SOME INEQUALITIES OF MEROMORPHIC $p$-VALENT FUNCTIONS ASSOCIATED WITH THE LIU-SRIVASTAVA OPERATOR

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Abstract. We derive several inequalities associated with a linear operator defined for a certain family of meromorphic $p$-valent functions. Also, we indicate relevant connections the various results present in this paper with those obtained in earlier work.

Keywords: analytic function, Schwarz function, generalized hypergeometric function, linear operator, Hadamard product, subordination.

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1. Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and meromorphic $p$-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} and 0 < |z| < 1\} = U\setminus\{0\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written symbolically as $f \prec g$ or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1(z \in U)$ such that $f(z) = g(w(z))(z \in U)$. In particular, if the function $g$ is univalent in $U$, we have the equivalence (see [6]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in \sum_p$ given by (1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of $f$ and $g$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z).$$
For real parameters \( a_1, \ldots, a_q \) and \( b_1, \ldots, b_s \) \( (b_j \in \mathbb{Z}_0 = \{0, -1, -2, \ldots\}; j = 1, \ldots, s) \), we now define the generalized hypergeometric function \( {}_q F_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) by (see [8])
\[
{}_q F_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_{k} \cdots (a_q)_{k}}{(b_1)_{k} \cdots (b_s)_{k}} \frac{z^k}{k!} \quad (q \leq s + 1; q, s \in \mathbb{N}; z \in U),
\]
where \( (\theta)_v \) is the Pochhammer symbol defined, in terms of the Gamma function \( \Gamma \), by
\[
(\theta)_v = \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\
(\theta(\theta + 1) \cdots (\theta + \nu - 1)) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}).
\end{cases}
\]
Corresponding to the function \( h_p(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) defined by
\[
h_p(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = z^{-p} {}_q F_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z),
\]
we consider a linear operator \( H_p(a_1, \ldots, a_q; b_1, \ldots, b_s) : \sum_p \to \sum_p \), which is defined by the following Hadamard product (or convolution):
\[
H_p(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = h_p(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ast f(z)
\]
(2) or, equivalently, by
\[
H_p(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a_1)_{k} \cdots (a_q)_{k}}{(b_1)_{k} \cdots (b_s)_{k}} \frac{a_{k-p}}{k!} z^{k-p}.
\]
(3)
If, for convenience, we write
\[
H_{p,q,s}(a_1) = H_p(a_1, \ldots, a_q; b_1, \ldots, b_s),
\]
(4) then one can easily verify from the definition (2) or (3) that (see [4])
\[
z \left( H_{p,q,s}(a_1) f(z) \right) = a_1 H_{p,q,s}(a_1 + 1) f(z) - (a_1 + p) H_{p,q,s}(a_1) f(z).
\]
(5)
The linear operator \( H_{p,q,s}(a_1) \) was investigated recently by Liu and Srivastava [4] and Aouf [2]. In particular, for \( s = 1, q = 2, a_1 > 0, b_1 > 0 \) and \( a_2 = 1 \), we obtain the linear operator
\[
\ell_p(a_1, b_1) f(z) = H_p(a_1, 1; b_1) f(z),
\]
(6) which was introduced and studied by Liu and Srivastava [5].
We note that, for any integer \( n > -p \) and \( f \in \sum_p \),
\[
H_{p,2,1}(n + p, 1; 1) f(z) = D^{n+p-1} f(z) = \frac{1}{z^p (1-z)^{n+p}} \ast f(z),
\]
(7) where \( D^{n+p-1} \) is the differential operator studied by Uralegaddi and Somanatha [9], Yang [10], and Aouf [11].
To establish our main results we need the following lemma.
Lemma 1. [6] Let $\Omega$ be a set in the complex plane $\mathbb{C}$ and let $c$ be a complex number satisfying $\Re(c) > 0$. Suppose that the function $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ satisfies the condition:

$$\psi(ix, y; z) \not\in \Omega$$

(8)

for all real $x, y \leq -\frac{|c - ix|^2}{2\Re(c)}$ and all $z \in U$. If the function $g(z)$ defined by $g(z) = c + c_1 z + c_2 z^2 + \ldots$ is analytic in $U$ and if $\psi(g(z), zg'(z); z) \in \Omega$, then $\Re(g(z)) > 0$ in $U$.

In this paper, we shall derive some inequalities involving the linear operator $H_{p,q,s}(a_1)$ defined on meromorphic $p$-valent functions.

2. Inequalities involving the operator $H_{p,q,s}(a_1)$

Theorem 1. Let the function $f = \sum_{p} f$ defined by (1) satisfies the following inequality:

$$\Re\left\{ \frac{H_{p,q,s}(a_1 + 1)f(z)}{H_{p,q,s}(a_1)f(z)} \right\} < 1 + \frac{1 - \alpha}{\alpha} (a_1 > 0; 0 \leq \alpha < 1; z \in U).$$

(9)

then

$$\Re\left\{ \left( z^p H_{p,q,s}(a_1)f(z) \right) \frac{1}{2\beta(1 - \alpha)} \right\} > 2 \frac{1}{\beta} (\beta \geq 1; z \in U).$$

The result is sharp.

Proof. Form (5) and (9), we have

$$\Re\left\{ \frac{z(H_{p,q,s}(a_1)f(z))}{H_{p,q,s}(a_1)f(z)} \right\} > p + \alpha - 1 (z \in U)$$

$$- \frac{1}{2(1 - \alpha)} \Re\left\{ \frac{z(H_{p,q,s}(a_1)f(z))}{H_{p,q,s}(a_1)f(z)} + p \right\} < \frac{z}{1 - z}. (10)$$

Let

$$g(z) = \left( z^p H_{p,q,s}(a_1)f(z) \right)^{-\frac{1}{2(1 - \alpha)}}.$$

Then (10) may be written as
\[
\ln g(z) < \ln \left( \frac{1}{1-z} \right). \tag{11}
\]

Using a well-known result [7] to (11), we find that
\[
g(z) = \left[ z^p \mathcal{H}_{p,q,s}(a_i) f(z) \right] \frac{1}{2(1-\alpha)} < \frac{1}{1-z},
\]
that is, that
\[
\left( z^p \mathcal{H}_{p,q,s}(a_i) f(z) \right) \frac{1}{2(1-\alpha)} = \left( \frac{1}{1-\omega(z)} \right)^{\frac{1}{\beta}}, \tag{12}
\]
where \( \omega(z) \) analytic function in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 (z \in U) \).

According to \( \Re \left( \frac{1}{1-\beta} \right) \geq \left( \Re(t) \right)^{\frac{1}{\beta}} \) for \( \Re(t) > 0 \) and \( \beta \geq 1 \), (12) yields
\[
\Re \left\{ z^p \mathcal{H}_{p,q,s}(a_i) f(z) \right\} \frac{1}{2\beta(1-\alpha)} \geq \left[ \Re \left( \frac{1}{1-\omega(z)} \right) \right]^{\frac{1}{\beta}} \geq 2^{\frac{1}{\beta}}.
\]

Further, we see that the result is sharp for the function
\[
f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(b_1)_k \ldots (b_s)_k (2\alpha - 2)_k}{(a_1)_k \ldots (a_q)_k} z^{k-p} (z \in U^*).
\]

This completes the proof of Theorem 1.

**Remark 1.** For \( q = 2, s = 1 \) and \( a_2 = 1 \), Theorem 1 yields the result which is obtained by Liu [3, Theorem 1].

Putting \( q = 2, s = 1, a_1 = n + p (n > -p, p \in \mathbb{N}), a_2 = 1 \) and \( b_1 = 1 \) in Theorem 1, we obtain the following corollary.

**Corollary 1.** Let the function \( f \in \sum_p \) defined by (1) satisfy the following inequality:
\[
\Re \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} < 1 + \frac{1-\alpha}{n+p} (0 \leq \alpha < 1),
\]
then
\[
\Re \left\{ \left( z^p D^{n+p-1} f(z) \right) \frac{1}{2\beta(1-\alpha)} \right\} > 2^{\frac{1}{\beta}} (\beta \geq 1).
\]

The result is sharp for the function
\[
f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(2\alpha - 2)_k}{(n+p)_k} z^{k-p} (z \in U^*).\]
Theorem 2. Let the function $f \in \sum_p$ defined by (1) satisfies the following inequality:

\[
\Re \left\{ (1 - \lambda) \frac{H_{p,q,s}(a_i + 1)f(z)}{H_{p,q,s}(a_i)f(z)} + \lambda \frac{H_{p,q,s}(a_i + 2)f(z)}{H_{p,q,s}(a_i + 1)f(z)} \right\} < \alpha
\]

\[
(a_i > 0; \alpha > 1; 0 \leq \lambda < a_i + 1),
\]

then

\[
\Re \left\{ \frac{H_{p,q,s}(a_i + 1)f(z)}{H_{p,q,s}(a_i)f(z)} \right\} < \beta,
\]

where $\beta \in (\alpha, +\infty)$ is the positive root of the equation

\[
2(a_i + 1 - \lambda)x^2 + [3\lambda - 2(a_i + 1)\alpha]x - \lambda = 0.
\]

Proof: Let

\[
\frac{H_{p,q,s}(a_i + 1)f(z)}{H_{p,q,s}(a_i)f(z)} = \beta + (1 - \beta)g(z),
\]

then $g(z)$ is analytic in $U$ and $g(0) = 1$. Differentiating (15) with respect to $z$ and using (5) we deduce that

\[
(1 - \lambda) \frac{H_{p,q,s}(a_i + 1)f(z)}{H_{p,q,s}(a_i)f(z)} + \lambda \frac{H_{p,q,s}(a_i + 2)f(z)}{H_{p,q,s}(a_i + 1)f(z)} =
\]

\[
= \beta + \frac{\lambda(1 - \beta)}{a_i + 1} + \frac{(1 - \beta)(a_i + 1 - \lambda)}{a_i + 1} g(z) + \frac{\lambda(1 - \beta)}{a_i + 1} \frac{zg' (z)}{\beta + (1 - \beta)g(z)} =
\]

\[
= \psi \left( g(z), zg' (z) \right),
\]

where

\[
\psi(r, s) = \beta + \frac{\lambda(1 - \beta)}{a_i + 1} + \frac{(1 - \beta)(a_i + 1 - \lambda)}{a_i + 1} r + \frac{\lambda(1 - \beta)}{a_i + 1} \frac{s}{\beta + (1 - \beta)r}.
\]

Using (13) and (16), we have

\[
\left\{ \psi \left( g(z), zg' (z) \right); z \in U \right\} \subset \Omega = \left\{ w \in \mathbb{C} : \Re(w) < \alpha \right\}.
\]

Now, for all real $x, y \leq -\frac{1 + x^2}{2}$, we obtain

\[
\Re \left\{ \psi(ix, y) \right\} = \Re \left\{ \beta + \frac{\lambda(1 - \beta)}{a_i + 1} + \frac{\lambda(1 - \beta)}{a_i + 1} \frac{\beta y}{\beta^2 + (1 - \beta^2)x^2} \right\} \geq
\]

\[\ \]
\[ \geq \beta + \frac{\lambda(1-\beta)}{a_1+1} - \frac{\lambda(1-\beta)}{2(a_1+1)} \frac{1+x^2}{\beta^2 + (1-\beta)^2 x^2} \geq \]

\[ \geq \beta + \frac{\lambda(1-\beta)}{a_1+1} - \frac{\lambda(1-\beta)}{2\beta(a_1+1)} = \]

\[ = \beta + \frac{\lambda(1-\beta)(2\beta-1)}{2\beta(a_1+1)} = \alpha, \]

where \( \beta \) is the positive root of (14).

Note that \( 0 \leq \lambda < a_1 + 1 \) and \( f(\alpha) = -\lambda(2\alpha - 1)(\alpha - 1) \leq 0 \), then we have \( \beta \in (\alpha, +\infty) \). Hence for each \( z \in U, \psi(ix, y) \notin \Omega \), by using Lemma 1, we get \( \Re(g(z)) > 0 \). This proves Theorem 2.

**Remark 2.** For \( q = 2, s = 1 \) and \( a_2 = 1 \), Theorem 2 yields the result which is obtained by Liu [3, Theorem 2].

Putting \( q = 2, s = 1, a_1 = n + p \) \( (n > -p, p \in \mathbb{N}), a_2 = 1 \) and \( b_1 = 1 \) in Theorem 2, we obtain the following corollary.

**Corollary 2.** Let the function \( f \in \sum_p \) defined by (1) satisfies the following inequality:

\[ \Re\left\{ (1-\lambda) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \lambda \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} \right\} < \alpha \ (\alpha > 1; 0 \leq \lambda < n + p + 1). \]

Then

\[ \Re\left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} < \beta, \]

where \( \beta \in (\alpha, +\infty) \) is the positive root of the equation

\[ 2(n + p + 1 - \lambda)x^2 + [3\lambda - 2(n + p + 1)\epsilon]x - \lambda = 0. \]

**Theorem 3.** Let \( \lambda \geq 0, \alpha > 1 \) and \( a_1 > 0 \). if the function \( f, g \in \sum_p \) satisfies the following inequalities:

\[ \Re\left\{ \frac{H_{p,q,s}(a_1)g(z)}{H_{p,q,s}(a_1+1)g(z)} \right\} > \delta \ (0 \leq \delta < 1), \]  \hspace{1cm} \text{(17)}

\[ \Re\left\{ (1-\lambda) \frac{H_{p,q,s}(a_1)f(z)}{H_{p,q,s}(a_1)g(z)} + \lambda \frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1+1)g(z)} \right\} < \alpha, \]

then

\[ \Re\left\{ \frac{H_{p,q,s}(a_1)f(z)}{H_{p,q,s}(a_1)g(z)} \right\} < \frac{2\alpha a_1 + \lambda\delta}{2a_1 + \lambda\delta} \ (z \in U). \]
Proof. Let \( \beta = \frac{2\alpha a_i + \lambda \delta}{2a_i + \lambda \delta} \) and consider the function

\[
\frac{H_{p,q,s}(a_1)f(z)}{H_{p,q,s}(a_1)g(z)} = \beta + (1 - \beta)u(z), \tag{19}
\]

where \( u(z) \) is analytic in \( U \) and \( u(0) = 1 \). Set \( B(z) = \frac{H_{p,q,s}(a_1)g(z)}{H_{p,q,s}(a_1 + 1)f(z)} \), then \( \Re(B(z)) > \delta \). Differentiating (19) with respect to \( z \) and using (5), we have

\[
(1 - \lambda)\frac{H_{p,q,s}(a_1)f(z)}{H_{p,q,s}(a_1)g(z)} + \lambda\frac{H_{p,q,s}(a_1 + 1)f(z)}{H_{p,q,s}(a_1 + 1)g(z)} = \]

\[
= \beta + (1 - \beta)u(z) + \frac{\lambda(1 - \beta)}{a_1}B(z)u(z).
\]

Let

\[
\psi(r,s) = \beta + (1 - \beta)r + \frac{\lambda(1 - \beta)}{a_1}B(z)s,
\]

then from (18), we deduce that

\[
\left\{ \psi\left(p(z), zp^{\prime}(z)\right) : z \in U \right\} \subset \Omega = \{ w \in \mathbb{C} : \Re(w) < \alpha \}.
\]

Now, for all real \( x, y \leq -\frac{1 + x^2}{2} \), we have

\[
\Re\left[\psi(ix, y)\right] = \beta + \frac{\lambda(1 - \beta)y}{a_1}\Re(B(z)) \geq \beta - \frac{\lambda(1 - \beta)y}{2a_1} \geq \]

\[
\geq \beta - \frac{\lambda(1 - \beta)\delta}{2a_1} = \alpha.
\]

Hence for each \( z \in U, \Re[\psi(ix, y)] \in \Omega \). Thus by using Lemma 1, \( \Re(u(z)) > 0 \) in \( U \). The Proof of Theorem 3 is completed.

Remark 3. For \( q = 2, s = 1 \) and \( a_2 = 1 \), Theorem 3 yields the result which is obtained by Liu [3, Theorem 3].

Putting \( q = 2, s = 1, a_i = n + p (n \geq -p, p \in \mathbb{N}), a_2 = 1 \) and \( b_1 = 1 \) in Theorem 3, we obtain the following corollary.

Corollary 3. Let \( n > -p, \lambda \geq 0 \) and \( \alpha > 1 \). If the function \( f, g \in \sum_p \) satisfies the following inequalities

\[
\Re\left\{ \frac{D^{a+p-1}g(z)}{D^{a+p}g(z)} \right\} > \delta (0 \leq \delta < 1; z \in U),
\]

\( \Re\left[\psi(ix, y)\right] = \beta + \frac{\lambda(1 - \beta)y}{a_1}\Re(B(z)) \geq \beta - \frac{\lambda(1 - \beta)y}{2a_1} \geq \]

\[
\geq \beta - \frac{\lambda(1 - \beta)\delta}{2a_1} = \alpha.
\]

Theorem 3 yields the result which is obtained by Liu [3, Theorem 3].
\[
\Re\left\{ (1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \right\} < \alpha \quad (z \in U),
\]
then
\[
\Re\left\{ \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \right\} < \frac{2\alpha(n+p)+\lambda \delta}{2(n+p)+\lambda \delta} \quad (z \in U).
\]

References

Некоторые неравенства для мероморфных $p$-валентных функций, связанных с оператором Лиу–Сривастава

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РЕЗЮМЕ

Выведены неравенства, связанные с линейным оператором, определенным для некоторого класса мероморфных $p$ – валентных функций. Кроме того, показывается связь результатов полученных в настоящей работе с результатами полученными в более ранних работах.

Ключевые слова: аналитическая функция, функция Шварца, обобщенная гипергеометрическая функция, линейный оператор, произведение Адамара, субординация.