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EXISTENCE AND BEHAVIOR OF SOLUTIONS FOR CONVECTION-DIFFUSION EQUATIONS WITH THIRD TYPE BOUNDARY CONDITION

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ABSTRACT. In this paper, we study a nonlinear convection-diffusion type equation with initial and third type boundary condition in the general case. We obtain the sufficient conditions under which the existence and uniqueness of the generalized solution for the considered problem is proved. Moreover, we investigate the long-time behavior of the solution and prove the existence of absorbing sets in two different spaces.

Keywords: convection-diffusion equations, third type boundary value problem, existence and uniqueness theorems, semiflow, absorbing set.

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1. Introduction

Consider the problem

$$\frac{\partial u}{\partial t} + Lu + g(x,t,u) = h(x,t),\ (x,t) \in Q_T, \tag{1}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{2}$$

$$\left(\frac{\partial u}{\partial \nu} + k(x,t)u\right)\bigg|_{\Gamma_T} = \varphi(x,t),\ (x,t) \in \Gamma_T, \tag{3}$$

here $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $Q_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times [0,T]$; T is a positive number; L denotes the second order linear elliptic operator in divergence form;

$$Lu := -\sum_{i,j=1}^{n} D_i(a_{ij}\,(x,t)\,D_j u) + \sum_{i=1}^{n} b_i\,(x,t)\,D_i u + c(x,t)u,$$

where a_{ij}, b_i and c are given coefficient functions, $(i,j = 1,...,n)$ and $\frac{\partial u}{\partial \nu}$ is defined according to L as following form;

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}\,(x,t)\,D_j u\ \nu_i,$$

where ν is unit normal vector.
$g : Q_T \times \mathbb{R} \longrightarrow \mathbb{R}$, $k : \Gamma_T \longrightarrow \mathbb{R}$ and $u_0 : \Omega \longrightarrow \mathbb{R}$ are given functions; h and φ are given generalized functions. Here $u(x,t)$ is an unknown function which can represent temperature for heat transfer, species concentration for mass transfer or in general the quantity of a substance.

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As known, many physical problems involving convection and diffusion processes occur in fields where mathematical modeling is important such as physics, engineering, astrophysics, oceanography and particularly in fluid dynamics and transport problems. Therefore, the existence and behavior of solutions for Convection-diffusion equations have been the subject of study of many authors (see, for instance [4, 5, 11, 13, 21]).

In [2], Boni considered the following

$$\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}\left(x\right)\frac{\partial u}{\partial x_j})+a(x)f(u)=0,$$

with nonlinear boundary condition $\frac{\partial u}{\partial \nu}+b(x)g(u)=0$ and investigated the asymptotic behavior of positive solutions. In [7], Li and Li considered similar nonlinear divergence form parabolic equation with nonhomogeneous Neumann boundary conditions. They established the conditions on nonlinearities to guarantee that the solution exists globally.

Convection Diffusion equations with homogeneous Dirichlet boundary conditions have been studied extensively: In [12], Porzio considered $u_t-div(a(x,t,u,\nabla u))=0$ with the Dirichlet boundary condition and showed the existence and uniqueness of the weak solution under the Leray-Lions structure conditions, also investigated the long time behavior of the solution in $L_r(\Omega)$ where $r>1$ and in $L_\infty(\Omega)$. In [18], Souplet considered

$$\frac{\partial u}{\partial t}-\Delta u=|u|^{p-1}u-b|\nabla u|^q,\quad p>1,q\geq 1,b>0,$$

with homogeneous Dirichlet boundary conditions. Sufficient conditions for blow-up, behavior of blowing-up solutions, global existence and stability are investigated depending on the relation between p and q. In [10], Ma considered the classical semilinear parabolic equation

$$u_t-\Delta u=|u|^{p-1}u,\quad 1<p<\frac{n+2}{n-2}\quad if\quad n\geq 3;\quad p<\infty\quad if\quad n=1,2,$$

with homogeneous Dirichlet boundary condition and the existence of global solutions is obtained. In [3], Chen, Fila and Guo dealt with equation $u_t=\Delta u+f(u)$ with the Dirichlet boundary condition and determined the sufficient conditions on f which guarantee that the solution is bounded if it exists. Also in [17], Song, He and Zhang investigated the long-time behavior for the nonlinear parabolic equation

$$u_t-d\Delta u+a|u|^{p-1}u+\sum_{j=1}^{p-1}b_j(x)u^j=0,\quad d>0,\ a>0,\quad 1\leq p\leq 3$$

with homogeneous Dirichlet boundary condition. Soltanov, in [14, 16], dealt with following equations

$$\frac{\partial\rho(u)}{\partial t}-\sum_{i=1}^{n}D_i[(a_i\left(t,x,u\right)D_iu)+b_i\left(t,x,u,Du\right)+c_i(t,x,u)]+f(t,x,u)=0,$$

with nonhomogeneous Dirichlet boundary condition and obtained the sufficient conditions for the solvability of the considered problems.

In our previous study [6], we considered problem (1)-(3) with homogeneous initial condition and proved the existence of the generalized solution by using a general result. Also, we proved the uniqueness of the solution only for a model case of the problem.

In this paper, unlike of the mentioned, we consider a nonhomogeneous, nonlinear convection-diffusion equation by taking mapping g in general form with nonhomogeneous third type (Robin) boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. In Section 2, we give some preparations for our consideration. In section 3, we give the existence result obtained by using same method as in [6]. In section 4, we obtain two results on the uniqueness of the solution under different conditions on the mapping g. Finally, in Section 5, we investigate the long time behavior of the solution. Firstly, we give some results on the behavior of the solution in $L_2(\Omega)$ when the equation and boundary condition are homogeneous; secondly, we prove the existence of the absorbing sets in the autonomous case of the problem.

2. Setting of the problem and the main conditions

We investigate problem (1)-(3) when generalized functions h and φ belong to

$$L_2(0,T;(W_2^1(\Omega))^*) + L_q(Q_T)$$

where $q > 1$ and $L_2(0,T;(W_2^{-\frac{1}{2}}(\partial\Omega))$, respectively.

Consider the following conditions:

(1) The coefficients $a_{ij} \in L_\infty(\overline{Q_T})$, $a_{ij} = a_{ji}$ is satisfied for all $i,j = 1,...,n$ and there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \theta\,|\xi|^2$$

holds for all $\xi \in \mathbb{R}^n$ and a.e. $(x,t) \in \overline{Q_T}$.

(2) The function g is a Caratheodary function in $Q_T \times \mathbb{R}$ such that there exist a number $\alpha > 1$ and some nonnegative functions $g_0 \in L_{\frac{\alpha+1}{\alpha}}(Q_T)$, $g_1 \in L_\infty(Q_T)$ which satisfy the following inequality for all $\tau \in \mathbb{R}$ and a.e. $(x,t) \in Q_T$:

$$|g(x,t,\tau)| \leq g_1(x,t)\,|\tau|^\alpha + g_0(x,t).$$

(3) The function k belongs to $L_\infty(0,T;L_{n-1}(\partial\Omega))$.

We introduce the following space:
$P_0 \equiv L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = u_0\}$.

A solution of the considered problem is understood as following:

Definition 2.1. *A function $u \in P_0$ is called the generalized solution of problem (1)-(3) if it satisfies the equality*

$$-\int_0^T\int_\Omega u\frac{\partial v}{\partial t}dxdt + \int_\Omega u(x,T)v(x,T)dx - \int_\Omega u(x,0)v(x,0)dx + \int_0^T\int_\Omega \sum_{i,j=1}^{n} a_{ij}\,(x,t)\,D_j u D_i v dxdt$$

$$+\int_0^T\int_\Omega \sum_{i=1}^{n} b_i\,(x,t)\,D_i u\; v dxdt + \int_0^T\int_\Omega c(x,t)uvdxdt + \int_0^T\int_\Omega g(x,t,u)vdxdt + \int_0^T\int_{\partial\Omega} k(x,t)uvdxdt$$

$$= \int_0^T\int_\Omega hvdxdt + \int_0^T\int_{\partial\Omega} \varphi vdxdt,$$

for all $v \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*)$.

3. Solvability of problem (1)-(3)

In this section, we give the existence result:

Theorem 3.1. *Let conditions (1)-(3) be fulfilled and the following conditions be satisfied:*

(i) *Let functions b_i and c belong to $L_{\frac{2(\alpha+1)}{\alpha-1}}(Q_T)$, for all $i = 1, ..., n$ and $L_{\frac{\alpha+1}{\alpha-1}}(Q_T)$, respectively.*

(ii) *There exist some numbers $\widetilde{g_1} > 0$ and $\widetilde{g_0} \geq 0$ such that*

$$g(x,t,\xi)\xi \geq \widetilde{g_1}\,|\xi|^{\alpha+1} - \widetilde{g_0}$$

holds for all $\xi \in \mathbb{R}$ and a.e. $(x,t) \in Q_T$.

(iii) *There exists a number $k_0 \geq 0$ such that $k(x,t) \geq -k_0 > -\frac{\theta_1}{c_3}$ holds for a.e. $(x,t) \in \Gamma_T$ where $0 < \theta_1 < \min\{\widetilde{\theta}, \widetilde{g}\}$ such that $0 < \widetilde{\theta} < \theta$, $0 < \widetilde{g} < \widetilde{g_1}$. (Here c_3 is a constant of Sobolev's embedding inequality[1] [1]).*

Then problem (1)-(3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$.

For the proof of this theorem, we apply the existence theorem from [15] (see, [Theorem 3, 15]) to problem (1)-(3). We need to make a shift to have the zero initial condition. By using the transformation $v(x,t) := u(x,t) - u_0(x)$, we can rewrite problem (1)-(3) in the following:

$$\frac{\partial(v+u_0)}{\partial t} + L(v+u_0) + g(x,t,v+u_0) = h(x,t),$$

$$v(x,0) = 0,$$

$$\left(\frac{\partial(v+u_0)}{\partial \nu} + k(x,t)(v+u_0)\right)\Bigg|_{\Gamma_T} = \varphi(x,t).$$

Now let define the corresponding mappings:

$$f := \{f_1, f_2\} : P_0 \longrightarrow L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T),$$

where

$$f_1(v) := L(v+u_0) + g(x,t,v+u_0),$$

$$f_2(v) := \frac{\partial(v+u_0)}{\partial \nu} + k(x,t)(v+u_0)$$

and

$$A \equiv Id : P_0 \longrightarrow P_0.$$

We show that the conditions of the existence theorem from [15] are satisfied by proving the following lemmas.

Lemma 3.1. *Mapping f is weakly compact[2] from P_0 to $L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)$, under the conditions of Theorem 3.1.*

[1] $\|u\|_{L_2(\partial\Omega)} \leq c_3\|u\|_{W_2^1(\Omega)}$

[2] Let X,Y be separable Banach spaces, $F : D(F) \subseteq X \to Y$ is weakly compact iff any sequence $\{x_n\} \subset D(F)$ weakly convergent in X has a subsequence $\{x_m\}$ such that $\{F(x_m)\}$ is weakly convergent in Y [15].

Proof. It is clear that linear parts of f are continuous since they are bounded. Let investigate the nonlinear part of mapping f. For the nonlinear function g, we have

$$\|g(x,t,v+u_0)\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}(Q_T)} \leq 2^{\frac{1}{\alpha}} \left(\|g_1\|^{\frac{\alpha+1}{\alpha}}_{L_\infty(Q_T)} \|v+u_0\|^{\alpha+1}_{L_{\alpha+1}(Q_T)} + \|g_0\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}(Q_T)} \right),$$

which means g is a bounded mapping from P_0 to $L_{\frac{\alpha+1}{\alpha}}(Q_T)$.
Let $\{v_m\} \subset P_0$ and $v_m \rightharpoonup v$ in P_0. Then $v_m \rightharpoonup v$ in $L_{\alpha+1}(Q_T)$. Since

$$L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega)^*)) \hookrightarrow L_2(Q_T) \quad (\textit{compact embedding } [9]),$$

then $\exists\ \{v_{m_l}\} \subset \{v_m\}$ such that $v_{m_l} \longrightarrow v$ almost everywhere in Q_T.
Using condition (2), we have $g(x,t,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Then according to a general result (see Lemma 1.3 in [8]), $\exists\ \{v_{m_r}\} \subset \{v_m\}$ such that

$$g(x,t,v_{m_r}) \underset{L_{\frac{\alpha+1}{\alpha}}(Q_T)}{\rightharpoonup} g(x,t,v).$$

So, we obtain that mapping g is weakly compact from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)$. □

Lemma 3.2. *Mappings f and A generate a coercive pair on $L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)$ under the conditions of Theorem 3.1.*

Proof. It is enough to see that mapping f is coercive on $L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)$, since A is an identity mapping. Using Hölder, Young and some Sobolev embedding inequalities [3] [1], we have the following:

$$\langle f(v), v\rangle_{Q_T} \geq \frac{\theta_1 - k_0 c_3}{2} \left[\|v+u_0\|^2_{L_2(0,T;W_2^1(\Omega))} + \|v+u_0\|^{\alpha+1}_{L_{\alpha+1}(Q_T)} \right] - K.$$

Here $\theta_1 := \min\{\theta - \varepsilon, \widetilde{g_1} - \varepsilon'\}$ where ε, ε' are small enough and
$K = K(\theta_1, k_0, c_1, c_3, T, \widetilde{g_0}, meas(\Omega), \|a_{ij}\|_{L_\infty(Q_T)}, \|Du_0\|_{L_2(Q_T)}, \|u_0\|_{L_{\alpha+1}(\Omega)}, \|b_i\|_{L_{\frac{2(\alpha+1)}{\alpha-1}}(Q_T)},$
$\|c\|_{L_{\frac{\alpha+1}{\alpha-1}}(Q_T)}, \|g_1\|_{L_\infty(Q_T)}, \|g_0\|_{L_{\frac{\alpha+1}{\alpha}}(Q_T)}, \|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))})$.

Thus, from the previous inequality and the conditions of Theorem 3.1, we have the following:

$$\frac{1}{\|v\|_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)}} \langle f(v), v\rangle_{Q_T} \to \infty \quad as \quad \|v\|_{L_2(0,T;(W_2^1(\Omega))) \cap L_{\alpha+1}(Q_T)} \to \infty\ .$$

□

Proof of Theorem 3.1. Using Lemma 1 and Lemma 2, we obtain that all conditions of the existence theorem ([Theorem 3, 15]) are satisfied for the mappings f and A. So applying the general theorem to the problem, we obtain that the problem is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$. Consequently, we find the solution of problem (1)-(3), $u(x,t) = v(x,t) + u_0(x)$. □

4. Results on the uniqueness of the solution

In this section, we investigate the uniqueness of solutions for problem (1)-(3). Assume that the following conditions are satisfied:

(i′) Let $b_i \in L_\infty(0,T;L_n(\Omega))$, $(\forall i = 1,...,n)$ and $c \in L_\infty(0,T;L_{\frac{n}{2}}(\Omega))$.

(ii′) Let one of the following conditions be satisfied:

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[3] $\|u\|^2_{W_2^1(\Omega)} \leq \|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} + \|Du\|^2_{L_2(\Omega)} + c'$, $\|u\|_{L_{\frac{2(n-1)}{n-2}}(\partial\Omega)} \leq c_1 \|u\|_{W_2^1(\Omega)}$

(a) There exists a number $\widetilde{c} > 0$ such that $c(x,t) \geq \widetilde{c}$ holds for a.e. $(x,t) \in Q_T$ and

$$\sigma c_0 + c_1 \|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))} \leq \widetilde{\theta} < \min\{\theta_1, \widetilde{c}\}$$

holds for some $\widetilde{\theta} > 0$.

(b) There exists a number $k_0 > 0$ such that $k(x,t) \geq k_0$ for a.e. $(x,t) \in \Gamma_T$ and

$$\sigma c_0 + c_0^2 \|c\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))} \leq \widetilde{\theta} < c_2 \min\{\theta_1, k_0\}$$

holds for some $\widetilde{\theta} > 0$.

Here $\theta_1 < \theta$, $\sigma \equiv \sup\limits_i \|b_i\|_{L_\infty(0,T;L_n(\Omega))}$ and c_0 is a constant of Sobolev's Embedding inequality[4][1].

Our main results are stated below. The result in Theorem 4.1 is obtained when nonlinear function g is taken as smooth function. Other result in Theorem 4.2 is obtained when function g satisfies the local Lipschitz condition.

Theorem 4.1. *Let conditions (1)-(3) and (i′), (ii′) be fulfilled. Furthermore, assume* $g_\xi \in L_\infty(0,T;L_{\frac{n}{2}}(\Omega))$ *and there exists a number* $\widetilde{g} > 0$ *such that* $g_\xi(x,t,\xi) \geq -\widetilde{g}$ *holds for all* $\xi \in \mathbb{R}$ *and a.e.* $(x,t) \in Q_T$. *Then the solution of problem (1)-(3) is unique in* P_0.

Proof. Let $u, v \in P_0$ be two different solutions of problem (1)-(3). Let define $w := u - v$. Then for w, we have the following problem:

$$\begin{cases} \frac{\partial w}{\partial t} + Lw + g(x,t,u) - g(x,t,v) = 0, \ (x,t) \in Q_T, \\ w(x,0) = 0, \ x \in \Omega, \\ \left(\frac{\partial w}{\partial \nu} + k(x,t)w\right)\big|_{\Gamma_T} = 0. \end{cases}$$

Multiplying the equation by w, then using the integration by parts and the conditions, we get:

$$\int_\Omega \frac{\partial w}{\partial t} w dx + \int_\Omega \sum_{i,j=1}^n a_{ij}(x,t) D_j w D_i w dx + \int_\Omega \sum_{i=1}^n b_i(x,t) D_i w w dx + \int_\Omega c(x,t) w^2 dx$$
$$+ \int_\Omega [g(x,t,u) - g(x,t,v)] w dx + \int_{\partial\Omega} k(x,t) w^2 dx = 0. \tag{4}$$

Case (a): Considering case (a) in (4) and using Young, Hölder inequalities, we have the following

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} \leq -(\widetilde{\theta} - \widetilde{g})\|w\|^2_{L_2(\Omega)},$$

where $\widetilde{\theta} := \min\{\theta, \widetilde{c}\} - \sigma c_0 - c_1\|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}$.

By solving the last inequality, we obtain

$$\|w(t)\|^2_{L_2(\Omega)} \leq \|w(0)\|^2_{L_2(\Omega)} e^{-2(\widetilde{\theta} - \widetilde{g})t}.$$

Here if we consider $w(0) = 0$, then we obtain $w = 0$.

Case (b): Considering case (b) in (4) and using Hölder, Young inequalities, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} \leq -(\widetilde{\theta} - \widetilde{g})\|w\|^2_{L_2(\Omega)},$$

where $\widetilde{\theta} := c_2 \min\{\theta, k_0\} - \sigma c_0 - c_0^2\|c\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}$.

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[4] $\|u\|_{L_{\frac{2n}{n-2}}(\Omega)} \leq c_0 \|u\|_{W_2^1(\Omega)}$

Thus we obtain

$$\|w(t)\|^2_{L_2(\Omega)} \leq \|w(0)\|^2_{L_2(\Omega)} e^{-2(\widetilde{\theta}-\widetilde{g})t}$$

which completes the proof of the theorem. □

Theorem 4.2. *Let conditions (1)-(3) and (i′), (ii′) be fulfilled. Furthermore, assume that*

$$|g(x,t,\xi) - g(x,t,\eta)| \leq g_1(x,t)(|\xi|^{\alpha-1} + |\eta|^{\alpha-1})|\xi - \eta|$$

holds for all $\xi, \eta \in \mathbb{R}$, $1 < \alpha \leq \frac{n}{n-2}$ *and* $g_1 \in L_\infty(Q_T)$. *Then the solution of problem (1)-(3) is unique in*

$$L_\infty(0,T; W_2^1(\Omega)) \cap W_2^1(0,T; (W_2^1(\Omega))^*) \cap \{u : u(x,0) = u_0\}.$$

Proof. For the proof of this theorem, we have the following estimation for the nonlinear part as distinct from the proof of Theorem 4.1:

$$\langle g(x,t,u) - g(x,t,v), w\rangle_\Omega \geq -\varepsilon\|w\|^2_{W_2^1(\Omega)} - \widetilde{g}\|w\|^2_{L_2(\Omega)},$$

where $\widetilde{g} := c(\varepsilon)c_0^2\|g_1\|^2_{L_\infty(Q_T)}(\|u\|^{\alpha-1}_{L_\infty(0,T;L_{(\alpha-1)n}(\Omega))} + \|v\|^{\alpha-1}_{L_\infty(0,T;L_{(\alpha-1)n}(\Omega))})^2$ and ε is small enough. If we use the last inequality in (4), we obtain the following inequality for the cases (a) and (b):

$$\|w(t)\|^2_{L_2(\Omega)} \leq \|w(0)\|^2_{L_2(\Omega)} e^{-2(\widetilde{\theta}-\widetilde{g})t},$$

that completes the proof of the theorem. □

From the obtained inequality in the proof of the above theorems, we get the following:

Corollary 4.1. *Under the conditions of Theorem 3.1 and Theorem 4.1, for any* $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ *and any* $T > 0$ *there exists a unique solution* $u \in P_0$ *of problem (1)-(3). The mapping* $u_0 \mapsto u(t)$ *is continuous in* $L_2(\Omega)$, *i.e. the solution depends on the initial data continuously.*

5. Behavior of solution of problem (1)-(3)

5.1. Homogeneous Case: $h(x,t) = 0$, $\varphi(x,t) = 0$.

Theorem 5.1. *Let* $h(x,t) = 0$ *and* $\varphi(x,t) = 0$ *for problem (1)-(3). Assume that the conditions of Theorem 3.1 and Theorem 4.1 are fulfilled such that condition (ii) in Theorem 3.1 is satisfied with* $\widetilde{g_0} = 0$. *Then the solution of problem (1)-(3) satisfies the following inequality:*

$$\|u\|^2_{L_2(\Omega)}(t) \leq \frac{2\|u_0\|^2_{L_2(\Omega)} e^{-K_0 t}}{\left[-\frac{K_1}{K_0}\|u_0\|^{\alpha-1}_{L_2(\Omega)}(e^{(\frac{1-\alpha}{2})K_0 t} - 1) + 2^{\frac{\alpha-1}{2}}\right]^{\frac{2}{\alpha-1}}},$$

where $K_0 = K_0(\theta, \sigma, \widetilde{c}, k_0, \|c\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}, \|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}, c_0, c_1, c_2) > 0$ *and* $K_1 = K_1(\widetilde{g_1}, c_5, \alpha) > 0$. *(Here,* c_5 *is constant of embedding inequality* [5]*.)*

Proof. Let make use of Lyapunov functional $F(t) := \frac{1}{2}\|v\|^2_{L_2(\Omega)}(t)$ where $v(t) \in L_2(\Omega)$, $\forall t \geq 0$. If we take u instead of v such that $u(x,t)$ is the solution of problem (1)-(3) and compute $F'(t) = \int\limits_\Omega u u_t dx$, then we obtain the following inequality by using integration by parts and making use of some calculations:

$$F'(t) \leq -\theta\|Du\|^2_{L_2(\Omega)} + \sum_{i=1}^n \|b_i\|_{L_n(\Omega)}\|D_i u\|_{L_2(\Omega)}\|u\|_{L_{\frac{2n}{n-2}}(\Omega)} - \widetilde{g_1}\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} -$$

[5] $\|u\|_{L_{\alpha+1}(\Omega)} \geq c_5\|u\|_{L_2(\Omega)}$

$$-\int_{\Omega} c(x,t)u^2dx - \int_{\partial\Omega} k(x,t)u^2dx$$

$$\leq -\theta\|Du\|^2_{L_2(\Omega)} + \sigma c_0\|u\|^2_{W^1_2(\Omega)} - \widetilde{g_1}c_5^{\alpha+1}\|u\|^{\alpha+1}_{L_2(\Omega)} - \int_{\Omega} c(x,t)u^2dx - \int_{\partial\Omega} k(x,t)u^2dx. \qquad (5)$$

Now inequality (5) will be considered separately due to condition (ii').

Case (a): Consider case (a) in (5). By using Hölder, Young inequalities and making some calculations we have

$$F'(t) \leq -K_2F(t) - K_1F(t)^{\frac{\alpha+1}{2}},$$

where $K_1 = 2\widetilde{g_1}c_5^{\alpha+1}$ and $K_2 = 2\left(min\{\theta, \widetilde{c}\} - \sigma c_0 - c_1\|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}\right)$.

Case (b): Consider case (b) in (5). Making use of similar calculations as in above we have the following

$$F'(t) \leq -K_3F(t) - K_1F(t)^{\frac{\alpha+1}{2}},$$

where $K_1 = 2\widetilde{g_1}c_5^{\alpha+1}$ and $K_3 = 2\left(c_2min\{\theta, k_0\} - \sigma c_0 - c_0^2\|c\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}\right)$.

For both of these cases, if we denote $K_0 := \max\{K_2, K_3\}$ and define $p := \frac{\alpha+1}{2} > 1$, we have the following differential inequality:

$$F'(t) \leq -K_0F(t) - K_1F(t)^p.$$

If we solve this inequality considering $F(0) = \frac{1}{2}\|u_0\|^2_{L_2(\Omega)}$ and making use of the substitution $v := F(t)^{1-p}$, then we have

$$v(t) \geq v(0)e^{(p-1)K_0t} - \frac{K_1}{K_0}(1 - e^{(p-1)K_0t}).$$

Thus we obtain following inequality

$$F^{\frac{1-\alpha}{2}}(t) \geq F^{\frac{1-\alpha}{2}}(0)e^{\frac{\alpha-1}{2}K_0t} - \frac{K_1}{K_0}(1 - e^{\frac{\alpha-1}{2}K_0t}),$$

which completes the proof of this theorem. □

Corollary 5.1. *Under the conditions of Theorem 5.1, for the solution of problem (1)-(3),*

$$\|u\|^2_{L_2(\Omega)}(t) \leq \|u_0\|^2_{L_2(\Omega)}e^{-K_0t}$$

is satisfied where $K_0 = K_0(\theta, k_0, \sigma, c_0, c_1, c_2, \|c\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}, \widetilde{c}, \|k\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}) > 0$.

Corollary 5.2. *Under the conditions of Theorem 5.1, for the solution of problem (1)-(3),*

$$\|u\|^2_{L_2(\Omega)}(t) \leq \frac{2^{\frac{\alpha+1}{\alpha-1}}\|u_0\|^2_{L_2(\Omega)}}{\left[(\alpha-1)\|u_0\|^{\alpha-1}_{L_2(\Omega)}K_1t + 2^{\frac{\alpha+1}{2}}\right]^{\frac{2}{\alpha-1}}}$$

is satisfied where $K_1 = K_1(\widetilde{g_1}, c_5, \alpha) > 0$.

Corollary 5.3. *Under the conditions of Theorem 5.1, the solution goes to zero as* $t \to \infty$ *independent of initial function* u_0.

5.2. **Nonhomogeneous Case.** We investigate the behavior of solution for problem (1)-(3) in autonomous case; i.e.

$$\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}D_i(a_{ij}(x)\,D_ju)+\sum_{i=1}^{n}b_i(x)\,D_iu+c(x)u+g(x,u)=h(x), \tag{6}$$

$$u(x,0)=u_0(x), \tag{7}$$

$$\left(\frac{\partial u}{\partial \nu}+k(x)u\right)\bigg|_{\Gamma_T}=\varphi(x). \tag{8}$$

Taking into account Theorem 3.1 (existence) and Theorem 4.1 (uniqueness), we obtain that for all $u_0\in W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)$ and $T>0$, there exists a continuous mapping $u_0\to u(t)$. It should be noted that this map satisfies the conditions of semiflow for the autonomous case (see [20]). In what follows, we always assume that $\{S(t)\}_{t\geq 0}$ is the semiflow generated by the solutions of problem (5.2)-(5.4) with $S(t)u_0=u(t)$.

5.2.1. *Existence of an absorbing set in $L_2(\Omega)$.*

Theorem 5.2. *We assume that the conditions of Theorem 3.1 and Theorem 4.1 are satisfied. The semiflow $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L_2(\Omega)$ for any $(h,\varphi)\in[(W_2^1(\Omega))^*+L_{\frac{\alpha+1}{\alpha}}(\Omega)]\times W_2^{-\frac{1}{2}}(\partial\Omega)$, i. e., there exists a bounded set ,*

$$B_K:=\{u\in L_2(\Omega):\|u\|_{L_2(\Omega)}\leq (K)^{\frac{1}{2}}\}$$

with $K=K(\widetilde{g_1},\widetilde{g_0},meas(\Omega),\|h\|,\|\varphi\|,\|k\|_{L_{n-1}(\partial\Omega)},\|c\|_{L_{\frac{n}{2}}(\Omega)},\alpha,\theta,k_0,\widetilde{c},\sigma,c_0,c_1,c_2,c_4)$[6] such that for any bounded subset B in $W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)$ and for any $\delta>0$ there exists a $t_0^\delta=t_0^\delta(B,\delta)>0$ such that $S(t)B\subset B_{K+\delta}$ for every $t\geq t_0^\delta$.

Proof. This proof is similar with the proof of Theorem 5.1 as making use of the same Lyapunov functional $F(t)$. We compute $F'(t)$ and use Hölder, Young inequalities, so we obtain the following

$$F'(t)\leq -\theta\|Du\|^2_{L_2(\Omega)}+\sigma c_0\|u\|^2_{W_2^1(\Omega)}-\widetilde{g_1}\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)}+\widetilde{g_0}meas(\Omega)-\int_\Omega c(x)u^2dx$$

$$-\int_{\partial\Omega}k(x)u^2dx+2\varepsilon_1\|u\|^2_{W_2^1(\Omega)}+2\varepsilon_1\|u\|^2_{L_{\alpha+1}(\Omega)}+c(\varepsilon_1)\|h\|^2+\varepsilon_2\|u\|^2_{W_2^1(\Omega)}+c(\varepsilon_2)c_4^2\|\varphi\|^2. \tag{9}$$

Case (a): Considering case (a) in (9), we have the following differential inequality:

$$F'(t)\leq -K_1F(t)+K_0,$$

where $K_1=2\left(min\{\theta,\widetilde{c}\}-\sigma c_0-c_1\|k\|_{L_{n-1}(\partial\Omega)}-2\varepsilon_1-\varepsilon_2\right)>0$ and
$K_0=\widetilde{g_1}+\widetilde{g_0}meas(\Omega)+c(\varepsilon_1)\|h\|^2+c(\varepsilon_2)c_4^2\|\varphi\|^2$.
By the Gronwall Lemma, we obtain the following inequality:

$$\|u\|^2_{L_2(\Omega)}(t)\leq\|u_0\|^2_{L_2(\Omega)}e^{-K_1t}+2\frac{K_0}{K_1}.$$

Consequently we can find a t_0^δ for given $\delta>0$ that is $t_0^\delta=\frac{1}{K_1}\ln(\frac{\|u_0\|^2_{L_2(\Omega)}}{\delta})$ such that

$$\|u\|^2_{L_2(\Omega)}(t)\leq\delta+2\frac{K_0}{K_1}$$

[6] $\|h\|\equiv\|h\|_{(W_2^1(\Omega))^*+L_{\frac{\alpha+1}{\alpha}}(\Omega)},\quad \|\varphi\|\equiv\|\varphi\|_{W_2^{\frac{-1}{2}}(\partial\Omega)}$

holds for all $t \geq t_0^\delta$. Thus we find that

$$B_{K+\delta} := \{u \in L_2(\Omega) : \|u\|_{L_2(\Omega)} \leq \sqrt{\delta + 2\frac{K_0}{K_1}}\},$$

is an absorbing set in $L_2(\Omega)$.

Case (b): Considering case (b) in (9), we obtain the following differential inequality:

$$F'(t) \leq -K_2 F(t) + K_0,$$

where $K_2 = 2\left(c_2 min\{\theta, k_0\} - \sigma c_0 - c_0^2 \|c\|_{L_{\frac{n}{2}}(\Omega)} - 2\varepsilon_1 - \varepsilon_2\right) > 0$. Using the same way we also obtain an absorbing set. □

5.2.2. *Existence of an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$.* In this subsection, we will investigate the existence of an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$. Here we take differential operator L by the following simple form with $b_i \equiv 0$, $\forall i = 1, ..., n$:

$$Lu := -\sum_{i=1}^{n} D_i(a_i\,(x)\,D_i u) + c(x)u.$$

Thus we rewrite the problem (6)-(8) in the following:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i(a_i\,(x)\,D_i u) + c(x)u + g(x,u) = h(x), \tag{10}$$

$$u(x,0) = u_0(x), \tag{11}$$

$$\left(\frac{\partial u}{\partial \nu} + k(x)u\right)\Big|_{\Gamma_T} = \varphi(x). \tag{12}$$

Here to investigate the posed question, we assume that all data functions possess some smoothness and solution u belongs to $W_2^1(0,T; W_2^1(\Omega)) \cap W_{\alpha+1}^1(0,T; L_{\alpha+1}(\Omega))$.

Theorem 5.3. *Assume that the conditions of Theorem 3.1 and Theorem 4.1 are fulfilled such that condition (ii) in Theorem 3.1 is satisfied with $g(x,0) = 0$ and for all $\xi \in \mathbb{R}$, $\xi \neq 0$,*

$$0 < (\alpha+1)G(x,\xi) \leq g(x,\xi)\xi$$

is satisfied where $G(x,u) := \int\limits_0^u g(x,\tau)d\tau$. Then the semiflow $\{S(t)\}_{t\geq 0}$ has an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ for any $h \in L_2(\Omega)$ and $\varphi \in L_2(\partial\Omega)$.

Proof. Assuming that the solution is smooth (as mentioned above), let multiply equation (10) by u_t and integrate on Ω. Then using integration by parts, (12) and the assumption of the theorem, we have

$$\|u_t\|_{L_2(\Omega)}^2 + \frac{d}{dt}\left[\frac{1}{2}\int\limits_\Omega \sum_{i=1}^{n} a_i\,(x)\,(D_i u)^2 dx + \frac{1}{2}\int\limits_\Omega c(x)u^2 dx + \int\limits_\Omega G(x,u)dx + \frac{1}{2}\int\limits_{\partial\Omega} k(x)u^2 dx\right.$$
$$\left. - \int\limits_\Omega h(x)u dx - \int\limits_{\partial\Omega} \varphi(x)u dx\right] = 0$$

which gives following inequality:

$$\frac{d}{dt}\left[\frac{1}{2}\int\limits_\Omega \sum_{i=1}^{n} a_i(x)(D_i u)^2 dx + \frac{1}{2}\int\limits_\Omega c(x)u^2 dx + \int\limits_\Omega G(x,u)dx + \frac{1}{2}\int\limits_{\partial\Omega} k(x)u^2 dx\right.$$

$$\left. -\int_{\Omega} h(x)udx - \int_{\partial\Omega} \varphi(x)udx \right] < 0. \tag{13}$$

Denoting by

$$y(t) := \frac{1}{2}\int_{\Omega}\sum_{i=1}^{n} a_i(x)(D_iu)^2dx + \frac{1}{2}\int_{\Omega} c(x)u^2dx + \int_{\Omega} G(x,u)dx +$$
$$+\frac{1}{2}\int_{\partial\Omega} k(x)u^2dx - \int_{\Omega} h(x)udx - \int_{\partial\Omega} \varphi(x)udx,$$

and considering (13), we obtain that $\frac{d}{dt}y(t) < 0$ for all $t \geq 0$.

Now let multiply (10) by u and integrate on Ω. Using integration by parts and (12), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \int_{\Omega}\sum_{i=1}^{n} a_i(x)(D_iu)^2dx + \int_{\Omega} c(x)u^2dx + \int_{\Omega} g(x,u)udx + \int_{\partial\Omega} k(x)u^2dx$$
$$-\int_{\Omega} h(x)udx - \int_{\partial\Omega} \varphi(x)udx = 0. \tag{14}$$

Case (a): Consider case (a) in (14). By using the assumptions of the theorem, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \int_{\Omega}\sum_{i=1}^{n} a_i(x)(D_iu)^2dx + \int_{\Omega} c(x)u^2dx + (\alpha+1)\int_{\Omega} G(x,u)dx$$
$$+\int_{\partial\Omega} k(x)u^2dx - \int_{\Omega} h(x)udx - \int_{\partial\Omega} \varphi(x)udx \leq 0.$$

Making some calculations, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \frac{1}{2}\theta\|Du\|^2_{L_2(\Omega)} + \frac{1}{2}\widetilde{c}\|u\|^2_{L_2(\Omega)} - \frac{1}{2}c_1\|k\|_{L_{n-1}(\partial\Omega)}\|u\|^2_{W_2^1(\Omega)} + y(t) \leq 0$$

that follows

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \frac{1}{2}\left(\min\{\theta,\widetilde{c}\} - c_1\|k\|_{L_{n-1}(\partial\Omega)}\right)\|u\|^2_{W_2^1(\Omega)} + y(t) \leq 0.$$

Since $\min\{\theta,\widetilde{c}\} - c_1\|k\|_{L_{n-1}(\partial\Omega)} > 0$ by the assumptions, we obtain

$$y(t) \leq |\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)}|.$$

Here using the inequality obtained in the proof of Theorem 5.2, we get

$$y(t) \leq \frac{1}{2}K_1\|u\|^2_{L_2(\Omega)} + K_0$$

that follows

$$y(t) \leq \frac{1}{2}K_1(\delta + K) + K_0$$

for all $t \geq t_0^{\delta}$. So, we get the existence of a constant $M_1 = M_1(K,\delta) > 0$ such that

$$y(t) \leq M_1, \quad \forall t \geq t_0^{\delta}. \tag{15}$$

Consequently, from (13) and (15), we obtain that y(t) is decreasing and bounded by a constant that is independent of t for $t \geq t_0^{\delta}$.

Making some calculations, using Hölder and Young inequalities, we get

$$\frac{1}{2}\theta\|Du\|^2_{L_2(\Omega)} + \frac{1}{2}\widetilde{c}\|u\|^2_{L_2(\Omega)} - \frac{1}{2}c_1\|k\|_{L_{n-1}(\partial\Omega)}\|u\|^2_{W_2^1(\Omega)} + g_3\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} - g_4 meas\Omega$$

$$\leq M_1 + \varepsilon_2 \|u\|^2_{W_2^1(\Omega)} + \varepsilon_1 \|u\|^2_{L_2(\Omega)} + c(\varepsilon_1)\|h\|^2_{L_2(\Omega)} + c(\varepsilon_2) c_3^2 \|\varphi\|^2_{L_2(\partial\Omega)}$$

where $g_3, g_4 > 0$ are some constants such that $G(x,u) \geq g_3|u|^{\alpha+1} - g_4$. Hence we obtain there exists a constant $M_2 = M_2(M_1, g_3, g_4, c_3, \|h\|_{L_2(\Omega)}, \|\varphi\|_{L_2(\partial\Omega)}) > 0$ such that

$$\left(min\{\theta, \widetilde{c} - 2\varepsilon_1\} - c_1\|k\|_{L_{n-1}(\partial\Omega)} - 2\varepsilon_2\right) \|u\|^2_{W_2^1(\Omega)} + 2g_3\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} \leq M_2$$

that follows

$$\|u\|_{W_2^1(\Omega)}(t) + \|u\|_{L_{\alpha+1}(\Omega)}(t) \leq M, \quad \forall t \geq t_0^\delta$$

holds for $M = M(M_2) > 0$. So we find that

$$B_0 = \{u : u \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega), \|u\|_{W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)} \leq M\}$$

is an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$.

Case (b): Consider case (b) in (14). By making similar operations and using Theorem 5.2, we have

$$y(t) \leq \frac{1}{2} K_2(\delta + K) + K_0$$

for all $t \geq t_0^\delta$. So, we get the existence of a constant $M_3 = M_3(K, \delta) > 0$ such that

$$y(t) \leq M_3, \quad \forall t \geq t_0^\delta. \tag{16}$$

Consequently, from (13) and (16), we obtain that y(t) is decreasing and bounded by a constant that is independent of t for $t \geq t_0^\delta$.

Making some calculations, we obtain the existence of a constant $M_3 = M_3(K, \delta, g_3, g_4, c_3, \|h\|_{L_2(\Omega)}, \|\varphi\|_{L_2(\partial\Omega)}) > 0$ such that

$$\left(c_2 min\{\theta, k_0 - 2\varepsilon_2\} - c_0\|c\|_{L_{\frac{n}{2}}(\Omega)} - 2\varepsilon_1\right) \|u\|^2_{W_2^1(\Omega)} + 2g_3\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} \leq M_3$$

holds for all $t \geq t_0^\delta$. So we find that for some constant $\widetilde{M} = \widetilde{M}(M_3) > 0$,

$$B_1 = \{u : u \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega), \|u\|_{W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)} \leq \widetilde{M}\}$$

is an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$. □

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References

[1] Adams, R.A., (1975), Sobolev Spaces, Academic Press, New York.

[2] Boni, T.K., (2011), On the asymptotic behavior of solutions for some parabolic and elliptic equations of second order with nonlinear boundary conditions, Nonlinear Analysis, 45, pp.895-908.

[3] Chen, X., Fila, M., Guo, J-S., (2008), Boundedness of global solutions of a supercritical parabolic equation, Nonlinear Analysis, 68, pp.621-628.

[4] Escobedo, M., Zuazua, E., (1991), Large time Behavior for convection diffusion equations in $\mathbb{R}^N$, Journal of Functional Analysis, 100, pp.119-161.

[5] Gmira, A., Veron, L., (1984), Large time behavior of the solutions of a semilinear parabolic equation in $\mathbb{R}^N$, Journal of Differential Equations, 53, pp.258-276.

[6] Kallı, K., Soltanov, K.N., (2011), Some BVP for a general semilinear parabolic equation in divergence form, Proceedings of the 8th Congress of the International Society for Analysis, its Applications and Computation, 2(16), pp.16-23.

[7] Li, F., Li, J., (2012), Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions, Journal of Mathematical Analysis and Applications, 385, pp.1005-1014.
[8] Lions, J.L., (1969), Quelques méthodes de resolution des problémes aux Limities non linéaires, Dunod, Gauthier-Villars, Paris.
[9] Lions, J.L., Magenes, E., (1972), Non-Homogeneous Boundary Value Problems and Applications, I, Springer-Verlag, Berlin.
[10] Ma, L., (2010), Boundary value problem for a classical semilinear parabolic equation, arXiv:1012.5861v1 [math.AP]
[11] Pao, C.V., (1978), Asymptotic behavior and nonexistence of global solutions for a class of nonlinear boundary value problems of parabolic type, Journal of Mathematical Analysis and Applications, 65, pp.616-637.
[12] Porzio, M.M., (2011), Existence, uniqueness and behavior of solutions for a class of nonlinear parabolic problems, Nonlinear Analysis, 74, pp.5359-5382.
[13] Soltanov K.N., (1993), Existence and nonexistence of the global solutions some nonlinear elliptic-parabolic equations, Differential Equations, 29(4), pp.550-563 (translation).
[14] Soltanov, K.N., (2001), On some nonlinear boundary value problems and embedding theorems, Proceeding of Nonlinear Analysis and Nonlinear Modeling, pp.76-85.
[15] Soltanov, K.N., (2003), On some modification on Navier-Stokes equations, Non-linear Analysis- Theory Methods and Applications, 52(3), pp.769-793.
[16] Soltanov, K.N., (2006), Some nonlinear equations of the nonstable filtration type and embedding theorems, Nonlinear Analysis, 65, pp.2103-2134.
[17] Song, L., He, Y., Zhang, Y., (2008), The existence of global attractors for semi-linear parabolic equation in H^k Spaces, Nonlinear Analysis, 68, pp.3541-3549.
[18] Souplet, P., (2001), Recent results and open problems on parabolic equations with gradient nonlinearities, Electronic Journal of Differential Equations, 2001(20), pp.1-19.
[19] Struwe, M., (1990), Variational Methods Applications to Nonlinear Partial Differential Equations and Hamiltonian, Springer-Verlag.
[20] Temam, R., (1997), Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York.
[21] Tsutsumi, M., (1972/73), Existence and nonexistence of global solutions for nonlinear parabolic equations, Publication of RIMS, Kyoto Univ., 8, pp.211-229.

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