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ON AN EXTREMUM PROBLEM IN THE METRIC SPACES

H.S. Akhundov[1]

[1]Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan,
e-mail: axund2005@rambler.ru

Abstract. In the present paper, we study unconditional extremum problem in metric spaces, obtained from conditional extremum problem by using the notion of covering operators. The necessary optimality conditions for the considered problem are obtained .

Keyword: Banach space, strictly differentiablity, optimality condition.

AMS Subject Classification: 46N10, 54E35.

1. Introduction

Our aim in this paper is solving the unconditional extremum problem in metric space. Using the distance function some theorems are obtained on the exact penalty, and high order necessary and sufficient conditions are derived within constraints. The exact penalty function is constructed [3] for the extremal problems with constraints using the distance functions, in the class of Lipschitz functions, which is defined for $(\alpha,\beta,\vartheta,\delta)$ and $\varphi-(\alpha,\beta,\vartheta,\delta)$ in [2]. In the point of Banach space, where some properties of the penalty functions corresponding are studied, and extremal problems with constraints are investigated.

2. Preliminary facts

Let (X,d_X), (Y,d_Y) be metric spaces, $F:\mathrm{X}\to\mathrm{Y}$ operator, $B_X(x,r)$ -sphere of radius r, with the center at the point x of the space X; $B_Y(y,z)$ -sphere of radius r, with the center at the point y of the space Y. Let $U\subset X$ be open set.

We say (see [1]), that the operator F covers on the set U with the constant $a>0$ if for any $B_X(x,r)\subset U$ the inclusion

$$F(B_X(x,r))\supset B_Y(F(x),ar)$$

is valid.

Let's define

$$d_M(x)=\inf\{d_X(x,y):y\in M\},$$

where $M\subset X$.

3. Main results

Lemma 1. Let $M \subset S \subset X$ and $\bar{x}$ be a minimum point of the function $f: X \to R$ on the set M, the function f satisfies to the Lipschitz condition on the set S with the constant K. Then for any $\bar{K} \ge K$ the function $g(y) = f(y) + \bar{K} d_M(y)$ reaches the minimum on the set S at the point $\bar{X}$ and if M is closed and $\bar{K} \ge K$ any point minimized $g(y)$ on set S belongs to M. Let $B(\Omega;\delta) = \{x \in X : d(x, \Omega) \le \delta\}$.

Proof. Let the first statement of lemma 1 be not satisfied, i.e. there exists $\bar{y} \in S$, such that $g(\bar{y}) \le g(\bar{x})$. Then there exists such $y \in S$ and $\varepsilon > 0$ that $f(y) + \bar{K}\rho(y) < f(\bar{x}) - \bar{K}\varepsilon$.

Suppose that the point $c \in M$ satisfies to the condition $d(y,c) \le \rho(y) + \varepsilon$. Then

$$f(c) \le f(y) + \bar{K} d(y,c) \le f(y) + \bar{K}(\rho(y) + \varepsilon) < f(x).$$

But this is contrary to the fact that $\bar{X}$ gives minimum to f in the set M.

Let $\bar{K} > K$ and $\bar{y}$ minimized the function g on S. Then

$$f(\bar{y}) + \bar{K}\rho(\bar{y}) = f(\bar{x}) \le f(\bar{y}) + \frac{K + \bar{K}}{2}\rho(\bar{y}).$$

From this we obtain that $g(\bar{y}) = 0$. Since M is closed set, $\bar{y} \in M$.
Lemma is proved.

Let us consider minimization of the function $f: X \to R$ on the set $M = \{x \in G : F(x) = F(x_0)\}$, where G is an open set in X and $F: G \to Y$ operator.

Theorem 1. Let Y be Banach space, G be an open set in X, X - full metric space, the continuous operator $F: G \to Y$ covers with a constant $a > 0$ on the open set $G \subset X$; $\Omega \subset G$ bounded set; there exists a number $\delta > 0$ such, that $B(\Omega;\delta) \subset G$, $\{x \in G : F(x) = F(x_0)\} = \{x \in \Omega : F(x) = F(x_0)\}$; f satisfies to the Lipschitz condition on the set Ω with the constant K and the point x_0 be a minimum point of the function f on the set $M = \{x \in \Omega : F(x) = F(x_0)\}$. Then there exists such number $L > 0$, that x_0 is a minimum point for the function $f(x) + L\|F(x) - F(x_0)\|$ on the set Ω.

Proof. It is clear that $M \subset \Omega$ and the conditions of lemma 1 are satisfied. Then as follows from lemma 1 X_0 gives minimum to the function $f(x) + \bar{K}\rho(x, M)$ on the set Ω, when $\bar{K} \ge K$.

On the base of lemma 44 [1] there exists a number L such, that the inequality

$$\rho(x, M) \le L\|F(x) - F(x_0)\|,$$

holds when $x \in \Omega$. Thus, the inequality

$$f(x) + \overline{K}\rho(x, M) \le f(x) + \overline{K}L\|F(x) - F(x_0)\|,$$

also holds when $x \in \Omega$. Since $x_0 \in M$, we obtain that

$$f(x_0) = f(x_0) + \overline{K}\rho(x_0, M) = f(x_0) + \overline{K}L\|F(x_0) - F(x_0)\|.$$

Then by $\forall x \in \Omega$

$$f(x_0) = f(x_0) + \overline{K}L\|F(x_0) - F(x_0)\| \le f(x) + \overline{K}L\|F(x) - F(x_0)\|.$$

Theorem is proved.

Let X and Y be Banach spaces. If $A: X \to Y$ is a linear continuous operator and $A(X) = Y$, it follows from of the theorem on the open mappings that there exists a number $a > 0$ such, that $A(B_X(x, r)) \supset B_Y(Ax, ar)$ at $x \in X$, $r > o$, the operator A covers on X with the constant a.

Lemma 2. If X and Y are Banach spaces, G - open set in X, $F: G \to Y$ is strictly differentiable, $\operatorname{Im} F'(x) = Y$ at $x \in G$ and $\left\|[(F'(x))^*]^{-1}\right\|$ is bounded in G, then F covers with the constant

$$\alpha = \frac{1}{\sup_{x \in G}\left\|[(F'(x))^*]^{-1}\right\|}$$

on the set G.

Proof. As follows from Lusternik-Grayvs theorem [4], for any $x \in G$ the exists $K(x) > 0$, such that $B(F(z), t) \in F(B(x)K(z)t))$ by enough close to x, z and small $t > 0$. It means that there exist $\delta_1(x) > 0$ and $\delta_2(x) > 0$, such, that

$$B(F(z, t) \subset F(B(z, K(x)t),$$

for $\|z - x\| \le \delta_1(x)$ and $0 < t \le \delta_2(x)$. Taking $\tau = K(x)t$ we have

$$B\left(F(z), \frac{1}{K(x)}\tau\right) \subset F(B(z, \tau)),$$

by $\|z - x\| \le \delta_1(x)$ and $0 < \tau \le K(x)\delta_2(x)$. Moreover, upper bound of $K(x)$ is equal to $\left\|[(F'(x))^*]^{-1}\right\|$. If we assume that $K(x) = \left\|[(F'(x))^*]^{-1}\right\|$ and $a = \dfrac{1}{\sup_{x \in G}\left\|[(F(x))^*]^{-1}\right\|}$, we get $B(F(z), a\tau) \subset F(B(z, \tau))$ for $\|z - x\| \le \delta_1(x)$ and $0 < \tau \le \overline{\delta}_2(x) = K(x)\delta_2(x)$.

Applying lemma 5 [43] or theorem 5-48 [4] we get the proof of lemma 2.

Lemma is proved.

Let X be Banach space

$f: X \to \overline{R} = R \cup \{\pm\infty\}$, $dom f = \{x \in X : |f(x)| < +\infty\}$, $x_0 \in dom f$.

Let's define (see [2]) $f^{+}(x_0;x)=\overline{\lim_{t\downarrow 0}}\dfrac{f(x_0+tx)-f(x_0)}{t}$,

$f^{-}(x_0;x)=\underline{\lim_{t\downarrow 0}}\dfrac{f(x_0+tx)-f(x_0)}{t}$, $f^{I}(x_0;x)=\max\left\{f^{+}(x_0;x),-f^{-}(x_0;-x)\right\}$.

Let $|f(x_0)|<+\infty$. Let's define (see [3])

$f^{\uparrow}(x_0;x)=\lim\limits_{\substack{(y,\alpha)\downarrow f^{x_0},\vartheta\to x\\ t\downarrow 0}}\dfrac{f(y+t\vartheta)-\alpha}{t}$, where the symbol $(y,\alpha)\downarrow f^{x_0}$ means, that

$$(y,\alpha)\in E(f)=\{(x,\alpha)\in X\times R: f(x)\le\alpha\},\quad y\to x_0,\ \alpha\to f(x_0).$$

We say, that the function f in the point $x_0\in dom f$ admits sublinear approximation $h(x)$, if $h(x)$ sublinear semi-continous from below function and $h(x)\ge f^{I}(x_0;x)$ at $x\in X$. Sublinear approximation h of the function f in the point x_0 refers to as the main approximation, if there not exists another sublinear approximation h_1, such, that $h(x)\ge h_1(x)$ at $x\in X$. Further we accept, that the main sublinear approximation $h(x)$ of the function f in the point x_0 additionally satisfies to the inequality: $h(x)\le f^{\uparrow}(x_0;x)$ by $x\in X$, and $\partial h(0)$ we call as a main approximate on subdifferential of the function f in the point x_0 and denote by $\partial^{гл} f(x_0)$ [2].

Let B^* be unit sphere in Y^*.

Theorem 2. If X and Y are Banach spaces, the condition of the theorem 1 are satisfied, $x_0\in\operatorname{int}\Omega$ and $F:X\to Y$ is strictly differentiable at the point x_0, then there exists a number $L>0$ such, that

$$0\in\partial^{m} f(x_0)+LB^*F'(x_0).$$

Proof. On the base of theorem 1 there exists such $L>0$, that the point x_0 is a minimum point for the function $g(x)=f(x)+L\|F(x)-F(x_0)\|$ on the set Ω. Since $x_0\in\operatorname{int}\Omega$, it is easy to cheek that $o\in\partial^{m} g(x_0)$. From theorem 1.2 [1] follows that

$$\partial^{m}\|F(x)-F(x_0)\|_{x=x_0}=B^*F'(x_0).$$

Under the conditions of theorem 2 the relation

$$\partial^{m} g(x_0)\subset\partial^{m} f(x_0)+LB^*F'(x_0),$$

is valid (see [2]). Therefore

$$0\in\partial^{m} f(x_0)+LB^*F'(x_0).$$

The theorem is proved.

Theorem 3. Let X, Y be Banach spaces, G-open set in X and $F: G \to Y$, operator, the point x_0 be a local minimum of the function f on the set $M = \{x \in G : F(x) = F(x_0)\}$, f satisfies to Lipschtz condition in the neighborhood of the point x_0, F is strictly differentiable at the point x_0 and $F'(x_0)X = Y$. Then there exists a number $L > 0$ such, that

$$0 \in \partial^m f(x_0) + LB^* F'(x_0) .$$

Proof. Consider the problem

$$f(x) \to \inf, \ F(x) - F(x_0) = 0 . \tag{1}$$

It is clear that x_0 is a local minimum point also for the problem (1), i.e. there exists such $\delta > 0$, that x_0 is a minimum point for the problem (1) in the δ neighborhood of this point. Taking $C = \{x \in B(x_0, \delta) : F(x) = F(x_0)\}$, on the base of the lemma 1 we get that the function $f(x) + Kd_c(x)$ reaches its minimum on $B(x_0, \delta)$ in the point x_0. Since F is strong differentiable in the point x_0 and $F'(x_0)X = Y$, then as follows from theorem 2.2 [1] there exists a neighborhood $O(x_0)$ of the point x_0, and $m > 0$ such holds that the inequality $d_c(x) \le m\|F(x) - F(x_0)\|$ holds for $x \in O(x_0)$. Therefore

$$f(x_0) \le f(x_0) + Km\|F(x_0)\| \le f(x) + Kd_c(x) \le f(x) + Km\|F(x) - F(x_0)\| ,$$

by $x \in B(x_0, \delta) \cap O(x_0)$, i.e. the point x_0 gives minimum to the function $f(x) + Km\|F(x) - F(x_0)\|$ on the set $x \in B(x_0, \delta) \cap O(x_0)$. Since F is strong differentiable in the point x_0, it follows from preposition 2.2.1 [3] that F satisfies to the Lipschitz condition in the neighborhood of the point x_0. Then from the theorem 1 we have that $0 \in \partial^m (f(\cdot) + Km\|F(\cdot) - F(x_0)\|)_{x = x_0}$. Applying the theorem 1.3 and lemma 1.8 [2], we obtain that $0 \in \partial^m (f(x_0) + KmB^* F'(x_0)$. The theorem is proved.

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Metrik fəzada ekstremum məsələsi

H.S. Axundov

XÜLASƏ

İşdə metrik fəzada şərti ekstremum məsələsinə baxılır. Əgər məhdudiyyət operatoru örtülmə xassəsinə malik olarsa, onda şərti ekstremum məsələsi şərtsiz ekstremum məsələsinə gətirilir. Şərti ekstremum məsələsinin həlli üçün zəruri şərt tapılır.

Açar sözlər: Banax fəzası, ciddi diferensiallanan optimallıq şərti.

Задача экстремума в метрическом пространстве

Х.С. Ахундов

РЕЗЮМЕ

В работе рассматривается задача экстремума в метрическом пространстве. Используя понятие накрывания операторов, задача на условной экстремум приводится к задаче на безусловный экстремум. Находится необходимое условие для решения задачи условного экстремума.

Ключевые слова: Пространство Банаха, строго дифференцируемость, условия оптимальности.