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EXISTENCE OF MILD SOLUTION FOR HYBRID DIFFERENTIAL EQUATIONS WITH ARBITRARY FRACTIONAL ORDER

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Abstract. We investigate in this article the existence problems of mild solutions for hybrid differential equations involving fractional Caputo derivative of arbitrary order. Different types of fixed point theorems are applied for solving the existence problem. An example is given to explain the applicability of all theorems.

Keywords: existence, uniqueness, hybrid fractional differential equations, fixed point theorems.

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1. Introduction

This paper deals with the existence and uniqueness of mild solutions for a class of hybrid differential equations of arbitrary fractional order of the form

$$\begin{cases} {}^{C}D_{t_0}^{q}\left(\frac{u(t)}{h(t,u(t))}\right) = f(t,u(t)),\ t \in J = [t_0, T], \\ \left.\left(\frac{u(t)}{h(t,u(t))}\right)^{(k)}\right|_{t=t_0} = b_k \in \mathbb{R},\ k = 0,1,2,...,n-1, \end{cases} \tag{1}$$

where ${}^{C}D_{t_0}^{q}$ denotes the Caputo fractional derivative of order $q \in (n-1,n), n = [q]+1$, $h : J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, and $f : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Fractional differential equations have recently played a significant role in the recent developments of special functions and integral transforms, biology, control theory, bioengineering and biomedical, economics, variational problems, etc. For further details, see [1, 2, 9, 13, 16] and references cited therein.

As a result of these investigations, the existence problem of solution for fractional differential equations of such models has gained an attention of many mathematical scientists. Therefore, many articles have been appeared in the literature on existence of solutions for initial, boundary and nonlocal fractional equations using different types of fractional derivatives (see [3-5, 8, 14, 15] and references therein). For some noteworthy papers are dealing with the integral operator and the arbitrary fractional order differential operator, see [3, 4, 14].

An interesting class of problems involves hybrid fractional differential equations appearing recently, we refer to [6, 7, 19, 20, 21] and the references cited therein for more details in this topic.

The basic tool for dealing with the nonlinear differential equations is using an appropriate fixed point theorem applied on an operator equation. Some cases, one needs to define more than one operator as in the case of Krasnoselskii's fixed point theorem for the sum of two

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operators. Recently, Dhage [10] established a new fixed point theorem involving the product of two operators.

Being motivated by the works in the literature, we consider a class of hybrid differential equations of arbitrary fractional order in the form of (1), and obtain sufficient conditions of the existence and uniqueness of their solutions in accordance with Dhage's, Banach's, and Schauder's fixed point theorems.

This paper is organized as follows. In Section 2, we recall some preliminaries about fractional calculus and the solution of problem (1). Section 3 deals with the existence and uniqueness of a solution for the problem (1). We close this article by an example to illustrate the applicability of the theorems.

2. Preliminaries

First of all, we fix our terminology and recall some basic ideas of fractional calculus following [12], and some preliminaries for the results in the sequel.

Definition 2.1. *The Riemann-Liouville fractional integral of order $q > 0$, for a continuous function h is defined as*

$$I_{t_0}^{q} h(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s) ds.$$

Definition 2.2. *The Caputo derivative of fractional order $q > 0$, for nth differentiable function h is defined as*

$$^{c}D_{t_0}^{q} h(t) = I_{t_0}^{n-q} h^{(n)}(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} h^{(n)}(s) ds, n-1 < q < n.$$

The next result is a recurrence relation indicating the nth derivative of a quotient function [17], [11].

Lemma 2.1. *The nth derivative of the quotient $\frac{u}{v}$ is given by*

$$\left(\frac{u}{v}\right)^{(k)} = \frac{1}{v}\left(u^{(k)} - k! \sum_{j=1}^{k} \frac{v^{(k+1-j)}}{(k+1-j)!} \frac{\left(\frac{u}{v}\right)^{(j-1)}}{(j-1)!}\right). \tag{2}$$

Let $C(J, \mathbb{R})$ be the Banach space of all continuous real valued functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$, and $C^n(J, \mathbb{R})$ be the Banach space of all n times continuously differentiable functions on J.

Definition 2.3. *A function $u \in C(J, \mathbb{R})$, is said to be a mild solution of (1) if it satisfies the equation $^{C}D_{t_0}^{q}\left(\frac{u(t)}{h(t,u(t))}\right) = f(t, u(t))$ on J, and the condition $\left(\frac{u(t)}{h(t,u(t))}\right)^{(k)}\Big|_{t=t_0} = b_k$, $k = 0, 1, 2, ..., n-1$.*

The integral form that is equivalent to problem (1) is given by the following.

Lemma 2.2. *Assume that $\frac{u(t)}{\widetilde{h}(t)} \in C^n(J, \mathbb{R})$, then the hybrid linear differential equation*

$$\begin{cases} ^{C}D_{t_0}^{q}\left(\frac{u(t)}{\widetilde{h}(t)}\right) = \widetilde{f}(t), \ t \in J, \\ \left(\frac{u(t)}{\widetilde{h}(t)}\right)^{(k)}\Big|_{t=t_0} = b_k, \ k = 0, 1, 2, ..., n-1, \end{cases} \tag{3}$$

is equivalent to

$$u(t) = \widetilde{h}(t)\sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k + \widetilde{h}(t)I_{t_0}^{q}\widetilde{f}(t),\ t\in J. \tag{4}$$

Proof. The nth differentiability of $\frac{u(t)}{\widetilde{h}(t)}$ implies the continuity of ${}^{C}D_{t_0}^{q}\left(\frac{u(t)}{\widetilde{h}(t)}\right)$, hence the continuity of the fractional integral $I_{t_0}^{q}\widetilde{f}(t)$ for any $t\in J$. Applying the fractional operator $I_{t_0}^{q}$ to the equation (3), and using the identity [12]

$$I_{t_0}^{q\,C}D_{t_0}^{q}x(t) = x(t) + \sum_{j=0}^{n-1}c_j\,(t-t_0)^j\,,$$

we have

$$\frac{u(t)}{\widetilde{h}(t)} + \sum_{j=0}^{n-1}c_j\,(t-t_0)^j = I_{t_0}^{q}\widetilde{f}(t),\ t\in J, \tag{5}$$

where the constant c_j can be evaluated using the given initial conditions. For this, Differentiating equation (5) k times and using the formula (2), we have

$$\frac{u^{(k)}(t)}{\widetilde{h}(t)} - \frac{k!}{\widetilde{h}(t)}\sum_{j=1}^{k}\frac{\widetilde{h}^{(k+1-j)}(t)}{(k+1-j)!}\frac{\left(\frac{u(t)}{\widetilde{h}(t)}\right)^{(j-1)}}{(j-1)!}$$
$$+\sum_{j=k}^{n-1}\frac{j!}{(j-k)!}c_j\,(t-t_0)^{j-k} = I_{t_0}^{q-k}\widetilde{f}(t).$$

In virtue of nth differentiability of $\frac{u(t)}{\widetilde{h}(t)}$, the functions $u^{(k)}(t)$, $\widetilde{g}^{(k)}(t)$, and $I_{t_0}^{q-k}\widetilde{f}(t)$ are continuous for any $t\in J$, and $k=0,1,2,...,n-1$. In accordance with initial conditions, substituting $t=t_0$, it follows that

$$\frac{u^{(k)}(t_0)}{\widetilde{h}(t_0)} - \frac{k!}{\widetilde{h}(t_0)}\sum_{j=1}^{k}\frac{\widetilde{h}^{(k+1-j)}(t_0)}{(k+1-j)!}\frac{\left.\left(\frac{u(t)}{\widetilde{h}(t)}\right)^{(j-1)}\right|_{t=t_0}}{(j-1)!} + k!c_k = 0,$$

which implies

$$c_k = -\frac{b_k}{k!}, k=0,1,2,...,n-1.$$

Substituting c_k in (5) leads to equation (4). On the other hand, applying the fractional derivative to equation (5), and noting that ${}^{C}D_{t_0}^{q}\left(\sum\limits_{j=0}^{n-1}c_j\,(t-t_0)^j\right) = 0$, we deduce equation (3). This finishes the proof. $\square$

3. Existence results

We obtain in this section the main results on existence theorems based on Dhage's, Schauder's and Banach's fixed point theorems. Consequently, we transform the integral form (4) into an operator equation by which we obtain the fixed point principle. Thus, this fixed point is the required solution of the problem (1).

The following fixed point theorem due to Dhage [10] is the essential tool for the proof of the first result.

Theorem 3.1. *Let Ω be a nonempty bounded closed convex subset of a Banach algebra X. Let $\Phi : \Omega \to X$ and $\Theta : X \to X$ be continuous operators satisfying:*
(a) Φ is completely continuous,
(b) Θ is Lipschitzian with a Lipschitz constants k_Θ,
(c) $x = \Phi y \Theta x$ implies $x \in \Omega$ for all $y \in \Omega$, and
(d)$Mk_\Theta < 1$, where $M = \sup\{\|\Phi x\| : x \in \Omega\}$.
Then the operator equation $x = \Phi x \Theta x$ has a solution in Ω.

The first result of existence problem is based on Theorem 3.1.

Theorem 3.2. *Assume that:*

(H1): *There exist positive continuous functions μ and ν with bounds $\|\mu\|$, and $\|\nu\|$ respectively, such that*

$$\begin{cases} |h(t,u) - h(t,v)| \le \mu(t) |u - v| , \\ |f(t,u)| \le \nu(t), \end{cases}$$

for $t \in J$, $u, v \in \mathbb{R}$.

Then the problem (1) has a solution on J, whenever

$$\omega = \left(\sum_{k=0}^{n-1} \frac{|b_k|}{k!} (T - t_0)^k + \frac{\|\nu\| (T - t_0)^q}{\Gamma(q+1)} \right) \|\mu\| < 1.$$

Proof. Define a subset Ω of $C(J, \mathbb{R})$ as $\Omega = \{u \in C(J, \mathbb{R}) : \|u\| \le r\}$, where

$$r \ge \frac{h_{\max}\omega}{\|\mu\| (1 - \omega)}.$$

Here $h_{\max} = \max_{t \in J} |h(t, 0)|$. Clearly Ω is closed, convex, and bounded subset of $C(J, \mathbb{R})$. By Lemma 2.2, the problem (1) has a solution given by

$$u(t) = h(t, u(t)) \left(\sum_{k=0}^{n-1} \frac{b_k}{k!} (t - t_0)^k + I_{t_0}^q f(t, u(t)) \right), \ t \in J. \tag{6}$$

Define the operators $\Phi : \Omega \to C(J, \mathbb{R})$ by

$$\Phi u(t) = \sum_{k=0}^{n-1} \frac{b_k}{k!} (t - t_0)^k + I_{t_0}^q f(t, u(t)), \ t \in J, u \in C(J, \mathbb{R}),$$

and $\Theta : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\Theta u(t) = h(t, u(t)), \ t \in J, \ u \in C(J, \mathbb{R}).$$

Then (6) is transformed into operator equation $u(t) = \Theta u(t) \Phi u(t)$, $t \in J$. We shall prove that the operators Φ, and Θ satisfy the conditions of Theorem 3.1. For the sake of clarity, we split the proof into a sequence of steps.

Step 1. We show that Φ is completely continuous. First we show that Φ is continuous on Ω. Let $\{u_m\}$ be a sequence converging to u in Ω. In virtue of dominated convergence theorem,

$$\begin{aligned}\lim_{m\to\infty}\Phi u_m(t) &= \lim_{m\to\infty}\left[\sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k + I_{t_0}^q f(t,u_m(t))\right]\\ &= \sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k + I_{t_0}^q\left(\lim_{m\to\infty} f(t,u_m(t))\right)\\ &= \sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k + I_{t_0}^q f(t,u(t)) = \Phi u(t),\end{aligned}$$

for all $t\in J$. Next we show that Φ is a compact operator on Ω. It is enough to show that $\Phi(\Omega)$ is uniformly bounded and equicontinuous in $C(J,\mathbb{R})$. In view of (H1), we have

$$\begin{aligned}|\Phi u(t)| &\le \sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k + I_{t_0}^q\,|f(t,u(t))|\\ &\le \sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k + \|\nu\|\frac{(t-t_0)^q}{\Gamma(q+1)},\end{aligned}$$

for all $t\in J$, $u\in\Omega$. Taking the supremum over J, we emphasize the uniform boundedness of Φ. For equicontinuity of Φ on Ω, let $t_1<t_2$ be both in J, then

$$\begin{aligned}|\Phi u(t_2)-\Phi u(t_1)| &\le \sum_{k=0}^{n-1}\frac{|b_k|}{k!}\left[(t_2-t_0)^k-(t_1-t_0)^k\right]\\ &\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)\nu(s)ds\\ &\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\,\nu(s)ds\\ &\le \sum_{k=0}^{n-1}\frac{|b_k|}{k!}\left[(t_2-t_0)^k-(t_1-t_0)^k\right]\\ &\quad+\frac{\|\nu\|}{\Gamma(q+1)}\left((t_2-t_0)^q-(t_1-t_0)^q\right).\end{aligned}$$

Obviously, the right hand side of the above inequality tends to zero independently of any $u\in\Omega$, as t_2 approaches to t_1, this shows the equicontinuity of Φ on Ω. Therefore, it follows from the Arzela Ascoli theorem that Φ is a completely continuous operator on Ω.

Step 2. We show that Θ is Lipschitzian with a Lipschitz constant $k_\Theta=\|\mu\|$. Indeed, for any $u,v\in C(J,\mathbb{R})$, and $t\in J$, we have

$$|\Theta u(t)-\Theta v(t)| = |h(t,u(t))-h(t,v(t)))| \le \mu(t)\,|u(t)-v(t)|\,,$$

which implies after taking the supremum over J,

$$\|\Theta u-\Theta v\| \le \|\mu\|\,\|u-v\|\,.$$

Step 3. We show that $u \in \Omega$, whenever $u = \Theta u \Phi v$ and $v \in \Omega$. Let $v \in \Omega$, then

$$\begin{aligned} |u(t)| &\leq |h(t,u(t)) - h(t,0) + h(t,0)| \left(\sum_{k=0}^{n-1} \frac{|b_k|}{k!} (t-t_0)^k + \left| I_{t_0}^q f(t,v(t)) \right| \right) \\ &\leq (\mu(t)|u(t)| + h(t,0)) \left(\sum_{k=0}^{n-1} \frac{|b_k|}{k!} (t-t_0)^k + \frac{\|\nu\| (t-t_0)^q}{\Gamma(q+1)} \right). \end{aligned}$$

Taking the supremum over J, we deduce that

$$\|u\| \leq \frac{h_{\max} \left(\sum\limits_{k=0}^{n-1} \frac{|b_k|}{k!} (T-t_0)^k + \frac{\|\nu\|(T-t_0)^q}{\Gamma(q+1)} \right)}{1 - \|\mu\| \left(\sum\limits_{k=0}^{n-1} \frac{|b_k|}{k!} (T-t_0)^k + \frac{\|\nu\|(T-t_0)^q}{\Gamma(q+1)} \right)} \leq r.$$

Hence $u \in \Omega$.

Step 4. Finally, we show that $Mk_\Theta < 1$, where $M = \sup\{\|\Phi u\| : u \in \Omega\}$. This is obvious, since

$$|\Phi u(t)| \, k_\Theta \leq \left(\sum_{k=0}^{n-1} \frac{|b_k|}{k!} (t-t_0)^k + \frac{\|\nu\| (t-t_0)^q}{\Gamma(q+1)} \right) \|\mu\| \leq \omega < 1, \ t \in J.$$

Thus all conditions of Theorem 3.1 are satisfied, hence the operator equation $u = \Phi u \Theta u$ has a solution in Ω. In consequence, the problem (1) has a solution on J. The proof is completed. □

The second result is based on the Banach's contraction principle.

Theorem 3.3. *Assume that:*

(H2): *For any $u, v \in \mathbb{R}$, and $t \in J$, there exist positive constants A_h, and A_f such that*

$$\begin{cases} |h(t,u) - h(t,v)| \leq A_h |u-v|, \\ |f(t,u) - f(t,v)| \leq A_f |u-v|. \end{cases}$$

(H3): *For any $(t,u) \in J \times \mathbb{R}$, there exist positive constants B_h, and B_f such that*

$$\begin{cases} |h(t,u))| \leq B_h, \\ |f(t,u)| \leq B_f. \end{cases}$$

Then there exists a unique solution for the problem (1), whenever

$$\gamma = A_h \sum_{k=0}^{n-1} \frac{|b_k|}{k!} (T-t_0)^k + \frac{A_h B_f (T-t_0)^q}{\Gamma(q+1)} + \frac{A_f B_h (T-t_0)^q}{\Gamma(q+1)} < 1.$$

.

Proof. Define the operator Ψ on $C(J,\mathbb{R})$ by

$$\Psi u(t) = h(t,u(t)) \sum_{k=0}^{n-1} \frac{b_k}{k!} (t-t_0)^k + h(t,u(t)) I_{t_0}^q f(t,u(t)). \tag{7}$$

Using the dominated convergence theorem, and the continuity of f and h imply the continuity of Ψ at any $x \in C(J,\mathbb{R})$. Hence $\Psi(C(J,\mathbb{R})) \subset C(J,\mathbb{R})$. Let $\mathbf{G} = \{u \in C(J,\mathbb{R}) : \|u\| \leq R\}$, where

$$R \geq \frac{h_{\max} \left(\sum\limits_{k=0}^{n-1} \frac{|b_k|}{k!} (T-t_0)^k + \frac{B_f (T-t_0)^q}{\Gamma(q+1)} \right)}{1 - A_h \left(\sum\limits_{k=0}^{n-1} \frac{|b_k|}{k!} (T-t_0)^k + \frac{B_f (T-t_0)^q}{\Gamma(q+1)} \right)}.$$

The fact that $A_h\left(\sum\limits_{k=0}^{n-1}\frac{|b_k|}{k!}(T-t_0)^k+\frac{B_f(T-t_0)^q}{\Gamma(q+1)}\right)<1$ follows from the assumption $\gamma<1$. Accordingly,

$$\begin{aligned}
|\Psi u(t)| &\leq (|h(t,u(t))-h(t,0)|+|h(t,0)|)\\
&\quad\times\left(\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k+I_{t_0}^q|(f(t,u(t))|\right)\\
&\leq (A_h|u(t)|+h_{\max})\left(\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k+\frac{B_f(t-t_0)^q}{\Gamma(q+1)}\right)\\
&\leq A_h|u(t)|\left(\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k+\frac{B_f(t-t_0)^q}{\Gamma(q+1)}\right)\\
&\quad+h_{\max}\left(\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k+\frac{B_f(t-t_0)^q}{\Gamma(q+1)}\right)\\
&\leq R.
\end{aligned}$$

This shows that Ψ maps $\mathbf{G}$ into itself. For the contraction principle, let $u,v\in C(J,\mathbb{R})$, and $t\in J$, we have

$$\begin{aligned}
|\Psi u(t)-\Psi v(t)| &\leq |h(t,u(t))-h(t,v(t))|\sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k\\
&\quad+\left|h(t,u(t))I_{t_0}^qf(t,u(t))-h(t,v(t))I_{t_0}^qf(t,v(t))\right|\\
&\leq |h(t,u(t))-h(t,u(t))|\sum_{k=0}^{n-1}\frac{b_k}{k!}(t-t_0)^k\\
&\quad+|h(t,u(t))-h(t,v(t))|\left|I_{t_0}^qf(t,u(t))\right|\\
&\quad+|h(t,v(t))|\left|I_{t_0}^qf(t,u(t))-I_{t_0}^qf(t,v(t))\right|\\
&\leq A_h|u(t)-v(t)|\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(t-t_0)^k+A_h|u(t)-v(t)|\frac{B_f(t-t_0)^q}{\Gamma(q+1)}\\
&\quad+A_f\|u-v\|\frac{B_h(t-t_0)^q}{\Gamma(q+1)}.
\end{aligned}$$

After taking the supremum over J, it follows that

$$\begin{aligned}
&\|\Psi u-\Psi v\|\\
&\leq \left(A_h\sum_{k=0}^{n-1}\frac{|b_k|}{k!}(T-t_0)^k+\frac{A_hB_f(T-t_0)^q}{\Gamma(q+1)}+\frac{A_fB_h(T-t_0)^q}{\Gamma(q+1)}\right)\|u-v\|\\
&\leq \gamma\|u-v\|.
\end{aligned}$$

As $\gamma<1$, we emphasize that Ψ satisfies the contraction principle, hence by Banach's fixed point Theorem, there exists a unique solution for the problem (1). This finishes the proof. □

Next result shows the completely continuity of the operator Ψ.

Lemma 3.1. *The operator Ψ defined by (7) is completely continuous.*

Proof. Let $\mathcal{U}$ be a bounded proper subset of $C(J,\mathbb{R})$, then the continuity of h and f imply that for any $t\in J$, and $u\in\mathcal{U}$, there exist positive constants L_h, and L_f such that $|h(t,u(t))|\leq L_f$

and $|f(t,u(t))| \le L_f$. The rest of the proof is similar to that in the step 1 of the proof of Theorem 3.2, hence we omit it. □

The last result is based on the Schauder's fixed point theorem ([18]).

Theorem 3.4. *If $\mathcal{U}$ is a closed , bounded, convex subset of a Banach space $\mathcal{X}$ and the mapping $\Delta : \mathcal{U} \to \mathcal{U}$ is completely continuous, then Δ has a fixed point in $\mathcal{U}$.*

Accordingly, if we define a closed , bounded, convex subset $\mathcal{U}$ of $C(J,\mathbb{R})$ on which Ψ, as defined by (7), is completely continuous, then the problem (1) has a solution.

Theorem 3.5. *Assume that:*

(H4): *There exist functions $\varphi_h, \varphi_f \in C(J,\mathbb{R}^+)$ and continuous nondecreasing functions $\rho_h, \rho_f : [0,\infty) \to (0,\infty)$ such that*

$$\begin{cases} |h(t,u)| \le \varphi_h(t)\rho_h(|u|) \\ |f(t,u)| \le \varphi_f(t)\rho_f(|u|) \end{cases}, \ (t,u) \in J \times \mathbb{R}.$$

Then the problem (1) has a solution.

Proof. Define a subset $\mathcal{U} \subset C(J,\mathbb{R})$ as $\mathcal{U} = \{u \in C(J,\mathbb{R}) : |u(t)| \le \beta, t \in J\}$. Hence, $\mathcal{U}$ is a closed, bounded, and convex subset of $C(J,\mathbb{R})$. If $u \in \mathcal{U}$, then

$$\begin{cases} |h(t,u(t))| \le \|\varphi_h\| \, \rho_h(\beta), \\ |f(t,u(t))| \le \|\varphi_f\| \, \rho_f(\beta), \end{cases} \tag{8}$$

for any $t \in J$. The estimates in (8) make the applicability of Lemma 3 on Ψ is valid. Therefore, Theorem 3.4 ensures that the problem (1) has a solution. This finishes the proof. □

We close this article by introducing the following example.

Example 3.1. *Consider the following fractional hybrid differential equation*

$$\begin{cases} {}^C D_0^{3.2} \left(\frac{3u(t)}{t \sin u(t)} \right) = \frac{t|u(t)|}{4(1+|u(t)|)}, t \in (0,1], \\ \left(\frac{3u(t)}{t \sin u(t)} \right)^{(k)} \Big|_{t=0} = 1, k = 0,1,2,3. \end{cases} \tag{9}$$

Here $q = 3.2$, $f(t,u) = \frac{t|u|}{4(1+|u|)}$, and $h(t,u) = \frac{1}{3} t \sin u$. In view of Theorem 3.2, we have $\mu(t) = \frac{t}{3}$, and $\nu(t) = \frac{t}{4}$ such that $\|\mu\| = \frac{1}{3}$ and $\|\nu\| = \frac{1}{4}$. Moreover,

$$\omega = \frac{1}{3} \left(\sum_{k=0}^{3} \frac{1}{k!} + \frac{1}{4\Gamma(4.2)} \right) \simeq 0.8996 < 1.$$

Then the problem (9) has a solution in $C([0,1],\mathbb{R})$. Next we apply Theorem 3.3. The functions h and f satisfy Lipschitz condition with constants $A_f = \frac{1}{4}$, and $A_h = \frac{1}{3}$. Moreover, the functions h and f are bounded with $B_h = \frac{1}{3}$, and $B_f = \frac{1}{4}$, therefore

$$\gamma = \frac{1}{3} \sum_{k=0}^{3} \frac{1}{k!} + \frac{2}{12\Gamma(4.2)} \simeq 0.91 < 1.$$

Hence, the uniqueness of this solution is obtained. Finally, the result of Theorem 3.5 follows, since $\varphi_h(t) = \frac{t}{3}$, $\varphi_f(t) = \frac{t}{4}$, and $\rho_h(|u|) = \rho_f(|u|) = 1$.

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