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COMMUTATOR OF NONSINGULAR INTEGRAL ON WEIGHTED ORLICZ SPACES

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Abstract. We show continuity in weighted Orlicz spaces $L_{\omega}^{\Phi}(R_{+}^{n})$ of commutator of nonsingular integral operator.

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1. Introduction.

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

The Orlicz space were first introduced by Orlicz in [26,27] as generalizations of Lebesgue spaces $L_p(R^n)$. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [2,3] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let T be a Calderon-Zygmund singular integral operator and $b \in BMO(R^n)$. A celebrated result of Coifman, Rochberg and Weiss [6] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L_p(R^n)$ for $1 < p < \infty$. The commutator of Calderon-Zygmund operators plays an important role in the study of regularity of solutions of elliptic partial differential equations of second order (see, for example, [4, 5, 10, 11, 13, 14]).

Consider the half-space $R_{+}^{n} = R^{n-1} \times (0, \infty)$. For $x = (x', x_n) \in R_{+}^{n}$, let $\tilde{x} = (x', -x_n)$ be the "reected point". Let $x \in R_{+}^{n}$. The nonsingular integral operator $\tilde{T}$ is defined by

$$\widetilde{T}f(x) = \int_{R^n_+} \frac{|f(y)|}{|\widetilde{x} - y|^n} dy, \quad \widetilde{x} = (x', -x_n). \tag{1}$$

Given a function b locally integrable on R^n and the nonsingular integral operator $\widetilde{T}$, we consider the linear commutator of nonsingular integral operator $[b, \widetilde{T}]$ defined by setting, for smooth, compactly supported functions f,

$$[b, \widetilde{T}](f) = b\widetilde{T}(f) - \widetilde{T}(bf) .$$

The operator $\widetilde{T}$ and its commutator $[b, \widetilde{T}]$ appear in [4, 5, 10, 21, 22, 23, 24, 25] in connection with boundary estimates for solutions to elliptic equations.

The main purpose of this paper is mainly to study the boundedness of the commutator of nonsingular integral operator $[b, \widetilde{T}]$ on weighted Orlicz spaces $L^\Phi_\omega(R^n_+)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $A \gtrsim B$, we write $A \approx B$ and say that A and B are equivalent.

2. Definitions and Preliminary Results.

Even though the $A_p(R^n)$ class is well known, for completeness, we offer the definition of A_p weight functions. Here and everywhere in the sequel $B(x, r)$ is the ball in R^n of radius r centered at x and $|B(x, r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in R^n. Let $\mathrm{B} = \{B(x, r) : x \in R^n, r > 0\}$.

Definition 2.1. *For, $1 < p < \infty$, a locally integrable function $\omega : R^n \to [0, \infty)$ is said to be an A_p weight if*

$$\sup_{B \in \mathrm{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty .$$

A locally integrable function $\omega : R^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B \omega(y) dy \le C\, \omega(x), \ \ a.e. \ x \in B$$

for some constant $C>0$. *We define* $A_{\infty}(R^n)=\bigcup_{p\geq 1}A_p(R^n)$.

For any $\omega\in A_{\infty}$ *and any Lebesgue measurable set* E, *we write* $\omega(E)=\int_E \omega(x)dx$.

We recall the definition of Young functions.

Definition 2.2. *A function* $\Phi:[0,\infty)\to[0,\infty]$ *is called a Young function, if* Φ *is convex, left-continuous,* $\lim_{r\to 0^+}\Phi(r)=\Phi(0)=0$ *and* $\lim_{r\to\infty}\Phi(r)=\infty$.

The convexity and the condition $\Phi(0)=0$ force any Young function to be increasing. In particular, if there exists $s\in(0,\infty)$ such that $\Phi(s)=\infty$, then it follows that $\Phi(r)=\infty$ for $r\geq s$.

Let Y be the set of all Young functions Φ such that

$$0<\Phi(r)<\infty \qquad \text{for} \qquad 0<r<\infty.$$

If $\Phi\in\mathrm{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself. For a Young function Φ and $0\leq s\leq\infty$, let

$$\Phi^{-1}(s)\equiv\inf\{r\geq 0:\Phi(r)>s\} \quad (\inf\emptyset=\infty).$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi\in\Delta_2$, if

$$\Phi(2r)\leq k\Phi(r), \quad r>0$$

for some $k>1$. If $\Phi\in\Delta_2$, then $\Phi\in\mathrm{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi\in\nabla_2$, if

$$\Phi(r)\leq\frac{1}{2k}\Phi(kr), \quad r\geq 0$$

for some $k>1$. The function $\Phi(r)=r$ satisfies the Δ_2-condition and it fails the ∇_2-condition. If $1<p<\infty$, then $\Phi(r)=r^p$ satisfies both the conditions. The function $\Phi(r)=e^r-r-1$ satisfies the ∇_2-condition but it fails the Δ_2-condition.

For a Young function Φ, the complementary function $\tilde{\Phi}(r)$ is defined by

$$\tilde{\Phi}(r)\equiv\begin{cases}\sup\{rs-\Phi(s):s\in[0,\infty)\} & \textit{if} \quad r\in[0,\infty)\\ \infty & \textit{if} \quad r=\infty.\end{cases}$$

The complementary function $\tilde{\Phi}$ is also a Young function and it satisfies $\tilde{\tilde{\Phi}}=\Phi$. Note that $\Phi\in\nabla_2$ if and only if $\tilde{\Phi}\in\Delta_2$.

It is also known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \quad r \ge 0. \tag{2}$$

We recall an important pair of indices used for Young functions. For any Young function Φ, write

$$h_\Phi(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \quad t > 0.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \quad \text{and} \quad I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t}$$

respectively.

A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $t \in [1,\infty)$ (resp. $t \in [0,1]$) and $s \in [0,\infty)$,

$$\Phi(st) \le Ct^p \Phi(s).$$

Remark 2.1. *It is well known that if Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\widetilde{\Phi}$ is of lower type p_1' and upper type p_0' and Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $\Phi \in \Delta_2 \cap \nabla_2$.*

Definition 2.3. *For a Young function Φ and $\omega \in A_\infty$, the set*

$$L_\omega^\Phi(R^n) \equiv \left\{ f - measurable: \int_{R^n} \Phi(k|f(x)|)\omega(x)dx < \infty \ \text{ for some } k > 0 \right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L_\omega^{\Phi,loc}(R^n)$ is defined as the set of all functions f such that $f\chi_B \in L_\omega^\Phi(R^n)$ for all balls $B \subset R^n$.

Note that $L_\omega^\Phi(R^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\omega^\Phi(R^n)} \equiv \|f\|_{L_\omega^\Phi} = \inf\left\{ \lambda > 0: \int_{R^n} \Phi\left(\frac{|f(x)|}{\lambda} \right) \omega(x)dx \le 1 \right\}$$

and

$$\int_{R^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\omega^\Phi}} \right) \omega(x)dx \le 1. \tag{3}$$

The following analogue of the Hölder inequality is known.

$$\left| \int_{R^n} f(x)g(x)\omega(x)dx \right| \le 2\|f\|_{L_\omega^\Phi} \|g\|_{L_\omega^{\widetilde{\Phi}}}. \tag{4}$$

For the proof of (2) and (4), see, for example [29].

For a weight ω, a measurable function f and $t>0$, let

$$m(\omega, f, t)=\omega(\{x\in R^n : |f(x)|>t\}).$$

Definition 2.4. *The weak weighted Orlicz space*

$$WL_\omega^\Phi(R^n)=\{f - measurable: \|f\|_{WL_\omega^\Phi}<\infty\}$$

is defined by the norm

$$\|f\|_{WL_\omega^\Phi(R^n)} \equiv \|f\|_{WL_\omega^\Phi} = \inf\left\{\lambda>0: \sup_{t>0}\Phi(t)m\left(\omega, \frac{f}{\lambda}, t\right)\le 1\right\}.$$

We can prove the following by a direct calculation:

$$\|\chi_B\|_{L_\omega^\Phi} = \|\chi_B\|_{WL_\omega^\Phi} = \frac{1}{\Phi^{-1}\left(\omega(B)^{-1}\right)}, \quad B\in \mathrm{B},$$

where χ_B denotes the characteristic function of the B.

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy, \quad x\in R^n$$

for a locally integrable function f on R^n.

Let $M^{\aleph}$ be the sharp maximal function defined by

$$M^{\aleph} f(x)=\sup_{r>0}|B(x,r)|^{-1}\int_{B(x,r)}\left|f(y)-f_{B(x,r)}\right|dy,$$

where $f_{B(x,t)}(x)=|B(x,t)|^{-1}\int_{B(x,r)} f(y)dy$.

Theorem 2.1. [20] *Let* $1<p<\infty$. *Then* $M: L_\omega^p(R^n)\to L_\omega^p(R^n)$ *if and only if* $\omega\in A_p(R^n)$.

Theorem 2.2. [17, Theorem 1] *Let* Φ *be a Young function with* $\Phi\in\Delta_2\cap\nabla_2$. *Assume in addition* $\omega\in A_{i_\Phi}(R^n)$. *Then, there is a constant* $C\ge 1$ *such that*

$$\int_{R^n}\Phi(Mf(x))\omega(x)dx\le C\int_{R^n}\Phi(|f(x)|)\omega(x)dx \tag{5}$$

for any locally integrable function f.

With [7, Remark 2.5] and [8, Remark 6.1.3] taken into account, the better boundedness result which was proved in [9] runs as follows.

Theorem 2.3. [9] *Let* Φ *be a Young function with* $\Phi\in\nabla_2$. *Assume in addition* $\omega\in A_{i_\Phi}(R^n)$. *Then the modular inequality (5) holds.*

Remark 2.2. *Note that the strong modular inequality (5) implies the corresponding norm inequality. Indeed, let (5) hold. Then, using the sublinearity of* M *, convexity of* Φ *and (3) we have*

$$\int_{R^n} \Phi\left(\frac{Mf(x)}{C\|f\|_{L^\Phi_\omega}} \right) \omega(x)dx = \int_{R^n} \Phi\left(M\left(\frac{f}{C\|f\|_{L^\Phi_\omega}} \right)(x) \right) \omega(x)dx$$

$$\leq C \int_{R^n} \Phi\left(\frac{|f(x)|}{C\|f\|_{L^\Phi_\omega}} \right) \omega(x)dx \leq 1,$$

where C *is the constant in (5). This implies* $\|Mf\|_{L^\Phi_\omega} \lesssim \|f\|_{L^\Phi_\omega}$.

The following theorem is valid (see, for example, [18, 28]).

Theorem 2.4. [25] *Let* $\tilde{T}$ *be a nonsingular integral operator, defined by (1),* $f \in L^p_\omega(R^n_+)$, $1 \leq p < \infty$ *and* $\omega \in A_p(R^n)$. *Then there exists a constant* C_p *independent of* f *, such that*

$$\|\tilde{T}f\|_{L^\Phi_\omega(R^n_+)} \leq C_p \|f\|_{L^\Phi_\omega(R^n_+)}, \quad 1 < p < \infty$$

and

$$\|\tilde{T}f\|_{WL^1_\omega(R^n_+)} \leq C_1 \|f\|_{L^1_\omega(R^n_+)}.$$

Theorem 2.5. [25] *Let* Φ *be a Young function,* $\omega \in A_{i_\Phi}(R^n)$ *and* $\tilde{T}$ *be a nonsingular integral operator, defined by (1). If* $\Phi \in \Delta_2$, *then the operator* $\tilde{T}$ *is bounded from* $L^\Phi_\omega(R^n_+)$ *to* $WL^\Phi_\omega(R^n_+)$ *and if* $\Phi \in \Delta_2 \cap \nabla_2$, *then the operator* $\tilde{T}$ *is bounded on* $L^\Phi_\omega(R^n_+)$.

3 . Commutatpr of nonsingular integral operators in the weighted Orlicz space $L^\Phi_\omega(R^n_+)$

We recall the de_nition of the space of $BMO(R^n)$.

Definition 3.1. *Suppose that* $b \in L^1_{loc}(R^n)$, *let*

$$\|b\|_* = \sup_{x \in R^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy.$$

Define

$$BMO(R^n) \equiv BMO = \{b \in L^1_{loc}(R^n) : \|b\|_* < \infty\}.$$

Lemma 3.1. [16] *Let* $b \in BMO$. *Then there is a constant* $C > 0$ *such that*

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_* \ln\frac{t}{r} \quad \textit{for} \quad 0 < 2r < t,$$

where C *is independent of* b, x, r *and* t.

Lemma 3.2. [15] *Let* $\omega \in A_\infty$, $b \in BMO$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$. *Then,*

$$\sup_{x \in R^n,\, r>0} \Phi^{-1}\left(\omega(B(x,r))^{-1}\right)\left\|b - b_{B(x,r)}\right\|_{L^\Phi_\omega(B(x,r))} \lesssim \|b\|_*.$$

Theorem 3.1. [1, Theorem 1.13] Let $b \in BMO(R^n)$. *Suppose that* X *is a Banach space of measurable functions defined on* R^n. *Moreover, assume that* X *satisfies the lattice property, that is*

$$0 \le g \le f \Rightarrow \|g\|_X \lesssim \|f\|_X$$

Assume that M *is bounded on* X. *Then the operator* M_b *is bounded on* X, *and the inequality*

$$\|M_b f\|_X \le C\|b\|_* \|f\|_X$$

holds with constant C *independent of* f.

Combining Theorems 2.3 and 3.1, we obtain the following statement.

Corollary 3.1. *Let* Φ *be a Young function with* $\Phi \in \nabla_2$ *and* $b \in BMO(R^n)$. *Assume in addition* $\omega \in A_{i_\Phi}(R^n)$, *then* M_b *is bounded on* $L^\Phi_\omega(B)$.

The space $L_p(R^n_+)$ *coincides with the space*

$$\left\{ f(x) : \left| \int_{R^n_+} f(y) g(y) dy \right| < \infty \quad \textit{for all} \quad g \in L_{p'}(R^n_+) \right\}$$

up to the equivalence of the norms

$$\|f\|_{L_p(R^n_+)} \approx \sup_{\|g\|_{L_{p'}} \le 1} \left| \int_{R^n_+} f(y) g(y) dy \right|. \tag{6}$$

The following statement holds:

Lemma 3.3. *Let* $1 \le p < \infty$. *Then for all* $f \in L_p(R^n_+)$ *and* $g \in L_{p'}(R^n_+)$ *there holds*

$$\left| \int_{R^n_+} f(y) g(y) dy \right| \le C \int_{R^n_+} M^{\aleph} f(y) Mg(y) dy$$

with a constant $C>0$ *not depending on* f *.*

Lemma 3.4. *Let* $1\le p<\infty$ *,* $\omega\in A_p$*. Then*

$$\|f\|_{L^p_\omega(R^n_+)}\le C\left\|M^{\aleph} f\right\|_{L^p_\omega(R^n_+)}$$

with a constant $C>0$ *not depending on* f *.*

Proof. By (6) we have

$$\|f\|_{L^p_\omega(R^n_+)}\le C\sup_{\|g\|_{L_{p'}(R^n_+)}\le 1}\left|\int_{R^n_+} f(y)g(y)\omega(y)dy\right|.$$

According to Lemma 3.3,

$$\|f\|_{L^p_\omega(R^n_+)}\le C\sup_{\|g\|_{L_{p'}(R^n_+)}\le 1}\int_{R^n_+} M^{\aleph} f(y)M(g\omega)(y)dy.$$

By the Hölder inequality and Theorem 2.1, we derive

$$\|f\|_{L^p_\omega(R^n_+)}\le C\sup_{\|g\|_{L_{p'}(R^n_+)}\le 1}\left\|M^{\aleph} f\right\|_{L^p_\omega(R^n_+)}\left\|\omega^{-1}M(g\omega)\right\|_{L_{p'}(R^n_+)}$$

$$\le C\sup_{\|g\|_{L_{p'}(R^n_+)}\le 1}\left\|M^{\aleph} f\right\|_{L^p_\omega(R^n_+)}\|g\|_{L_{p'}(R^n_+)}\le C\left\|M^{\aleph} f\right\|_{L^p_\omega(R^n_+)}.$$

Theorem 3.2. *Let* $\tilde{T}$ *be a nonsingular integral operator,* $b\in BMO$ *,* $1<p<\infty$ *and* $\omega\in A_p$*. Then the commutator operator* $\left[b,\tilde{T}\right]$ *is bounded on the space* $L^p_\omega(R^n_+)$*.*

Proof. We are going to adapt an idea of Stromberg (see [30, pp. 417-418]). Observe that it is enough to prove

$$M^{\aleph}\left(\left[b,\tilde{T}\right]f\right)(x)\le C\|b\|_*\left(\left(M\left|\tilde{T}f\right|^r\right)^{\frac{1}{r}}(x)+\left(M|f|^r\right)^{\frac{1}{r}}(x)\right)\tag{7}$$

for all $r>1$, $x\in R^n$.

To see this choose $1<r<p$, then (7) combined with Lemmas 2.1 and 3.4 and with the L^p_ω estimate on $\tilde{T}$ implies

$$\left\|\left[b,\tilde{T}\right]f\right\|_{L^p_\omega}\le C\|b\|_*\left(\left\|\tilde{T}f\right\|_{L^p_\omega}+\|f\|_{L^p_\omega}\right)\le C\|b\|_*\|f\|_{L^p_\omega}.$$

From this result and [19, Theorem 2.7], we have the following boundedness of $\left[b,\tilde{T}\right]$ on $L^{\Phi}_\omega(R^n_+)$.

Theorem 3.3. *Let* Φ *be a Young function,* $\omega \in A_{i_\Phi}$ *and* $\tilde{T}$ *be a nonsingular integral operator, defined by (1). If* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $b \in BMO$, *then the commutator operator* $\left[b, \tilde{T}\right]$ *is bounded on* $L^{\Phi}_{\omega}\left(R^n_+\right)$.

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