-mappings \( S \) and \( H \) on \( Y \):
\[
G(Sa, Sb, Sc) \leq \alpha G(Ha, Hb, Hc) \quad \text{for all} \quad a, b, c \in Y \quad \text{where} \quad \alpha \in (0, 1). \tag{2}
\]

**Definition 7.** [4] let \( f \) and \( g \) be two self-mappings on a G-metric space \((Y, G)\). The two mappings are said to be compatible if \( \lim_{n \to \infty} G(fga_n, gfa_n) = 0 \), whenever \( \{a_n\} \) is a sequence in \( Y \) such that \( \lim_{n \to \infty} fa_n = \lim_{n \to \infty} ga_n = z \) for some \( z \in Y \).

In 2002, Branciari in [5] introduced a general contractive condition of integral type as follows.

**Theorem 1.** [5] Let \((Y, d)\) be a complete metric space, \( \alpha \in (0, 1) \), and \( f : Y \to Y \) is a mapping such that for all \( x, y \in Y \),
\[
\int_{\epsilon}^{d(f(x), f(y))} \varphi(t) dt \leq \alpha \int_{0}^{d(x, y)} \varphi(t) dt,
\]
where \( \varphi : [0, +\infty) \to [0, +\infty) \) is nonnegative and Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of \([0, +\infty)\) such that for each \( \epsilon > 0 \), \( \int_{0}^{\epsilon} \varphi(t) dt \), then \( f \) has a unique fixed point \( a \in Y \), such that
\[
\lim_{n \to \infty} f^n(x) = a.
\]
The aim of this research paper is to carry the above idea of integral type contractive mappings to G-metric spaces.
3. Main results

In this section, we prove some common fixed point results in the setting of $G$-metric spaces by using the idea of integral type contractive mappings. Our first main result is stated as:

**Theorem 2.** Let $(Y, G)$ be a complete $G$-metric space and $f, g$ be two self-mappings on $(Y, G)$ satisfies the following conditions:

1. $f(Y) \subseteq g(Y)$,
2. $f$ or $g$ is continuous,
3. \[
\int_0^t \varphi(t)dt \leq \alpha \int_0^t \varphi(t)dt + \beta \int_0^t \varphi(t)dt + \gamma \int_0^t \varphi(t)dt.
\]

For every $a, b, c \in Y$ and $\alpha, \beta, \gamma \geq 0$ with $0 \leq \alpha + 3\beta + 3\gamma < 1$ and $\varphi: [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. Then the mappings $f$ and $g$ have a unique common fixed point in $Y$ provided $f$ and $g$ are compatible maps.

**Proof.** Let $a_0$ be arbitrary in $Y$. Choose $a_1 \in Y$ such that $fa_0 = ga_1$. In general we can choose $a_{n+1}$ such that $b_n = fa_n = ga_{n+1}$, $n = 0, 1, 2, \ldots$.

From (5), we have

\[
\int_0^t \varphi(t)dt \leq \alpha \int_0^t \varphi(t)dt + \beta \int_0^t \varphi(t)dt + \gamma \int_0^t \varphi(t)dt
\]

\[
= \alpha \int_0^t \varphi(t)dt + \beta \int_0^t \varphi(t)dt + \gamma \int_0^t \varphi(t)dt
\]

\[
= (\beta + \gamma) \int_0^t \varphi(t)dt.
\]
By use of (G5) and Proposition 3, we have
\[ G(f_{a_{n-1}}, f_{a_n}, f_{a_{n+1}}) \leq G(f_{a_{n-1}}, f_{a_n}, f_{a_n}) + G(f_{a_n}, f_{a_{n+1}}) \]
\[ \leq G(f_{a_{n-1}}, f_{a_n}, f_{a_n}) + 2G(f_{a_n}, f_{a_{n+1}}, f_{a_{n+2}}). \]

Then,
\[ \int_0^1 \varphi(t) dt \leq (\beta + \gamma) \int_0^1 \varphi(t) dt \]
\[ \leq \{ (\beta + \gamma) \int_0^1 \varphi(t) dt \}
\]
\[ + 2G(f_{a_n}, f_{a_{n+1}}, f_{a_{n+2}}) \]
\[ \leq (\beta + \gamma) \int_0^1 \varphi(t) dt \]
\[ + (2\beta + 2\gamma) \int_0^1 \varphi(t) dt \]
\[ \leq \frac{(\beta + \gamma)}{1 - 2\beta - 2\gamma} \int_0^1 \varphi(t) dt \]
\[ \leq l \int_0^1 \varphi(t) dt. \]

where \( l = \frac{(\beta + \gamma)}{1 - 2\beta - 2\gamma} < 1. \)

Continuing this process, we get
\[ \int_0^1 \varphi(t) dt \leq l^n \int_0^1 \varphi(t) dt \]

For all \( n, m \in \mathbb{N}, n < m, \) we have
\[ \int_0^1 \varphi(t) dt \leq \int_0^1 \varphi(t) dt + \int_0^1 \varphi(t) dt \]
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\[
\begin{align*}
&+ \cdots + \int_0^{G(b_{m+1}, b_n, b_m) G(b_t, b, b_1)} \varphi(t) dt, \\
&\leq \left( l^n + l^{n+1} + \ldots + l^{m-1} \right) \int_0^{G(b_t, b, b_1)} \varphi(t) dt \\
&\leq \frac{l^n}{(1 - l)} \int_0^{G(b_t, b, b_1)} \varphi(t) dt \to 0 \text{ as } n, m \to \infty.
\end{align*}
\]

Thus,
\[
\lim_{n, m \to \infty} G(b_n, b_m, b_m) = 0.
\]

This means that \( \{b_n\} \) is a G-Cauchy sequence in \( Y \). Since \((Y, G)\) is complete G-metric space, therefore, there exists a point \( p \in Y \) such that
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} f a_n = \lim_{n \to \infty} g a_{n+1} = p.
\]

As the mapping \( f \) or \( g \) is continuous, so we can assume that \( g \) is continuous, therefore \( \lim_{n \to \infty} g f a_n = \lim_{n \to \infty} g g a_n = gp \).

Also \( f \) and \( g \) are compatible, therefore, \( \lim_{n \to \infty} G(f g a_n, g f a_n, g f a_n) = 0 \), this implies \( \lim_{n \to \infty} f g a_n = gp \).

From (5), we have
\[
\begin{align*}
\int_0^{G(f g a_n, f a_n, f a_n)} \varphi(t) dt &\leq \alpha \int_0^{G(f g a_n, g a_n, g a_n)} \varphi(t) dt + \beta \int_0^{G(g a_n, f a_n, f a_n)} \varphi(t) dt \\
&+ \gamma \int_0^{G(g a_n, g a_n, f a_n)} \varphi(t) dt.
\end{align*}
\]

Taking limit as \( n \to \infty \), we have \( gp = p \).

Again from condition (5), we have
\[
\begin{align*}
\int_0^{G(f a_n, f_p, f_p)} \varphi(t) dt &\leq \alpha \int_0^{G(f a_n, g_p, g_p)} \varphi(t) dt + \beta \int_0^{G(g a_n, f_p, f_p)} \varphi(t) dt \\
&+ \gamma \int_0^{G(g a_n, g_p, f_p)} \varphi(t) dt.
\end{align*}
\]

By taking limit as \( n \to \infty \), we have \( p = fp \). Therefore, we have \( gp = fp = p \). Thus \( p \) is a common fixed point of \( f \) and \( g \).

For uniqueness, we suppose that \( p \neq p \) be another common fixed point of \( f \) and \( g \). Then
This arise contradiction and hence $p_1 = p$. The proof is completed.

**Corollary 1.** Let $(Y, G)$ be a complete $G$-metric space and $f, g$ be two compatible self-mappings on $(Y, G)$ satisfies assertions $(3), (4)$ and the following condition:

\[
\int_0^l \phi(t) dt \leq \int_0^{G(a, b, c)} \phi(t) dt + \int_0^{G(a, b, c)} \phi(t) dt + \int_0^{G(a, b, c)} \phi(t) dt,
\]

for all $a, b, c \in Y$ and $0 < l < 1$. Then $f$ and $g$ have a unique common fixed point in $Y$.

**Theorem 3.** Let $f$ and $g$ be two weakly compatible self-mappings of a $G$-metric space $(Y, G)$ satisfying conditions $(3)$ and $(5)$ and any one of the subspace $f(Y)$ or $g(Y)$ is complete. Then $f$ and $g$ have a unique common fixed point in $Y$.

**Proof.** From the main result 3, we conclude that $\{b_n\}$ is a $G$-Cauchy sequence in $Y$. Since either $f(Y)$ or $g(Y)$ is complete, we assume that $g(Y)$ is complete subspace of $Y$ then the subsequence of $\{b_n\}$ must get a limit in $g(Y)$ be $p$. Let $v \in g^{-1}p$. Then $gv = p$ as $\{b_n\}$ is a $G$-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{b_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence. Now we can show that $fv = p$.

Setting $a = v, b = a_n$ and $p = a_n$, in condition $(5)$, we have

\[
\int_0^{G(p, p_1, p_1)} \phi(t) dt = \int_0^{G(p, p_1, p_1)} \phi(t) dt \\
\leq \int_0^{G(p, p_1, p_1)} \phi(t) dt + \int_0^{G(p, p_1, p_1)} \phi(t) dt \\
+ \int_0^{G(p, p_1, p_1)} \phi(t) dt, \]

As $n \to \infty$ in above inequality, we get

\[
G(p, p_1, p_1) \int_0^l \phi(t) dt = \int_0^l \phi(t) dt
\]

This arise contradiction and hence $p_1 = p$. The proof is completed.
Implies that $f v = p$.

Therefore, $f v = g v = p$, that is, $v$ is a coincident point of two mappings $f$ and $g$. Since the two mappings $f$ and $g$ are weakly compatible, it follows that $f g v = g f v$, that is, $f p = g p$.

Next we show that $f p = p$. Further we assume that $f p \neq p$.

From condition (5), we set $a = p$, $b = v$, $p = v$, we have

$$G(f p, p, p) \int_0^l \varphi(t) \, dt \leq \alpha G(f v, p, p) \int_0^l \varphi(t) \, dt + \beta G(g p, f v, g v) \int_0^l \varphi(t) \, dt + \gamma G(g p, g v, f v) \int_0^l \varphi(t) \, dt.$$

Which is contradiction and hence $f p = p$. Therefore, $f p = g p = p$ that is, $p$ is common fixed point of mappings $f$ and $g$. We can show the uniqueness as above easily. The proof is completed.

We now give an example to illustrate Theorem 2.

**Example 2.** Suppose that $Y = [0,1]$ and also assume that $G$ be the $G$-metric on $Y \times Y \times Y$ defined as $G(a, b, c) = |a - b| + |b - c| + |c - a| \forall a, b, c \in Y$.

Then $(Y, G)$ be a $G$-metric space. We define $f a = \frac{a}{6}$ and $g a = \frac{a}{2}$. Also we noted that, the mapping $f$ is continuous and $f(Y) \subseteq g(Y)$. Also,

$$G(f a, f b, f c) \int_0^l \varphi(t) \, dt \leq l G(g a, g b, g c) \int_0^l \varphi(t) \, dt,$$

holds for all $a, b, c \in Y$, $\frac{1}{3} \leq l < 1$ and $0$ is the unique common fixed point of $f$ and $g$. 

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G-metrik fəzalarda inteqral tipli sıxilmalarla uzlaşma inikaslar tərpənmə cəvər nöqtə haqqında bəzi ümumi teoremlər

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XÜLASƏ

Məqalədə G-metrik fəzalarda inteqral tip sıxilmalarından istifadə etməyən tərpənmə cəvər nöqtələr haqqında bəzi ümumi teoremlər isbat edilir. Alkınan nəticələri illüstrasiya edən məsələlər verilmişdir.

Açar sözər: G-metrik fəzalar, ümumi tərpənmə cəvər, uzlaşan inikaslar, inteqral tip sıxilmiş inikas.

Некоторые общие теоремы о неподвижных точках совместимых отображений с интегрального типа сжатиями в G-метрических пространствах

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РЕЗЮМЕ

В данной работе мы докажем некоторые общие теоремы о неподвижной точке в G-метрическом пространстве, используя понятие сжатия интегрального типа. Дается пример для иллюстрации наших результатов.

Ключевые слова: G-метрические пространства, общие неподвижные точки, совместимые отображения, отображения с интегрального типа сжатиями