ON AN EXTREMUM PROBLEM IN THE METRIC SPACES

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Abstract. In the present paper, we study unconditional extremum problem in metric spaces, obtained from conditional extremum problem by using the notion of covering operators. The necessary optimality conditions for the considered problem are obtained.

Keyword: Banach space, strictly differentiability, optimality condition.

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1. Introduction

Our aim in this paper is solving the unconditional extremum problem in metric space. Using the distance function some theorems are obtained on the exact penalty, and high order necessary and sufficient conditions are derived within constraints. The exact penalty function is constructed [3] for the extremal problems with constraints using the distance functions, in the class of Lipschitz functions, which is defined for \((\alpha, \beta, \varphi, \delta)\) and \(\varphi - (\alpha, \beta, \varphi, \delta)\) in [2]. In the point of Banach space, where some properties of the penalty functions corresponding are studied, and extremal problems with constraints are investigated.

2. Preliminary facts

Let \((X,d_X), (Y,d_Y)\) be metric spaces, \(F: X \rightarrow Y\) operator, \(B_X(x,r)\) -sphere of radius \(r\), with the center at the point \(x\) of the space \(X\); \(B_Y(y,z)\) -sphere of radius \(r\), with the center at the point \(y\) of the space \(Y\). Let \(U \subset X\) be open set.

We say (see [1]), that the operator \(F\) covers on the set \(U\) with the constant \(a > 0\) if for any \(B_X(x,r) \subset U\) the inclusion

\[ F(B_X(x,r)) \supset B_Y(F(x),ar) \]

is valid.

Let’s define

\[ d_M(x) = \inf \{d_X(x,y) : y \in M\}, \]

where \(M \subset X\).
3. Main results

**Lemma 1.** Let \( M \subseteq S \subseteq X \) and \( \bar{x} \) be a minimum point of the function \( f : X \to R \) on the set \( M \), the function \( f \) satisfies to the Lipschitz condition on the set \( S \) with the constant \( K \). Then for any \( \bar{K} \geq K \) the function \( g(y) = f(y) + \bar{K}d_M(y) \) reaches the minimum on the set \( S \) at the point \( \bar{x} \) and if \( M \) is closed and \( \bar{K} \geq K \) any point minimized \( g(y) \) on set \( S \) belongs to \( M \). Let \( B(\Omega; \delta) = \{ x \in X : d(x, \Omega) \leq \delta \} \).

**Proof.** Let the first statement of lemma 1 be not satisfied, i.e. there exists \( \bar{y} \in S \), such that \( g(\bar{y}) \leq g(\bar{x}) \). Then there exists such \( y \in S \) and \( \varepsilon > 0 \) that
\[
 f(y) + \bar{K}\rho(y) < f(\bar{x}) - \bar{K}\varepsilon.
\]
Suppose that the point \( c \in M \) satisfies to the condition \( d(y, c) \leq \rho(y) + \varepsilon \).
Then
\[
 f(c) \leq f(y) + \bar{K}d(y, c) \leq f(y) + \bar{K}(\rho(y) + \varepsilon) < f(x).
\]
But this is contrary to the fact that \( \bar{x} \) gives minimum to \( f \) in the set \( M \).

Let \( \bar{K} > K \) and \( \bar{y} \) minimized the function \( g \) on \( S \). Then
\[
 f(\bar{y}) + \bar{K}\rho(\bar{y}) = f(\bar{x}) \leq f(\bar{y}) + \frac{K + \bar{K}}{2} \rho(\bar{y}).
\]
From this we obtain that \( g(\bar{y}) = 0 \). Since \( M \) is closed set, \( \bar{y} \in M \). Lemma is proved.

Let us consider minimization of the function \( f : X \to R \) on the set \( M = \{ x \in G : F(x) = F(x_0) \} \), where \( G \) is an open set in \( X \) and \( F : G \to Y \) operator.

**Theorem 1.** Let \( Y \) be Banach space, \( G \) be an open set in \( X \), \( X \) - full metric space, the continuous operator \( F : G \to Y \) covers with a constant \( a > 0 \) on the open set \( G \subseteq X ; \ \Omega \subseteq G \) bounded set; there exists a number \( \delta > 0 \) such, that \( B(\Omega; \delta) \subseteq G \), \( \{ x \in G : F(x) \neq F(x_0) \} = \{ x \in \Omega : F(x) = F(x_0) \} \); \( f \) satisfies to the Lipschitz condition on the set \( \Omega \) with the constant \( K \) and the point \( x_0 \) be a minimum point of the function \( f \) on the set \( M = \{ x \in \Omega : F(x) = F(x_0) \} \). Then there exists such number \( L > 0 \), that \( x_0 \) is a minimum point for the function \( f(x) + L\| F(x) - F(x_0) \| \) on the set \( \Omega \).

**Proof.** It is clear that \( M \subseteq \Omega \) and the conditions of lemma 1 are satisfied. Then as follows from lemma 1 \( x_0 \) gives minimum to the function \( f(x) + \bar{K}\rho(x, M) \) on the set \( \Omega \), then \( \bar{K} \geq K \).

On the base of lemma 44 [1] there exists a number \( L \) such, that the inequality
\[
 \rho(x, M) \leq L\| F(x) - F(x_0) \|.
\]
holds when \( x \in \Omega \). Thus, the inequality
\[
f(x) + K_\rho(x,M) \leq f(x) + K \| F(x) - F(x_0) \|
\]
also holds when \( x \in \Omega \). Since \( x_0 \in M \), we obtain that
\[
f(x_0) = f(x_0) + K_\rho(x_0,M) = f(x_0) + K \| F(x_0) - F(x_0) \|
\]
Then by \( \forall x \in \Omega \)
\[
f(x_0) = f(x_0) + K \| F(x_0) - F(x_0) \| \leq f(x) + K \| F(x) - F(x_0) \|
\]
Theorem is proved.

Let \( X \) and \( Y \) be Banach spaces. If \( A: X \to Y \) is a linear continuous operator and \( A(X) = Y \), it follows from the theorem on the open mappings that there exists a number \( a > 0 \) such that \( A(B_X(x,r)) \supseteq B_Y(Ax,ar) \) at \( x \in X \, \, r > 0 \), the operator \( A \) covers on \( X \) with the constant \( a \).

**Lemma 2.** If \( X \) and \( Y \) are Banach spaces, \( G \) - open set in \( X \), \( F: G \to Y \) is strictly differentiable, \( \text{Im} F'(x) = Y \) at \( x \in G \) and \( \| F'(x) \|^{-1} \) is bounded in \( G \), then \( F \) covers with the constant
\[
\alpha = \frac{1}{\sup_{x \in G} \| (F'(x))^* \|^{-1}}
\]
on the set \( G \).

**Proof.** As follows from Lusternik-Grayvs theorem [4], for any \( x \in G \) the exists \( K(x) > 0 \), such that \( B(F(z),t) \subseteq B(F(z),K(z)t) \) by enough close to \( x \), \( z \) and small \( t > 0 \). It means that there exist \( \delta_1(x) > 0 \) and \( \delta_2(x) > 0 \), such that
\[
B(F(z),t) \subseteq B(F(z),K(x)t),
\]
for \( \| z - x \| \leq \delta_1(x) \) and \( 0 < t \leq \delta_2(x) \). Taking \( \tau = K(x)t \) we have
\[
B \left( F(z), \frac{1}{K(x)} \tau \right) \subseteq B(B(z,\tau))
\]
by \( \| z - x \| \leq \delta_1(x) \) and \( 0 < \tau \leq K(x)\delta_2(x) \). Moreover, upper bound of \( K(x) \) is equal to \( \| (F'(x))^* \|^{-1} \). If we assume that \( K(x) = \| (F'(x))^* \|^{-1} \) and
\[
a = \frac{1}{\sup_{x \in G} \| (F'(x))^* \|^{-1}},
\]
we get \( B(F(z),a\tau) \subseteq B(B(z,\tau)) \) for \( \| z - x \| \leq \delta_1(x) \) and
\[
0 < \tau \leq \frac{1}{a} \delta_2(x) = K(x)\delta_2(x).
\]

Lemma is proved.

Let \( X \) be Banach space
\[
f: X \to \overline{R} = R \cup \{ \pm \infty \}, \, \text{dom}f = \{ x \in X : \| f(x) \| < \infty \}, \, x_0 \in \text{dom}f.
\]
Let’s define (see [2]) \( f^+(x_0; x) = \lim_{t \downarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t} \),
\( f^−(x_0; x) = \lim_{t \downarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t} \), then
\( f^+(x_0; x), \) \( f^−(x_0; x) \) = \( \max \{ f^+(x_0; x), f^−(x_0; x) \} \).

Let \( |f(x_0)| < +\infty \). Let’s define (see [3])
\[ f^\uparrow(x_0; x) = \lim_{(y, \alpha) \downarrow f^\uparrow, \alpha \to x} \frac{f(y + t \alpha) - \alpha}{t}, \]
where the symbol \((y, \alpha) \downarrow f^\uparrow, \alpha \to x\) means, that
\( (y, \alpha) \in E(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}, \)
\( y \to x_0, \alpha \to f(x_0). \)

We say, that the function \( f \) in the point \( x_0 \in \text{dom} f \) admits sublinear approximation \( h(x) \), if \( h(x) \) sublinear semi-continuous from below function and \( h(x) \geq f^\downarrow(x_0; x) \) at \( x \in X \). Sublinear approximation \( h \) of the function \( f \) in the point \( x_0 \) refers to as the main approximation, if there not exists another sublinear approximation \( h_1 \), such, that \( h(x) \geq h_1(x) \) at \( x \in X \). Further we accept, that the main sublinear approximation \( h(x) \) of the function \( f \) in the point \( x_0 \) additionally satisfies to the inequality: \( h(x) \leq f^\uparrow(x_0; x) \) by \( x \in X \), and \( \partial h(0) \) we call as a main approximate on subdifferential of the function \( f \) in the point \( x_0 \) and denote by \( \partial^2 f(x_0) \) [2].

Let \( B^* \) be unit sphere in \( Y^* \).

**Theorem 2.** If \( X \) and \( Y \) are Banach spaces, the condition of the theorem 1 are satisfied, \( x_0 \in \text{int} \Omega \) and \( F : X \to Y \) is strictly differentiable at the point \( x_0 \), then there exists a number \( L > 0 \) such, that
\[ 0 \in \partial^m f(x_0) + LB^*F'(x_0). \]

**Proof.** On the base of theorem 1 there exists such \( L > 0 \), that the point \( x_0 \) is a minimum point for the function \( g(x) = f(x) + L\|F(x) - F(x_0)\| \) on the set \( \Omega \). Since \( x_0 \in \text{int} \Omega \), it is easy to cheek that \( o \in \partial^m g(x_0) \). From theorem 1.2 [1] follows that
\[ \partial^m\|F(x) - F(x_0)\|_{x=x_0} = B^*F'(x_0). \]

Under the conditions of theorem 2 the relation
\[ \partial^m g(x_0) \subset \partial^m f(x_0) + LB^*F'(x_0), \]
is valid (see [2]). Therefore
\[ 0 \in \partial^m f(x_0) + LB^*F'(x_0). \]

The theorem is proved.
Theorem 3. Let $X, Y$ be Banach spaces, $G$-open set in $X$ and $F : G \to Y$, operator, the point $x_0$ be a local minimum of the function $f$ on the set $M = \{x \in G : F(x) = F(x_0)\}$, $f$ satisfies to Lipschitz condition in the neighborhood of the point $x_0$, $F$ is strictly differentiable at the point $x_0$ and $F'(x_0)X = Y$. Then there exists a number $L > 0$ such, that

$$0 \in \partial^m f(x_0) + LB^* F'(x_0).$$

Proof. Consider the problem

$$f(x) \to \inf, F(x) - F(x_0) = 0. \tag{1}$$

It is clear that $x_0$ is a local minimum point also for the problem (1), i.e. there exists such $\delta > 0$, that $x_0$ is a minimum point for the problem (1) in the $\delta$ neighborhood of this point. Taking $C = \{x \in B(x_0, \delta) : F(x) = F(x_0)\}$, on the base of the lemma 1 we get that the function $f(x) + Kd_c(x)$ reaches its minimum on $B(x_0, \delta)$ in the point $x_0$. Since $F$ is strong differentiable in the point $x_0$ and $F'(x_0)X = Y$, then as follows from theorem 2.2 [1] there exists a neighborhood $O(x_0)$ of the point $x_0$, and $m > 0$ such holds that the inequality $d_c(x) \leq m\|F(x) - F(x_0)\|$ holds for $x \in O(x_0)$. Therefore

$$f(x_0) \leq f(x_0) + Km\|F(x_0)\| \leq f(x) + Kd_c(x) \leq f(x) + Km\|F(x) - F(x_0)\|,$$

by $x \in B(x_0, \delta) \cap O(x_0)$, i.e. the point $x_0$ gives minimum to the function $f(x) + Km\|F(x) - F(x_0)\|$ on the set $x \in B(x_0, \delta) \cap O(x_0)$. Since $F$ is strong differentiable in the point $x_0$, it follows from preposition 2.2.1 [3] that $F$ satisfies to the Lipschitz condition in the neighborhood of the point $x_0$. Then from the theorem 1 we have that $0 \in \partial^m (f(\cdot) + Km\|F(\cdot) - F(x_0)\|)_{x=x_0}$. Applying the theorem 1.3 and lemma 1.8 [2], we obtain that $0 \in \partial^m (f(x_0) + KmB^* F'(x_0))$. The theorem is proved.

References


Metrik fəzada ekstremum məsələsi

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XÜLASƏ

İşdə metrik fəzada şərti ekstremum məsələsinə baxılır. Əgər məhdudüyət operatoru örtüləş xəsəsinə malik olarsa, onda şərti ekstremum məsələsi şərtsiz ekstremum məsələsinə gətirilir. Şərtli ekstremum məsələsinin həllin üçün zəruri şərt tapılır.

Açar sözər: Banax fəzası, ciddi diferensiallanan optimallıq şərti.

Задача экстремума в метрическом пространстве

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РЕЗЮМЕ

В работе рассматривается задача экстремума в метрическом пространстве. Используя понятие накрывания операторов, задача на условной экстремум приводится к задаче на безусловный экстремум. Находится необходимое условие для решения задачи условного экстремума.

Ключевые слова: Пространство Банаха, строго дифференцируемость, условия оптимальности.