TWMS J . Pure Appl. Math. V.8, N.2, 2017, pp.198-208

ON HIGHER ORDER (p, q)-FROBENIUS-EULER POLYNOMIALS

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Abstract. The main purpose of this paper is to introduce (p, q)-Frobenius-Euler numbers and polynomials, and to investigate their some identities and properties including addition property, difference equation, derivative property, recurrence relationships. We also obtain integral representation, explicit formulae and relations for these polynomials and numbers. Furthermore, we consider some relationships for (p, q)-Frobenius-Euler polynomials of order α associated with (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials.

Keywords: (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials, (p, q)-Genocchi polynomials, (p, q)-Frobenius-Euler polynomials, generating function, Cauchy product.

AMS Subject Classification: Primary 05A30, Secondary 11B68, 11B73.

1. Introduction

Let

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}$$

denote (p, q)-numbers. We see here that $[n]_{p,q} = p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ stands for q-number known as $[n]_{q/p} = \frac{(q/p)^n-1}{(q/p)-1}$. One can see that (p, q)-number is closely related to q-number with this relation $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p}}$. By appropriately using this obvious relation between the q-notation and its variant, the (p, q)-notation, most (if not all) of the (p, q)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved.

The (p, q)-derivative is defined by

$$D_{p,q}f(x) := \frac{\partial}{\partial_{p,q}x} f(x) = \frac{f(px) - f(qx)}{(p - q)x}. \tag{1}$$

The (p, q)-binomial coefficients $\binom{n}{k}_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!} \;\; (n \geq k) \;\; \textit{and} \;\; [n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q} \quad (n \in \mathbb{N}).$$

Then, the (p, q)-power basis is defined by

$$\begin{aligned}(x+a)^n_{p,q} &= \begin{cases} (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \textit{if } n \geq 1, \\ 1, & \textit{if } n = 0, \end{cases} \\ &= \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}.\end{aligned}$$

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Manuscript received February 2017.

The (p,q)-exponential functions

$$e_{p,q}(x)=\sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!}\ \textit{and}\ E_{p,q}(x)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{[n]_{p,q}!}$$

hold the identities

$$e_{p,q}(x)E_{p,q}(-x)=1\ \textit{and}\ e_{p,q}(x)E_{p,q}(y)=\sum_{n=0}^{\infty}\frac{(x+y)_{p,q}^n}{[n]_{p,q}!}, \tag{2}$$

and have the (p,q)-derivatives

$$D_{p,q}e_{p,q}(x)=e_{p,q}(px)\ \textit{and}\ D_{p,q}E_{p,q}(x)=E_{p,q}(qx). \tag{3}$$

The definite (p,q)-integral is also defined by

$$\int_0^a f(x)\,d_{p,q}x=(p-q)\,a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right)$$

with

$$\int_a^b f(x)\,d_{p,q}x=\int_0^b f(x)\,d_{p,q}x-\int_0^a f(x)\,d_{p,q}x\ \text{(see [19])}. \tag{4}$$

We refer the reader to see for more information about (p,q)-calculus, *e.g.,* $[1,5,8,9,10,15,19]$. When $p=1$, all the notations of the (p,q)-calculus reduce to the notations of the q-calculus, see [13].

The (p,q)-Bernoulli polynomials, the (p,q)-Euler polynomials and the (p,q)-Genocchi polynomials are defined by means of the following generating functions:

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_n(x:p,q)\frac{z^n}{[n]_{p,q}!} &= \frac{z}{e_{p,q}(z)-1}e_{p,q}(xz) \quad (|z|<2\pi),\\
\sum_{n=0}^{\infty}\mathcal{E}_n(x:p,q)\frac{z^n}{[n]_{p,q}!} &= \frac{[2]_{p,q}}{e_{p,q}(z)+1}e_{p,q}(xz) \quad (|z|<\pi),\\
\sum_{n=0}^{\infty}\mathcal{G}_n(x:p,q)\frac{z^n}{[n]_{p,q}!} &= \frac{[2]_{p,q}\,z}{e_{p,q}(z)+1}e_{p,q}(xz) \quad (|z|<\pi),
\end{aligned}$$

respectively. Upon setting $x=0$ above, we then have

$$\begin{aligned}
\mathcal{B}_n(0:p,q) &:= \mathcal{B}_n(p,q)\ \ (\,(p,q)\text{-Bernoulli numbers}),\\
\mathcal{E}_n(0:p,q) &:= \mathcal{E}_n(p,q)\ \ (\,(p,q)\text{-Euler numbers}),\\
\mathcal{G}_n(0:p,q) &:= \mathcal{G}_n(p,q)\ \ (\,(p,q)\text{-Genocchi numbers}),
\end{aligned}$$

for further details, see [8].

Some special polynomials including Euler, Bernoulli, Genocchi, Frobenius-Euler polynomials and many kind of their generalizations have been studied by many mathematicians, see [1-4, 6-12, 14-18, 20-25].

The classical Frobenius-Euler numbers and polynomials of order α are defined by the following Taylor series expansions at $z=0$:

$$H^{(\alpha)}(z) = \sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(u)\frac{z^n}{n!} = \left(\frac{1-u}{e^z-u}\right)^{\alpha}, \tag{5}$$

$$H^{(\alpha)}(x:z) = \sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x;u)\frac{z^n}{n!} = \left(\frac{1-u}{e^z-u}\right)^{\alpha} e^{xz}, \tag{6}$$

where α is suitable (real or complex) parameter and $u\in\mathbb{C}$ with $u\neq 1$, see [2-4, 6, 12, 14, 16-18, 20, 25].

By Eqs. (5) and (6), for $n\geq 0$, we note that

$$H^{(\alpha)}\left(\frac{\partial}{\partial x}\right)x^n = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{(\alpha)}(u)\,x^{n-k} = \mathcal{H}_n^{(\alpha)}(x;u)$$

and

$$H^{(\alpha)}\left(\frac{\partial}{\partial x}\right)e^{xz} = H^{(\alpha)}(x:z)\,,\text{ see [2], [14] and [20].}$$

The following section provides some identities and properties for (p,q)-Frobenius-Euler number of order α involving addition property, difference equation, derivative property, recurrence relationships. We also provide integral representation, explicit formulas and relations for mentioned polynomials and numbers. By using generating function of the polynomial stated in Definition 2.1, we derive some relationships for (p,q)-Frobenius-Euler polynomials of order α related to (p,q)-Bernoulli polynomials, (p,q)-Euler polynomials and (p,q)-Genocchi polynomials.

2. (p,q)-Frobenius-Euler polynomials and numbers

We now begin with the following Definition 2.1.

Definition 2.1. *The (p,q)-Frobenius-Euler polynomials $\mathcal{H}_n^{(\alpha)}(x,u:p,q)$ of order α are defined by means of the following Taylor series expansion about $z=0$:*

$$H_{p,q}^{(\alpha)}(x:z) = \sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(x;u:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz)\,,$$

where α is suitable (real or complex) parameter, $p,q\in\mathbb{C}$ with $0<|q|<|p|\leq 1$ and $u\in\mathbb{C}/\{1\}$.

Let

$$H_{p,q}^{(\alpha)}(0:z) := H_{p,q}^{(\alpha)}(z) = \sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(u:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}$$

be generating function of (p,q)-Frobenius-Euler numbers of order α denoted by $\mathcal{H}_n^{(\alpha)}(u:p,q)$. Obviously that

$$\begin{aligned}
\mathcal{H}_n^{(1)}(x;u:p,q) &:= \mathcal{H}_n(x;u:p,q)\,,\\
\mathcal{H}_n^{(\alpha)}(x;u:p,q)\Big|_{p=1} &:= \mathcal{H}_{n,q}^{(\alpha)}(x;u)\ \text{ (see [14], [17], [20])},\\
\lim_{\substack{q\to 1^-\\ p=1}}\mathcal{H}_n^{(\alpha)}(x;u:p,q) &:= \mathcal{H}_n^{(\alpha)}(x;u)\quad\text{(see [2], [6], [25])}.
\end{aligned}$$

From Definition 2.1, we give the following Lemma 2.1.

Lemma 2.1. *The following relationship holds true*

$$H_{p,q}^{(\alpha)}\left(D_{p,q}\right)x^{n}=\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)x^{n-k}.$$

Proof. By means of the (p,q)-derivative operator $D_{p,q}$, we have

$$\begin{aligned}H_{p,q}^{(\alpha)}\left(D_{p,q}\right)x^{n} &= H_{p,q}^{(\alpha)}\left(\frac{\partial}{\partial_{p,q}x}\right)x^{n}=\sum_{k=0}^{\infty}\frac{\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)}{[k]_{p,q}!}\left(\frac{\partial}{\partial_{p,q}x}\right)^{k}x^{n}\\ &= \sum_{k=0}^{n}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)\frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}x^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)x^{n-k}.\end{aligned}$$

□

Here we state a relationship of (p,q)-Frobenius-Euler polynomials of order α and (p,q)-Frobenius-Euler numbers of order α.

Lemma 2.2. *We have*

$$\mathcal{H}_{n}^{(\alpha)}\left(x;u:p,q\right)=\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)x^{n-k}.$$

Proof. Using Definition 2.1, we observe that

$$\begin{aligned}H_{p,q}^{(\alpha)}\left(x:z\right) &= \sum_{n=0}^{\infty}\mathcal{H}_{n}^{(\alpha)}\left(x;u:p,q\right)\frac{z^{n}}{[n]_{p,q}!}=\left(\frac{1-u}{e_{p,q}\left(z\right)-u}\right)^{\alpha}e_{p,q}\left(xz\right)\\ &= \sum_{n=0}^{\infty}\mathcal{H}_{n}^{(\alpha)}\left(u:p,q\right)\frac{z^{n}}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^{n}\frac{z^{n}}{[n]_{p,q}!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right)x^{n-k}\right)\frac{z^{n}}{[n]_{p,q}!},\end{aligned}$$

which gives the desired result by comparing the coefficients $\frac{z^{n}}{[n]_{p,q}!}$ of the both sides. □

As a result of Lemma 2.2, we obtain the following Corollary 2.1.

Corollary 2.1. *In the case $x=1$ in Lemma 2.2, we have the following formula*

$$\mathcal{H}_{n}^{(\alpha)}\left(1;u:p,q\right)=\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_{k}^{(\alpha)}\left(u:p,q\right). \tag{7}$$

We remark that Eq. (7) is (p,q)-generalization of the following familiar formula:

$$\mathcal{H}_{n}^{(\alpha)}\left(1;u\right)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{k}^{(\alpha)}\left(u\right).$$

Note that

$$\mathcal{H}_{n}^{(0)}\left(x;u:p,q\right)=p^{\binom{n}{2}}x^{n}.$$

We now give the following Lemmas without proof, because one can easily derive by using the Eq. (2) and Definition 2.1.

Lemma 2.3. *(Addition Property) We have*

$$\mathcal{H}_n^{(\alpha)}\left((x+y)_{p,q};u:p,q\right)=\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k^{(\alpha)}(x;u:p,q)\,y^{n-k}q^{\binom{n-k}{2}}.$$

Lemma 2.4. *(Difference equation) We have*

$$(1-u)\,\mathcal{H}_n^{(\alpha-1)}(u:p,q)=\mathcal{H}_n^{(\alpha)}(1;u:p,q)-u\mathcal{H}_n^{(\alpha)}(u:p,q).$$

Lemma 2.5. *(Derivative property) We have*

$$\frac{\partial}{\partial_{p,q}x}\mathcal{H}_n^{(\alpha)}(x;u:p,q)=[n]_{p,q}\,\mathcal{H}_{n-1}^{(\alpha)}(px;u:p,q).$$

Lemma 2.6. *Let α and β be suitable (real or complex) parameters. Then (p,q)-Frobenius-Euler polynomials satisfy the following relation:*

$$\mathcal{H}_n^{(\alpha+\beta)}(x;u:p,q)=\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k^{(\alpha)}(x;u:p,q)\,\mathcal{H}_{n-k}^{(\beta)}(u:p,q).$$

Lemma 2.7. *(Recurrence relationship) $\mathcal{H}_n^{(\alpha)}(x;u:p,q)$ fulfills the following equality:*

$$\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(x;u:p,q)-u\mathcal{H}_n^{(\alpha)}(x;u:p,q)=(1-u)\,\mathcal{H}_n^{(\alpha-1)}(x;u:p,q).$$

When $\alpha=1$ in Lemma 2.7, we have

$$\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k(x;u:p,q)-u\mathcal{H}_n(x;u:p,q)=(1-u)\,x^np^{\binom{n}{2}}. \tag{8}$$

We provide now the following explicit formula for (p,q)-Frobenius-Euler polynomials of order α.

Theorem 2.1. *(p,q)-Frobenius-Euler polynomials of order α hold the following relation:*

$$\mathcal{H}_n^{(\alpha)}(x;u:p,q)=$$
$$=\frac{1}{1-u}\sum_{k=0}^{n}\binom{n}{k}_{p,q}\left(\sum_{s=0}^{k}\binom{k}{s}_{p,q}p^{\binom{k-s}{2}}\mathcal{H}_s(x;u:p,q)-u\mathcal{H}_k(x;u:p,q)\right)\mathcal{H}_{n-k}^{(\alpha)}(u:p,q).$$

Proof. Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(x;u:p,q)\frac{z^n}{[n]_{p,q}!}&=\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)\\&=\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{z^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(u:p,q)\frac{z^n}{[n]_{p,q}!}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{k}{2}}x^k\mathcal{H}_{n-k}^{(\alpha)}(u:p,q)\frac{z^n}{[n]_{p,q}!}.\end{aligned}$$

It remains to use Eq. (8) in order to complete the proof of this theorem. □

The (p,q)-integral representation of $\mathcal{H}_n^{(\alpha)}(x;u:p,q)$ is given by the following theorem.

Theorem 2.2. *The following (p,q)-integral is valid:*

$$\int_a^b \mathcal{H}_n^{(\alpha)}(x;u:p,q)\, d_{p,q}x = \frac{\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{b}{p};u:p,q\right) - \mathcal{H}_{n+1}^{(\alpha)}\left(\frac{a}{p};u:p,q\right)}{[n+1]_{p,q}}. \tag{9}$$

Proof. Since

$$\int_a^b \frac{\partial f(x)}{\partial_{p,q}x} d_{p,q}x = f(b) - f(a) \ \text{ (see [19])}$$

in terms of Lemma 2.5 and Eqs. (3) and (4), we arrive at the asserted result

$$\begin{aligned}\int_a^b \frac{\partial}{\partial_{p,q}x}\mathcal{H}_n^{(\alpha)}(x;u:p,q)\, d_{p,q}x &= \frac{1}{[n+1]_{p,q}} \int_a^b \mathcal{H}_{n+1}^{(\alpha)}\left(\frac{x}{p};u:p,q\right) d_{p,q}x \\ &= \frac{\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{b}{p};u:p,q\right) - \mathcal{H}_{n+1}^{(\alpha)}\left(\frac{a}{p};u:p,q\right)}{[n+1]_{p,q}}.\end{aligned}$$

This completes the proof of this theorem. □

The integral identity (9) is (p,q)-generalization of the formula

$$\int_a^b \mathcal{H}_n^{(\alpha)}(x;u)\, dx = \frac{\mathcal{H}_{n+1}^{(\alpha)}(b;u) - \mathcal{H}_{n+1}^{(\alpha)}(a;u)}{n+1}, \text{ see [14].}$$

Here is a recurrence relation of (p,q)-Frobenius-Euler polynomials by the following theorem.

Theorem 2.3. *We have*

$$\mathcal{H}_{n+1}(x;u:p,q) = xp^n\mathcal{H}_n(x;u:p,q) - \sum_{k=0}^{n}\binom{n}{k}_{p,q} q^k p^{n-k}\mathcal{H}_k(x;u:p,q)\,\mathcal{H}_{n-k}(1;u:p,q).$$

Proof. By inspring the proof way of Kurt used in [17], for $\alpha = 1$, applying the (p,q)-derivative to $\mathcal{H}_n(x;u:p,q)$ with respect to z yields to

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial_{p,q}z}\mathcal{H}_n(x;u:p,q)\frac{z^n}{[n]_{p,q}!} = (1-u)\frac{\partial}{\partial_{p,q}z}\left\{\frac{e_{p,q}(xz)}{e_{p,q}(z)-u}\right\},$$

From (p,q)-division rule, we get

$$\begin{aligned} &= (1-u)\frac{(e_{p,q}(qz)-u)\frac{\partial}{\partial_{p,q}z}e_{p,q}(xz) - e_{p,q}(qxz)\frac{\partial}{\partial_{p,q}z}(e_{p,q}(z)-u)}{(e_{p,q}(pz)-u)(e_{p,q}(qz)-u)} \\ &= x\frac{1-u}{e_{p,q}(pz)-u}e_{p,q}(xpz) - \frac{1-u}{e_{p,q}(pz)-u}e_{p,q}(pz)\frac{1-u}{e_{p,q}(qz)-u}e_{p,q}(qxz) \\ &= x\sum_{n=0}^{\infty}\mathcal{H}_n(x;u:p,q)p^n\frac{z^n}{[n]_{p,q}!} - \sum_{n=0}^{\infty}\mathcal{H}_n(1;u:p,q)p^n\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{H}_n(x;u:p,q)q^n\frac{z^n}{[n]_{p,q}!}.\end{aligned}$$

By using the method of Cauchy product, comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have the desired result. □

We now state a new identity for $\mathcal{H}_n(x;u:p,q)$ as given below.

Theorem 2.4. *We have*

$$(2u-1)\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k(u:p,q)\,\mathcal{H}_{n-k}(x;1-u:p,q) \tag{10}$$
$$=u\mathcal{H}_n(x;u:p,q)-(1-u)\,\mathcal{H}_n(x;1-u:p,q).$$

Proof. By utilizing the same method of Kurt's study [17], we first consider the identity

$$\frac{2u-1}{(e_{p,q}(z)-u)\,(e_{p,q}(z)-(1-u))}=\frac{1}{e_{p,q}(z)-u}-\frac{1}{e_{p,q}(z)-(1-u)},$$

then

$$(2u-1)\,\frac{(1-u)\,e_{p,q}(xz)\,(1-(1-u))}{(e_{p,q}(z)-u)\,(e_{p,q}(z)-(1-u))}$$
$$=u\frac{(1-u)\,e_{p,q}(xz)}{e_{p,q}(z)-u}-\frac{(1-u)\,e_{p,q}(xz)\,(1-(1-u))}{e_{p,q}(z)-(1-u)}.$$

Hence, we see

$$(2u-1)\sum_{n=0}^{\infty}\mathcal{H}_n(u:p,q)\,\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{H}_n(x;1-u:p,q)\,\frac{z^n}{[n]_{p,q}!}$$
$$=u\sum_{n=0}^{\infty}\mathcal{H}_n(x;u:p,q)\,\frac{z^n}{[n]_{p,q}!}-(1-u)\sum_{n=0}^{\infty}\mathcal{H}_n(x;1-u:p,q)\,\frac{z^n}{[n]_{p,q}!}.$$

Checking against the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have the asserted result (10). □

The relation (10) is a (p,q)-extension of Carlitz's result Eq. (2.19) in [7].

Theorem 2.5. *The following relationship holds true for (p,q)-Frobenius-Euler polynomials:*

$$\mathcal{H}_n(x;u:p,q)=u\sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\binom{n-k}{2}}(-1)^{n-k}\,\mathcal{H}_k(x;u:p,q)+(1-u)\,(x-1)_{p,q}^{n}.$$

Proof. This proof is obtained by considering the proof method in Kurt's work [17]. From the property (2) and the identity

$$\frac{u}{(e_{p,q}(z)-u)\,e_{p,q}(z)}=\frac{1}{e_{p,q}(z)-u}-\frac{1}{e_{p,q}(z)},$$

we can write

$$\frac{u\,(1-u)\,e_{p,q}(xz)}{(e_{p,q}(z)-u)\,e_{p,q}(z)}=\frac{(1-u)\,e_{p,q}(xz)}{e_{p,q}(z)-u}-\frac{(1-u)\,e_{p,q}(xz)}{e_{p,q}(z)},$$
$$\frac{u\,(1-u)\,e_{p,q}(xz)}{(e_{p,q}(z)-u)}E_{p,q}(-z)=\frac{(1-u)\,e_{p,q}(xz)}{e_{p,q}(z)-u}-(1-u)\,e_{p,q}(xz)\,E_{p,q}(-z)$$

which gives

$$u\sum_{n=0}^{\infty}\mathcal{H}_n(x;u:p,q)\,\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}q^{\binom{n}{2}}(-1)^n\,\frac{z^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\mathcal{H}_n(x;u:p,q)\,\frac{z^n}{[n]_{p,q}!}-(1-u)\sum_{n=0}^{\infty}(x-1)_{p,q}^{n}\,\frac{z^n}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{z^n}{[n]_{p,q}}$, we derive the asserted result. □

Now we are in a position to present some relationships for (p,q)-Frobenius-Euler polynomials of order α related to (p,q)-Bernoulli polynomials, (p,q)-Euler polynomials and (p,q)-Genocchi polynomials as given in Theorems 2.6-2.8.

Theorem 2.6. *The following recurrence relations are valid:*

$$\mathcal{H}_n^{(\alpha)}(x;u:p,q)$$
$$=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\left\{\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}(x:p,q)\,p^{\binom{k}{2}}-\mathcal{B}_s(x:p,q)\right\}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}(u:p,q)}{[n+1]_{p,q}}$$
$$=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\left\{\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}(p,q)\,p^{\binom{k}{2}}-\mathcal{B}_s(p,q)\right\}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}(x;u:p,q)}{[n+1]_{p,q}}.$$

Proof. Indeed,

$$\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)$$
$$=\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}\frac{z}{e_{p,q}(z)-1}\frac{e_{p,q}(z)-1}{z}e_{p,q}(xz)$$
$$=\frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{B}_{n-k}(x:p,q)\,p^{\binom{k}{2}}\right)\frac{z^n}{[n]_{p,q}!}-\sum_{n=0}^{\infty}\mathcal{B}_n(x:p,q)\frac{z^n}{[n]_{p,q}!}\right]$$
$$\cdot\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(u:p,q)\frac{z^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left[\binom{n}{s}\right]_{p,q}\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}(x:p,q)\,p^{\binom{k}{2}}-\sum_{s=0}^{n}\binom{n}{s}_{p,q}\mathcal{B}_s(x:p,q)\right]$$
$$\cdot\mathcal{H}_{n-s}^{(\alpha)}(u:p,q)\frac{z^{n-1}}{[n]_{p,q}!},$$

By using Cauchy product and comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$, the proof is completed. □

Theorem 2.7. *We have*

$$\begin{aligned}\mathcal{H}_n^{(\alpha)}(x;u:p,q) &= \sum_{s=0}^{n}\binom{n}{s}_{p,q}\left\{\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{E}_k(x:p,q)\,p^{\binom{s-k}{2}}+\mathcal{E}_s(x:p,q)\right\}\frac{\mathcal{H}_{n-s}^{(\alpha)}(u:p,q)}{[2]_{p,q}}\\ &= \sum_{s=0}^{n}\binom{n}{s}_{p,q}\left\{\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{E}_k(p,q)\,p^{\binom{s-k}{2}}+\mathcal{E}_s(p,q)\right\}\frac{\mathcal{H}_{n-s}^{(\alpha)}(x;u:p,q)}{[2]_{p,q}}.\end{aligned}$$

Proof. The proof of this theorem is based on the following equalities

$$\begin{aligned}\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz) &= \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)\frac{[2]_{p,q}}{e_{p,q}(z)+1}\frac{e_{p,q}(z)+1}{[2]_{p,q}},\\ \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz) &= \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha}\frac{[2]_{p,q}}{e_{p,q}(z)+1}\frac{e_{p,q}(z)+1}{[2]_{p,q}}e_{p,q}(xz)\end{aligned}$$

and is similar to that of Theorem 2.6. □

Theorem 2.8. *Each of the following relationships holds true:*

$$\mathcal{H}_n^{(\alpha)}(x;u:p,q)$$

$$= \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \left\{ \sum_{k=0}^{s} \binom{s}{k}_{p,q} \mathcal{G}_{s-k}(x:p,q)\, p^{\binom{k}{2}} + \mathcal{G}_s(x:p,q) \right\} \frac{\mathcal{H}_{n+1-s}^{(\alpha)}(u:p,q)}{[2]_{p,q}[n+1]_{p,q}}$$

$$= \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \left\{ \sum_{k=0}^{s} \binom{s}{k}_{p,q} \mathcal{G}_{s-k}(p,q)\, p^{\binom{k}{2}} + \mathcal{G}_s(p,q) \right\} \frac{\mathcal{H}_{n+1-s}^{(\alpha)}(x;u:p,q)}{[2]_{p,q}[n+1]_{p,q}}.$$

Proof. By making use of the following equalities

$$\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz) = \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz) \frac{[2]_{p,q}\, z}{e_{p,q}(z)+1} \frac{e_{p,q}(z)+1}{[2]_{p,q}\, z},$$

$$\left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz) = \left(\frac{1-u}{e_{p,q}(z)-u}\right)^{\alpha} \frac{[2]_{p,q}\, z}{e_{p,q}(z)+1} \frac{e_{p,q}(z)+1}{[2]_{p,q}\, z} e_{p,q}(xz),$$

the proof of this theorem is completed similar to that of Theorem 2.6. □

3. Conclusions

In this paper, we have introduced (p,q)-Frobenius-Euler polynomials and numbers of order α and investigated their some identities and properties involving in addition property, difference equation, derivative property, recurrence relationships. We have also given integral representation, explicit formulas and relations for these polynomials and numbers. Moreover, we have obtained some relationships for (p,q)-Frobenius-Euler polynomials of order α associated with (p,q)-Bernoulli polynomials, (p,q)-Euler polynomials and (p,q)-Genocchi polynomials. Thereafter, we have discovered (p,q)-extensions of Carlitz's result [7]. When $p = 1$, the results obtained here reduce to known properties of q-polynomials. Also, in the case $q \to p = 1$, all our results reduce to ordinary results for Frobenius-Euler polynomials and numbers.

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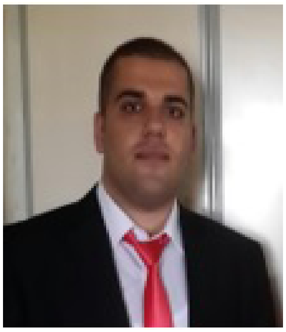

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