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SOLVING LINEAR FUZZY FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND VIA ITERATIVE METHOD AND SIMPSON QUADRATURE RULE: A REVIEW

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Abstract. In this paper, first, we survey some methods to solve fuzzy Fredholm integral equation such as iterative trapezoidal quadrature rule, hybrid of block-pulse functions and Taylor series, iterative fuzzy wavelet like operator method. Then, we introduce a new method of successive approximations based on the Simpson quadrature rule for solving linear fuzzy Fredholm integral equations of the second kind (LFFIE-2). Moreover, we present the convergence analysis. Also, we present numerical stability analysis of the proposed method. Furthermore, we give a stopping criterion. Finally, to illustrate the applicability of the proposed method, we present some numerical tests.

Keywords: integral equation, iterative method, Simpson quadrature rule, convergence analysis, numerical stability analysis.

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1. Introduction

For the first time in 1982, Dubios and Prade [12] introduced a concept of fuzzy integration. Also, Goetschel and Voxman [19], Kaleva [24] and others presented alternative approaches. Since many fuzzy-valued problems in engineering can be brought in the form of fuzzy differential and integral equations, it is important that we discuss them. For this reason in recent years, numerical solution of fuzzy integral and differential equations have been studied by many authors [3, 6-8, 10, 11, 13-21, 23, 24, 26, 28-39]. In [36], Wu and Ma investigated application of fuzzy integration for solving fuzzy Fredholm integral equation of the second kind. Bede and Gal in [9] introduced quadrature rules for integrals of fuzzy-number-valued function. Babolian *et al.* [6] presented a numerical solution of LFFIE-2 by Adomian method. Parandin and Fariborzi Araghi in 2009, proposed approximate solution of LFFIE-2 by using iterative interpolation [34]. In [32], Homotopy Analysis Method (HAM) is used for solving LFFIE. Ziari *et al.* [39] used Haar wavelet to solve fuzzy linear integral equation. Also, in [16] Ezzati and Ziari presented numerical solution of nonlinear FFIE using iterative method. After this, Ziari and Bica in [37] presented a new error estimate in the iterative numerical method for nonlinear fuzzy Hammerstein-Fredholm integral equations. In 2014, Hosseini Fadravi *et al.* [21] solved LFFIEs-2 by artificial neural networks. Recently, solving FFIE using sinc method and double exponential transformation was done by Fariborzi *et al.* [17].

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Here, first, we propose an iterative method via the Simpson quadrature formula for solving the following LFFIE

$$F(t) = f(t) \oplus (FR) \int_a^b K(s,t) \odot F(s)ds, \tag{1}$$

where $K(s,t)$ is a positive crisp kernel function over the square $s, t \in [a, b]$, $f(t)$ is a fuzzy valued function. Then, we prove the convergence analysis of the proposed method. Also, we present numerical stability analysis for the presented method based on the choice of the first iteration.

The paper is organized as follows: In Section 2, we give basic information about the fuzzy set theory. In Section 3, we state some of the proposed methods by authors of [7, 10, 16, 30] for solving FFIE. In Section 4, we present the proposed method to obtain numerical solution of LFFIE (1) based on iterative method and Simpson quadrature rule. In Section 5 and Section 6, the convergence analysis, stopping criterion and the numerical stability analysis are proved, respectively. In Section 7, we apply the proposed method for some examples and show the efficiency of the proposed method by comparing the numerical results with the exact solutions and the method of [16].

2. Preliminaries

In this section, we review the most basic notations used in the fuzzy calculus.

Definition 2.1 ([1, 5]). *Let $u : \mathbb{R} \to [0, 1]$ with the following properties:*

(1) *u is normal, i.e. $\exists\, x_0 \in \mathbb{R}$; $u(x_0) = 1$.*

(2) *$u(\eta x + (1-\eta)y) \geq \min\{u(x), u(y)\}$, $\forall\, x, y \in \mathbb{R}, \forall\, \eta \in [0, 1]$ (u is called a convex fuzzy subset).*

(3) *u is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \epsilon > 0$, $\exists$ neighborhood $V(x_0)$: $u(x) \leq u(x_0) + \epsilon$, $\forall\, x \in V(x_0)$.*

(4) *The set $\overline{supp(u)}$ is compact in $\mathbb{R}$ where $supp(u) := \{x \in \mathbb{R}; u(x) > 0\}$.*

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$.

Definition 2.2 ([2, 5]). *For $0 < r \leq 1$ and $u \in \mathbb{R}_F$ define $[u]^r := \{x \in \mathbb{R} : \ u(x) \geq r\}$ and*

$$[u]^0 := \overline{\{x \in \mathbb{R} : \ u(x) > 0\}}.$$

Then it is well known that for each $r \in [0, 1]$, $[u]^r$ is a closed and bounded interval of $\mathbb{R}$. For $\tilde{u}$, $\tilde{v} \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $\tilde{u} \oplus \tilde{v}$ and the product $\lambda \odot \tilde{u}$ by

$$[\tilde{u} \oplus \tilde{v}]^r = [\tilde{u}]^r + [\tilde{v}]^r, \quad [\lambda \odot \tilde{u}]^r = \lambda[\tilde{u}]^r, \qquad \forall\ r \in [0, 1],$$

where $[\tilde{u}]^r + [\tilde{v}]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[\tilde{u}]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$. Notice $1 \odot \tilde{u} = \tilde{u}$ and it holds $\tilde{u} \oplus \tilde{v} = \tilde{v} \oplus \tilde{u}$, $\lambda \odot \tilde{u} = \tilde{u} \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[\tilde{u}]^{r_2} \subseteq [\tilde{u}]^{r_1}$. Actually $[\tilde{u}]^r = [\tilde{u}_-^{(r)}, \tilde{u}_+^{(r)}]$, where $\tilde{u}_-^{(r)} \leq \tilde{u}_+^{(r)}$, $\tilde{u}_-^{(r)}$, $\tilde{u}_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0, 1]$. For $\lambda > 0$ one has $\lambda \tilde{u}_\pm^{(r)} = (\lambda \odot \tilde{u})_\pm^{(r)}$, respectively.

Definition 2.3 ([5]). *Define $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+$ by*

$$\begin{aligned} D(\tilde{u}, \tilde{v}) &:= \sup_{r \in [0,1]} \max \left\{ \left| \tilde{u}_-^{(r)} - \tilde{v}_-^{(r)} \right|, \left| \tilde{u}_+^{(r)} - \tilde{v}_+^{(r)} \right| \right\} \\ &= \sup_{r \in [0,1]} \textit{Hausdorff distance}\, ([\tilde{u}]^r, [\tilde{v}]^r), \end{aligned}$$

where $[\tilde{v}]^r = [\tilde{v}_-^{(r)}, \tilde{v}_+^{(r)}]$; $\tilde{u}$, $\tilde{v} \in \mathbb{R}_F$. *We have that* D *is a metric on* $\mathbb{R}_F$. *Then* $(\mathbb{R}_F, D)$ *is a complete metric space, with the properties*

$$\begin{aligned} D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) &= D(\tilde{u}, \tilde{v}), \ \ \forall\, \tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{R}_F, \\ D(k' \odot \tilde{u}, k' \odot \tilde{v}) &= \left|k'\right| D(\tilde{u}, \tilde{v}), \ \ \forall\, \tilde{u}, \tilde{v} \in \mathbb{R}_F, \forall\, k' \in \mathbb{R}, \\ D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) &\leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \ \ \forall\, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in \mathbb{R}_F. \end{aligned}$$

Definition 2.4 ([5]). *Let* $f, g : \mathbb{R} \to \mathbb{R}_F$ *be fuzzy number valued functions. The distance between* f, g *is defined by*

$$D^*(f, g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

Lemma 2.1 ([5],[25]). (1) *If we denote* $\tilde{0} := \chi_{\{0\}}$, *then* $\tilde{0} \in R_F$ *is the neutral element with respect to* $\oplus$, *i.e.,* $\tilde{u} \oplus \tilde{0} = \tilde{0} \oplus \tilde{u} = \tilde{u}$, $\forall\, \tilde{u} \in \mathbb{R}_F$.

(2) *With respect to* $\tilde{0}$, *none of* $\tilde{u} \in \mathbb{R}_F$, $\tilde{u} \neq \tilde{0}$ *has opposite in* $\mathbb{R}_F$.

(3) *Let* α, $\beta \in \mathbb{R} : \alpha.\beta \geq 0$, *and any* $\tilde{u} \in R_F$, *we have* $(\alpha + \beta) \odot \tilde{u} = \alpha \odot \tilde{u} \oplus \beta \odot \tilde{u}$. *For general* α, $\beta \in \mathbb{R}$, *the above property is false.*

(4) *For any* $\gamma \in \mathbb{R}$ *and any* $\tilde{u}$, $\tilde{v} \in \mathbb{R}_F$, *we have* $\gamma \odot (\tilde{u} \oplus \tilde{v}) = \gamma \odot \tilde{u} \oplus \gamma \odot \tilde{v}$.

(5) *For any* γ, $\eta \in \mathbb{R}$ *and any* $\tilde{u} \in \mathbb{R}_F$, *we have* $\gamma \odot (\eta \odot \tilde{u}) = (\gamma \odot \eta) \odot \tilde{u}$.

If we denote $\|\tilde{u}\|_F := D(\tilde{u}, \tilde{0})$, $\forall\, \tilde{u} \in \mathbb{R}_F$, *then* $\|.\|_F$ *has the properties of a usual norm on* $\mathbb{R}_F$, *i.e.,*

$$\begin{gathered} \|\tilde{u}\|_F = 0 \textit{ iff } \tilde{u} = \tilde{0}, \|\lambda \odot \tilde{u}\|_F = |\lambda| \,.\, \|\tilde{u}\|_F\,, \\ \|\tilde{u} \oplus \tilde{v}\|_F \leq \|\tilde{u}\|_F + \|\tilde{v}\|_F\,,\ \|\tilde{u}\|_F - \|\tilde{v}\|_F \leq D(\tilde{u}, \tilde{v}). \end{gathered}$$

Notice that $(\mathbb{R}_F, \oplus, \odot)$ *is not a linear space over* $\mathbb{R}$, *and consequently* $(\mathbb{R}_F, \|.\|_F)$ *is not a normed space. Here* $\sum^*$ *denoted the fuzzy summation.*

Definition 2.5 ([5]). *A fuzzy valued function* $f : [a, b] \to \mathbb{R}_F$ *is said to be continuous at* $x_0 \in [a, b]$, *if for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $D(f(x), f(x_0)) < \epsilon$, *whenever* $x \in [a, b]$ *and* $|x - x_0| < \delta$. *We say that* f *is fuzzy continuous on* $[a, b]$ *if* f *is continuous at each* $x_0 \in [a, b]$, *and denote the space of all such functions by* $C_F[a, b]$.

Definition 2.6 ([9]). *Let* $f : [a, b] \to \mathbb{R}_F$ *be a bounded mapping. Then the function* $\omega_{[a,b]}(f, .) : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$

$$\omega_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y)); x, y \in [a, b], |x - y| \leq \delta\},$$

is called the modulus of oscillation of f *on* $[a, b]$.

If $f \in C_F[a, b]$ *(i.e.* $f : [a, b] \to \mathbb{R}_F$ *is continuous on* $[a, b]$*), then* $\omega_{[a,b]}(f, \delta)$ *is called uniform modulus of continuity of* f.

The following properties will be very useful in what follows.

Theorem 2.1 ([9]). *The following statements, concerning the modulus of oscillation, are true:*

(1) $D(f(x), f(y)) \leq \omega_{[a,b]}(f, |x - y|)$, $\forall\, x, y \in [a, b]$,

(2) $\omega_{[a,b]}(f, \delta)$ *is a nondecreasing mapping in* δ,

(3) $\omega_{[a,b]}(f, 0) = 0$,

(4) $\omega_{[a,b]}(f, \delta_1 + \delta_2) \leq \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$, $\forall\, \delta_1$, $\delta_2 \geq 0$,

(5) $\omega_{[a,b]}(f, n\delta) \leq n\omega_{[a,b]}(f, \delta)$, $\forall\, \delta \geq 0$, $n \in \mathbb{N}$,

(6) $\omega_{[a,b]}(f,\eta\delta) \leq (\eta+1)\omega_{[a,b]}(f,\delta)$, $\forall\, \delta,\, \eta \geq 0$.

Definition 2.7 ([5]). *Let $f : [a,b] \to \mathbb{R}_F$. We say that f is fuzzy-Riemann integrable to $I \in \mathbb{R}_F$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(p) < \delta$, we have*

$$D\left(\sum_{p}^{*}(v-u)\odot f(\xi), I\right) < \epsilon,$$

where $\sum^{*}$ denotes the fuzzy summation. We choose to write

$$I := (FR)\int_a^b f(x)dx.$$

We also call an f as above (FR)-integrable.

Lemma 2.2 ([5]). *If $f, g : [a,b] \subseteq R \to \mathbb{R}_F$ are fuzzy continuous functions, then the function $F : [a,b] \to \mathbb{R}_+$ defined by $F(x) := D(f(x), g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx.$$

Definition 2.8 ([9]). *A function $f : [a,b] \to \mathbb{R}_F$ is said to be Lipschitz if*

$$D(f(x), f(y)) \leq L\,|x-y|\,, \tag{2}$$

for any $x, y \in [a,b]$.

Theorem 2.2 ([9]). *Let $f : [a,b] \to \mathbb{R}_F$ be a Henstock integrable, bounded mapping. Then, for any division $a = x_0 < x_1 < \cdots < x_n = b$ and any points $\xi_i \in [x_{i-1}, x_i]$ we have*

$$D\left((FH)\int_a^b f(t)dt, \sum_{i=1}^{n}(x_i - x_{i-1})\odot f(\xi_i)\right) \leq \sum_{i=1}^{n}(x_i - x_{i-1})\omega_{[x_{i-1},x_i]}(f, x_i - x_{i-1}).$$

By the above theorem, the following results hold:

Corollary 2.1 ([9]). *Let $f : [a,b] \to \mathbb{R}_F$ be a Henstock integrable, bounded mapping. Then*

(1) $D\left((FH)\displaystyle\int_a^b f(t)dt, \frac{b-a}{2}\odot\Big(f(a)\oplus f(b)\Big)\right) \leq \frac{b-a}{2}\omega_{[a,b]}\left(f, \frac{b-a}{2}\right)$,

(2)

$$D\left((FH)\int_a^b f(t)dt, \frac{b-a}{6}\odot\left(f(a)\oplus 4\odot f\left(\frac{a+b}{2}\right)\oplus f(b)\right)\right) \leq 3(b-a)\omega_{[a,b]}\left(f, \frac{b-a}{6}\right).$$

Theorem 2.3 ([4]). *Let $f \in C_F^{n,1}[a,b]$, $n \geq 1$, $[\alpha,\beta] \subseteq [a,b] \subseteq \mathbb{R}$. Then*

$$f(\beta) = f(\alpha) \oplus (\beta - \alpha) \odot f'(\alpha) \oplus \cdots \oplus \frac{(\beta-\alpha)^{n-1}}{(n-1)!} \odot f^{(n-1)}(\alpha)$$
$$\oplus \frac{1}{(n-1)!} \odot (FR) \int_{\alpha}^{\beta} (\beta - t)^{n-1} \odot f^{(n)}(t)dt. \tag{3}$$

The integral remainder is a fuzzy continuous function in β.

3. A review of solving FFIE using iterative method

In past, for solving FFIE several methods by many authors were presented. In this section, we briefly review some presented methods for solving FFIE.

3.1. **Numerical solution of NFFIEs by successive approximations and trapezoidal quadrature rule.** Consider the nonlinear fuzzy Fredholm integral equation (NFFIEs)

$$F(t) = f(t) \oplus (FR) \int_a^b K(t,s,F(s))ds, \; t \in [a,b], \tag{4}$$

such that the functions $f : [a,b] \to \mathbb{R}_F$ and $K : [a,b] \times [a,b] \times \mathbb{R}_F \to \mathbb{R}_F$ are continuous. Also, suppose that K is uniformly continuous with respect to t. Assume that there exists $M > 0$ such that

$$\|K(s,t,u)\|_F \leq M, \; \forall t,s \in [a,b], \forall u \in \mathbb{R}_F.$$

In [10], author generalized the quadrature formula to approximate

$$(FR) \int_a^b f(t)dt \;\; by \;\; \sum_{i=0}^{n-1} (FR) \int_{t_i}^{t_{i+1}} f(t)dt$$

for partition

$$\Delta : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

and presented the following result

$$D\Bigg((FR) \int_a^b f(t)dt, \sum_{i=0}^{n-1} \frac{(t_{i+1} - t_i)}{2} \odot [f(t_i) \oplus f(t_{i+1})] \Bigg) \leq \frac{L(b-a)^2}{4n}. \tag{5}$$

In the following theorem, the existence and uniqueness of the solution (4) was showed by using the Banach fixed point principle.
Consider the space of functions

$$X = \{f : [a,b] \to \mathbb{R}_F \mid f \text{ is continuous}\},$$

with the metric $D^*(f,g) = \sup\limits_{a \leq t \leq b} D(f(t), g(t))$. Recall the fact that (X, D^*) is complete metric space (see [24]).

Theorem 3.1 ([10]). *Assume that the functions f and K are continuous. Moreover, K is uniformly continuous with respect to t and there exist $L > 0$, $M > 0$ such that*

$$\|K(s,t,u)\|_F \leq M, \ \forall t, s \in [a,b], \forall u \in \mathbb{R}_F,$$

and

$$D(K(t,s,u), K(t,s,v)) \leq L.D(u,v), \ \forall t, s \in [a,b], \forall u, v \in \mathbb{R}_F.$$

If $L(b-a) < 1$ then Eq. (4) has a unique solution $F^* \in X$, which can be obtained by means of the method of successive approximations starting by any element of X. Furthermore, in the approximation of the solution by the terms of the sequence of successive approximations, $(F_m)_{m\in\mathbb{N}}$,

$$F_0(t) = f(t),$$

$$F_{m+1}(t) = f(t) \oplus (FR)\int_a^b K(t,s,F_m(s))ds, \ \forall t \in [a,b], \forall m \in \mathbb{N}^*,$$

the error estimate is

$$D(F^*(t), F_m(t)) \leq \frac{[L(b-a)]^m}{1-L(b-a)}.M(b-a), \ t \in [a,b], \forall m \in \mathbb{N}^*. \quad \square \tag{6}$$

Also, author of [10] presented the successive approximations and iterative quadrature rule on the knots of the partition Δ for solving FFIE (4) as follows:

$$u_0(t_i) = f(t_i), \quad \forall i = 0, \cdots, n$$

$$u_m(t_i) = f(t_i) \oplus \sum_{j=0}^{n-1} \frac{(b-a)}{2n} \odot [K(t_i, t_j, u_{m-1}(t_j)) \oplus K(t_i, t_{j+1}, u_{m-1}(t_{j+1}))]. \tag{7}$$

Finally, in the following theorem, author of [10] proved the convergence analysis for the proposed method.

Theorem 3.2 ([10]). *Assume that*

(i) The functions f and K are continuous;

(ii) There is $M > 0$ such that $\|K(t,s,u)\|_F \leq M$, $\forall t, s \in [a,b]$, $\forall u \in \mathbb{R}_F$;

(iii) There is $\delta > 0$ such that $D(f(s'), f(s'')) \leq \delta\,|s' - s''|$, $\forall s', s'' \in [a,b]$;

(iv) There is $\beta > 0$, $\gamma > 0$, $L > 0$ such that

$$D(K(t',s,u), K(t'',s,u)) \leq \beta\left|t' - t''\right|, \ \forall t', t'', s \in [a,b], \forall u \in \mathbb{R}_F,$$

and

$$D(K(t,s',u), K(t,s'',v)) \leq \gamma\left|s' - s''\right| + L.D(u,v), \ \forall t, s', s'' \in [a,b], \forall u, v \in \mathbb{R}_F; \tag{8}$$

(v) $L(b-a) < 1$.

Then the unique solution F^* of Eq. (4) is approximated on the knots t_i, $i = 0, \cdots, n$ of the partition Δ by the sequence $(u_m(t_i))_{m\in\mathbb{N}}$, given in Eq. (7), and the error estimate is

$$D(F^*(t_i), u_m(t_i)) \leq \frac{[L(b-a)]^m}{1-L(b-a)}.M(b-a) + \frac{L'(b-a)^2}{4n[1-L(b-a)]}, \tag{9}$$
$$\forall m \in \mathbb{N}^*, \forall i = 0, \cdots, n,$$

where

$$L' = \max\{\gamma + L\delta, \gamma + L[\delta + \beta(b-a)]\}. \qquad \square$$

3.2. **Numerical solution of Hammerstein NFFIE by successive approximations and trapezoidal quadrature rule.** Consider the NFFIE

$$F(t) = f(t) \oplus (FR)\int_a^b K(s,t) \odot G(F(s))ds,\ t \in [a,b], \tag{10}$$

where $K(s,t)$ is a positive crisp kernel function over the square $a \leq s,t \leq b$, $F(t)$ is a fuzzy valued function and $G : \mathbb{R}_F \to \mathbb{R}_F$ is continuous.

In [16], authors introduced a numerical method based on the iterative method and quadrature rule to solve Eq. (10) as follows:

$$\begin{aligned} u_0(t) &= f(t), \\ u_m(t) &= f(t) \\ &\oplus \sum_{j=0}^{n-1} \frac{h}{2} \odot [K(t_j,t) \odot G(u_{m-1}(t_j)) \oplus K(t_{j+1},t) \odot G(u_{m-1}(t_{j+1}))], m \geq 1, \end{aligned} \tag{11}$$

where $t_i = a + ih$, $i = 0, 1, \cdots, n$, and $h = \frac{b-a}{n}$. In the following theorem, authors o [16] proved the existence and uniqueness solution of Eq. (10) by using the Banach fixed point theorem.

Theorem 3.3 ([16]). *Let the function $K(s,t)$ be continuous and positive for $a \leq s,t \leq b$, and function $f(t)$ be a fuzzy continuous in $[a,b]$. Moreover, suppose that there exists $L > 0$, with*

$$D(G(F_1(u)), G(F_2(v))) \leq L.D(F_1(u), F_2(v)),\ \forall u, v \in [a,b].$$

If $C = ML(b-a) < 1$ then the fuzzy integral Eq. (10) has a unique solution $F^* \in X$, and it can be obtained by the following successive approximations method:

$$F_0(t) = f(t),$$
$$F_m(t) = f(t) \oplus (FR)\int_a^b K(s,t) \odot G(F_{m-1}(s))ds,\ \forall t \in [a,b], m \geq 1. \tag{12}$$

Moreover, the sequence of successive approximations, $(F_m)_{m\geq 1}$, converges to the solution F^*. Furthermore, the following error bound holds:

$$D(F^*(t), F_m(t)) \leq \frac{C^{m+1}}{L(1-C)} M_0,\ \forall t \in [a,b], m \geq 1, \tag{13}$$

where $M_0 = \sup_{a\leq t\leq b} \|G(f(t))\|_F$. $\square$

Now, in the following theorem we review the error estimation for the proposed method in [16].

Theorem 3.4 ([16])**.** *Consider the nonlinear fuzzy Fredholm Eq.* (10) *with continuous kernel* $K(s,t)$ *having positive sign on* $[a,b]\times[a,b]$, G *continuous on* $\mathbb{R}_F$, f *continuous on* $[a,b]$. *Besides, we have* $L>0$ *such that*

$$D(G(F_1(u)),G(F_2(v)))\leq L.D(F_1(u),F_2(v)),\ \forall u,v\in[a,b].$$

If $C=ML(b-a)<1$ *where* $M=\max_{s,t\in[a,b]}|K(s,t)|$, *then the iterative procedure* (11) *converges to the unique solution of Eq.* (10), F^*, *and its error estimate is as follows:*

$$\begin{aligned} D^*(F^*,u_m)\leq{}& \frac{C}{2(1-C)}\omega_{[a,b]}(f,h)+\frac{C^{m+1}L_1}{L(1-C)}\\ &+\frac{C^2+2C}{2LM(1-C)}(L_1\omega_s(k,h)+L_2\omega_t(k,h)). \quad \square \end{aligned} \tag{14}$$

3.3. **Numerical solution of NFFIEs using hybrid of block-pulse functions and Taylor series.** Firstly, we review some definitions of the hybrid block-pulse functions and Taylor series and we generalize them to the fuzzy setting which were written by authors of [7].

Definition 3.1 ([22])**.** *Block-pulse functions* $\phi_i(t)$, $i=1,\cdots,N$, *on the interval* $[0,1)$, *are defined as*

$$\phi_i(t)=\begin{cases}1, & \frac{i-1}{N}\leq t<\frac{i}{N},\\ 0, & otherwise,\end{cases}$$

where N *is an arbitrary positive integer.*

The block-pulse functions on $[0,1)$ *are disjoint, so for* $i,j=1,2,\cdots,N$, *we have* $\phi_i(t)\phi_j(t)=\delta_{ij}\phi_i(t)$, *where* δ_{ij} *is the Kronecker delta, also these functions have the property of orthogonality on* $[0,1)$.

Consider the set of Taylor polynomials $T_m(t)=t^m$, $m=0,1,2,\cdots$. *For* M *being an arbitrary positive integer, hybrid Taylor block-pulse functions are defined as follows.*

Definition 3.2 ([27, 28])**.** *The set of hybrid Taylor block-pulse functions* h_{ij}, $i=1,\cdots,N$; $j=0,\cdots,M$, *on the interval* $[0,1)$ *are defined as*

$$h_{ij}(t)=\begin{cases} T_j(Nt-(i-1)), & \frac{i-1}{N}\leq t<\frac{i}{N},\\ \\ 0, & otherwise\end{cases} \tag{15}$$

where i *and* j *are the order of block-pulse functions and Taylor polynomials, respectively.*

In [7], first, authors defined function approximation by using fuzzy hybrid of block-pulse functions and Taylor series and then, in Theorems 3.5 and 3.6, found the error estimation for the proposed method.

3.3.1. *Function approximation.* For $f\in C_F^{l-1}[0,1]$, let us consider a fuzzy hybrid polynomial of degree $l-1$,

$$T_M^F(f)(t)=\sum_{i=1}^{N}\sum_{j=0}^{l-1}f_{ij}\odot h_{ij}(t)=F^T\odot H(t), \tag{16}$$

where $h_{ij}(t)$ is defined in Eq. (15) and f_{ij} are given by

$$f_{ij} = \frac{1}{N^j j!} f^{(j)}(t)|_{t=\frac{i-1}{N}} \tag{17}$$

and also

$$F = [f_{10}, \cdots, f_{1(l-1)}, f_{20}, \cdots, f_{2(l-1)}, \cdots, f_{N0}, \cdots, f_{N(l-1)}]^T,$$
$$H(t) = [h_{10}(t), \cdots, h_{1(l-1)}(t), h_{20}(t), \cdots, h_{2(l-1)}(t), \cdots, h_{N0}(t), \cdots, h_{N(l-1)}(t)]^T.$$

Theorem 3.5 ([7])**.** *If $f \in C_F^{l,1}[0,1]$ and we have $K > 0$, where $K = \sup_{0\leq t\leq 1} \left\| f^{(l)}(t) \right\|_F$, then*

$$D^*(T_l^F(f), f) \leq \frac{K}{l! N^l}. \quad \square \tag{18}$$

This shows that

$$\lim_{l,N\to\infty} D^*(T_l^F(f), f) = 0. \quad \square$$

Consider NFFIE (10), where $K(s,t)$ is a positive crisp kernel function over the square $(s,t) \in [a,b]$, $F(t)$ is a fuzzy-valued function and $G : \mathbb{R}_F \to \mathbb{R}_F$ is continuous. Authors of [7] suppose that K is continuous and therefore it is uniformly continuous with respect to t and there exists $M > 0$, such that $M = \max_{a\leq s,t\leq b} |K(s,t)|$. To approximate the solution of this equation, authors of [7] introduced a new approach as follows:

$$\begin{aligned} u_0(t) &= f(t), \\ u_m(t) &= f(t) \oplus \sum_{i=1}^{N} \sum_{j=0}^{l-1} H_{ij}(t) \odot g_{ij}^{(m-1)}, \ \forall t \in [0,1], m \geq 1, \end{aligned} \tag{19}$$

where $t_i = ih$, $h = \frac{1}{N}$, $g_{ij}^{(m-1)}$ is a fuzzy number defined by

$$g_{ij}^{(m-1)} = \frac{1}{N^j j!} \left(\frac{d^j G(u_{m-1}(t))}{dt^j} \right)|_{t=\frac{i-1}{N}}, \ i = 1, 2\cdots, N, j = 0, 1, \cdots, l-1,$$

and

$$H_{ij}(t) = \int_0^1 k(s,t) h_{ij}(s) ds.$$

The following theorem presents the convergence of the iterative procedure (19) proposed for the solution of Eq. (10).

Theorem 3.6 ([7])**.** *Suppose that Eq.* (10) *satisfies the following conditions:*

(i) $f : [a,b] \to \mathbb{R}_F$ is fuzzy continuous.

(ii) $K : [0,1]\times[0,1] \to \mathbb{R}^+$ is continuous and there exists $M > 0$, such that $M = \max_{0\leq s,t\leq 1} |K(s,t)|$.

(iii) $G : \mathbb{R}_F \to \mathbb{R}_F$ is fuzzy differentiable of order l, for positive integer number l. In addition, there exists $L > 0$ such that

$$D(G(F_1(u)), G(F_2(v))) \leq L.D(F_1(u), F_2(v)), \ \forall u, v \in [0,1],$$

where $L < M^{-1}$ and $F_1, F_2 : [0,1] \to \mathbb{R}_F$. Then the iterative procedure Eq. (19) *convergence to the unique solution of Eq.* (10)*, F^*, and its error estimate is as follows:*

$$D^*(F^*, u_m) \leq \frac{M}{1 - ML}\left(M_0 (ML)^m + \frac{K'}{N^l l!}\right),$$

where $M_0 = \sup\limits_{0 \leq t \leq 1} \|G(f(t))\|_F$,

$$K' = \max\left\{\sup_{0 \leq t \leq 1}\left\|\frac{d^l G(u_n(t))}{dt^l}\right\|_F ; 0 \leq n \leq m-1\right\}. \quad \square \tag{20}$$

3.4. **The numerical solution of nonlinear Hammerstein fuzzy integral equations by using fuzzy wavelet like operator.** In [30], authors proposed a numerical procedure based on the fuzzy wavelet like operator for solving NHFIE

$$x(t) = g(t) \oplus (FR)\int_a^b H(t,s) \odot f(s, x(s))ds, \tag{21}$$

where $H(t,s)$ is an arbitrary kernel function over the square $a \leq s, t \leq b$ and $g(t)$ is a fuzzy valued function of t.

In the following theorem, first, Mokhtarnejad and Ezzati [30] reviewed the function approximate by using fuzzy wavelet like operator and also they recalled the pointwise and uniformly convergence of the proposed method.

Theorem 3.7 ([4])**.** *Let $f \in C_F(\mathbb{R})$ and the scaling function $\phi(x)$ a real-valued bounded function with supp $\phi(x) \subseteq [-a, a]$, $0 < a < +\infty$, $\phi(x) \geq 0$, such that*

$$\sum_{j=-\infty}^{+\infty} \phi(x - j) = 1,$$

on $\mathbb{R}$. For $k \in \mathbb{Z}$, $x \in \mathbb{R}$, put

$$(B_k f)(x) := \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \odot \phi(2^k x - j),$$

which a fuzzy wavelet like operator. Then

$$D((B_k f)(x), f(x)) \leq \omega_{\mathbb{R}}\left(f, \frac{a}{2^k}\right),$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. If $f \in C_F^U(\mathbb{R})$, then as $k \to +\infty$ one gets $\omega_{\mathbb{R}}\left(f, \frac{a}{2^k}\right) \to 0$ and $\lim\limits_{k \to +\infty} B_k f = f$, pointwise and uniformly with rates. $\square$

As [11], consider the following conditions:

(i) $g \in C_F([a,b])$, $f \in C_F([a,b] \times \mathbb{R}_F)$ and $H \in C^2([a,b])$, $H(t,s) \geq 0$, $\forall t, s \in [a,b]$;

(ii) there exist $\alpha, \gamma \geq 0$ such that

$$D(f(s,u), f(s',v)) \leq \gamma\left|s - s'\right| + \alpha D(u,v),$$

for all $s, s' \in [a,b]$, $u, v \in \mathbb{R}_F$;

(iii) $\alpha M_H(b-a) < 1$, $\beta M_H(b-a) < 1$ where $M_H \geq 0$ is such that $|H(t,s)| \leq M_H$.

(iv) there exist $\beta \geq 0$ such that

$$D(g(t), g(t')) \leq \beta \left| t - t' \right|, \ \forall t, t' \in [a,b];$$

(v) there exist $\mu \geq 0$ such that

$$\left| H(t,s) - H(t',s) \right| \leq \mu \left| t - t' \right|, \ \forall t, t' \in [a,b];$$

(vi) there exist $\delta \geq 0$ such that

$$\left| H(t,s) - H(t,s') \right| \leq \delta \left| s - s' \right|, \ \forall s, s' \in [a,b];$$

Now, in Theorem 3.8 and 3.9, authors of [30] explained the conditions of the existence and uniqueness of the solution Eq. (21) and then they were presented an error estimation for the proposed method.

Theorem 3.8 ([30]). *(a) Under the conditions (i)–(iii) the integral equation* (21) *has a unique solution in $C_F([a,b])$, $x^* \in C_F([a,b])$ and sequence of successive approximations $(x_k)_{k\in\mathbb{N}} \subset C_F([a,b])$,*

$$x_k(t) = g(t) \oplus (FR) \int_a^b H(t,s) \odot f(s, x_k(s)) ds, \ \forall t \in [a,b], k \geq 1, \tag{22}$$

converges to x^* in $C_F([a,b])$ for any choice of $x_0 \in C_F([a,b])$. In addition, the following error estimates hold:

$$D(x^*(t), x_k(t)) \leq \frac{(\alpha M_H(b-a))^k}{1 - \alpha M_H(b-a)} D(x_1(t), x_0(t)), \ \forall t \in [a,b], k \geq 1. \tag{23}$$

Choosing $x_0 \in C_F([a,b])$, $x_0 = g_0$ the inequality upper results in

$$D(x^*(t), x_k(t)) \leq \frac{(\alpha M_H(b-a))^k}{1 - \alpha M_H(b-a)} M_0 M_H(b-a), \ \forall t \in [a,b], k \geq 1,$$

where $M_0 \geq 0$. Moreover, the sequence of successive approximations (22) *is uniformly bounded, that is, there exists a constant $R \geq 0$ such that $D(x_k, \tilde{0}) \leq R$, for all $t \in [a,b]$, $k \geq 1$, solution x^* is bounded too.*

(b) Under the conditions (i)–(v), the sequence of successive approximations (22) *is uniformly Lipschitz, that is, there exist a constant $L_0 \geq 0$ such that*

$$D(x_{k-1}(t), x_{k-1}(t')) \leq L_0 \left| t - t' \right|, \ \forall t, t' \in [a,b], k \geq 1.$$

[30] Under the conditions (i)–(vi), for arbitrary fixed $t \in [a,b]$, we can obtain

$$D(H(t,s) f(s, x_k(s)), H(t,s') f(s', x_k(s'))) \leq L \left| s - s' \right|, \ \forall s, s' \in [a,b], k \geq 1,$$

for any fixed $t \in [a,b]$.

Theorem 3.9 ([30]). *Under the hypotheses of Theorem 3.8, we consider the following iterative procedure*

$$Y_{0,m}(t) = g(t),$$

$$Y_{k,m}(t) = g(t) \oplus (FR) \int_a^b H(s,t) \odot f\left(s, \sum_{j=-\infty}^{+\infty} Y_{k-1,m}\left(\frac{j}{2^m}\right) \odot \phi(2^m s - j)\right) ds, \quad k \geq 1,$$

where $Y_{k,m} \in C_F(\mathbb{R})$, $k \geq 0$, $m \in \mathbb{Z}$, the scaling function $\phi(x)$ a real valued bounded function with supp$\phi(s) \subseteq [-a, a]$, $0 < a < +\infty$, $\phi(s) \geq 0$, $s \in \mathbb{R}$ such that

$$\sum_{j=-\infty}^{+\infty} \phi(s-j) \equiv 1,$$

on $\mathbb{R}$. Then the above iterative procedure converges to the unique solution of (21), x, as $m, k \to \infty$, and the following error estimate holds true:

$$D(x(t), Y_{k,m}(t)) \leq \frac{(\alpha M_H(b-a))^k}{1 - \alpha M_H(b-a)} D(x_1(t), x_0(t)) + \frac{\alpha M_H(b-a)}{1 - \alpha M_H(b-a)} \omega\left(Y_{max}, \frac{a}{2^m}\right),$$

where

$$\omega\left(Y_{max}, \frac{a}{2^m}\right) = \max\left\{\omega\left(Y_{0,m}, \frac{a}{2^m}\right), \omega\left(Y_{1,m}, \frac{a}{2^m}\right), \cdots, \omega\left(Y_{k,m}, \frac{a}{2^m}\right)\right\},$$

and

$$D^*(x, Y_{k,m}) \leq \frac{(\alpha M_H(b-a))^k}{1 - (\alpha M_H(b-a))^k} D^*(x_1, x_0) + \frac{\alpha M_H(b-a)}{1 - \alpha M_H(b-a)} \omega\left(Y_{max}, \frac{a}{2^m}\right).$$

4. The numerical solution of FFIEs by using iterative method and Simpson quadrature rule

In this section, we present a new iterative method for solving LFFIE (1) where $K(s,t)$ is a positive crisp kernel function over the square $s, t \in [a, b]$, $f(t)$ is a fuzzy real valued function. Also, we suppose that $K(s,t)$ is continuous function, so it is uniformly continuous with respect to t and there exists $M > 0$ such that $M = \max_{t \in [a,b]} |K(s,t)|$. In [16], authors proved the existence and uniqueness of the solution of (1) by the following successive approximations method

$$F_0(t) = f(t),$$

$$F_m(t) = f(t) \oplus (FR) \int_a^b K(s,t) \odot F_{m-1}(s) ds, \ \ \forall\, t \in [a,b], \ m \geq 1.$$

Here, we propose an iterative method to solve (1). To this end, first, we assume the uniform partition of the interval $[a, b]$

$$\Delta : a = s_0 < s_1 < \cdots < s_{2n-1} < s_{2n} = b, \tag{24}$$

with $s_i = a + ih$, where $h = \frac{b-a}{2n}$. Then we present the following iterative procedure to the approximate the solution of (1) in point t

$$\begin{aligned} &u_0(t) = f(t), \\ &u_m(t) = f(t) \oplus \tfrac{h}{3} \odot \textstyle\sum_{i=1}^{n} \Big(K(s_{2i-2}, t) \odot u_{m-1}(s_{2i-2}) \\ &\oplus 4K(s_{2i-1}, t) \odot u_{m-1}(s_{2i-1}) \oplus K(s_{2i}, t) \odot u_{m-1}(s_{2i}) \Big). \end{aligned} \tag{25}$$

5. Convergence analysis

In this section, we get an error estimate between the exact solution and the approximate solution of LFFIE (1) obtained by (25).

Theorem 5.1. *Consider the LFFIE* (1) *with continuous kernel $K(s,t)$ having positive sign on $[a,b] \times [a,b]$, f continuous on $[a,b]$. If $C = M(b-a) < 1$ where $M = \max\limits_{s,t \in [a,b]} |K(s,t)|$, then the iterative procedure (25) converges to the unique solution of* (1), *F^*, and its error estimate is as follows*

$$\begin{aligned} D^*(F^*, u_m) &\le D^*(F^*, F_m) + D^*(F_m, u_m) \\ &= \frac{C^{m+1}}{1-C} M_0 + \frac{4C}{1-C} \Big(\omega_{[a,b]}(f,h) + \\ &+ \frac{C}{(1-C)M} \|f\| \, \omega_t(k,h) + \frac{1}{(1-C)M} \|f\| \, \omega_s(k,h) \Big), \end{aligned} \tag{26}$$

where

$$\omega_s(K,h) = \sup_{a \le t \le b} \{ |K(s_1,t) - K(s_2,t)| : |s_1 - s_2| \le h \}, \tag{27}$$

$$\omega_s(K,2h) = \sup_{a \le t \le b} \{ |K(s_1,t) - K(s_2,t)| : |s_1 - s_2| \le 2h \}, \tag{28}$$

$$\omega_t(K,h) = \sup_{a \le s \le b} \{ |K(s,t_1) - K(s,t_2)| : |t_1 - t_2| \le h \}. \tag{29}$$

Proof. Since $F_1(t) = f(t) \oplus (FR) \int\limits_a^b K(s,t) \odot F_0(s)ds$, we have

$$\begin{aligned} &D(F_1(t), u_1(t)) = D(f(t), f(t)) + D\Big((FR) \int_a^b K(s,t) \odot F_0(s)ds, \\ &\frac{h}{3} \odot \sum_{i=1}^{n} \Big(K(s_{2i-2}, t) \odot u_0(s_{2i-2}) \oplus 4K(s_{2i-1}, t) \odot u_0(s_{2i-1}) \\ &\qquad \oplus K(s_{2i}, t) \odot u_0(s_{s_{2i}}) \Big) \Big) \le \\ &\sum_{i=1}^{n} D\Big((FR) \int_{s_{2i-2}}^{s_{2i}} K(s,t) \odot f(s)ds, \frac{h}{3} \odot \Big(K(s_{2i-2}, t) \odot f(s_{2i-2}) \\ &\oplus 4K(s_{2i-1}, t) \odot f(s_{2i-1}) \oplus K(s_{2i}, t) \odot f(s_{2i}) \Big) \Big) \le \\ &\sum_{i=1}^{n} 3(s_{2i} - s_{2i-2}) \omega_{[s_{2i-2}, s_{2i}]} \Big(Kf, \frac{h}{3} \Big) \le 4(b-a) \omega_{[a,b]}(Kf, h). \end{aligned}$$

By using Definition 2.6, we have

$$\sup_{[a,b]} D\bigg(f(\alpha)\odot K(\alpha,t), f(\beta)\odot K(\beta,t)\bigg) \leq D\bigg(f(\alpha)\odot K(\alpha,t), f(\beta)\odot K(\alpha,t)\bigg)$$
$$+D\bigg(f(\beta)\odot K(\alpha,t), f(\beta)\odot K(\beta,t)\bigg) \leq |K(\alpha,t)|\, D\bigg(f(\alpha), f(\beta)\bigg)$$
$$+\left|K(\alpha,t)-K(\beta,t)\right| D\bigg(f(\beta),\tilde{0}\bigg) \leq M\omega_{[a,b]}(f,h)+\omega_s(K,h)\left\|f\right\|,$$

where $\alpha,\beta\in[a,b]$ and $|\alpha-\beta|\leq h$. Therefore

$$\omega_{[a,b]}(Kf,h)\leq M\omega_{[a,b]}(f,h)+\left\|f\right\|\omega_s(K,h).$$

So, we conclude, that

$$D(F_1(t),u_1(t))\leq 4(b-a)\bigg(M\omega_{[a,b]}(f,h)+\left\|f\right\|\omega_s(K,h)\bigg)$$
$$=4M(b-a)\omega_{[a,b]}(f,h)+4(b-a)\left\|f\right\|\omega_s(K,h).$$

Now, since $F_2(t)=f(t)\oplus(FR)\int\limits_a^b K(s,t)\odot F_1(s)ds$, we have

$$D(F_2(t),u_2(t))=D\bigg(f(t)\oplus(FR)\int_a^b K(s,t)\odot F_1(s)ds,$$
$$f(t)\oplus\frac{h}{3}\odot\sum_{i=1}^{n}\bigg(K(s_{2i-2},t)\odot u_1(s_{2i-2})\oplus 4K(s_{2i-1},t)\odot u_1(s_{2i-1})$$
$$\oplus K(s_{2i},t)\odot u_1(s_{2i})\bigg)\bigg)$$
$$=D\bigg(\sum_{i=1}^{n}(FR)\int_{s_{2i-2}}^{s_{2i}} K(s,t)\odot F_1(s)ds, \frac{h}{3}\odot\sum_{i=1}^{n}\bigg(K(s_{2i-2},t)\odot u_1(s_{2i-2})$$
$$\oplus 4K(s_{2i-1},t)\odot u_1(s_{2i-1})\oplus K(s_{2i},t)\odot u_1(s_{2i})\bigg)\bigg)$$
$$\leq\sum_{i=1}^{n}D\bigg((FR)\int_{s_{2i-2}}^{s_{2i}} K(s,t)\odot F_1(s)ds, \frac{h}{3}\odot\bigg(K(s_{2i-2},t)\odot F_1(s_{2i-2})$$
$$\oplus 4K(s_{2i-1},t)\odot F_1(s_{2i-1})\oplus K(s_{2i},t)\odot F_1(s_{2i})\bigg)\bigg)$$
$$+\sum_{i=1}^{n}D\bigg(\frac{h}{3}\odot K(s_{2i-2},t)\odot F_1(s_{2i-2}), \frac{h}{3}\odot K(s_{2i-2},t)\odot u_1(s_{2i-2})\bigg)$$
$$+\sum_{i=1}^{n}D\bigg(\frac{4h}{3}\odot K(s_{2i-1},t)\odot F_1(s_{2i-1}), \frac{4h}{3}\odot K(s_{2i-1},t)\odot u_1(s_{2i-1})\bigg)$$
$$+\sum_{i=1}^{n}D\bigg(\frac{h}{3}\odot K(s_{2i},t)\odot F_1(s_{2i}), \frac{h}{3}\odot K(s_{2i},t)\odot u_1(s_{2i})\bigg)$$
$$\leq 4(b-a)\omega_{[a,b]}(KF_1,h)+\sum_{i=1}^{n}\frac{h}{3}MD^*(F_1,u_1)$$
$$+\sum_{i=1}^{n}\frac{4h}{3}MD^*(F_1,u_1)+\sum_{i=1}^{n}\frac{h}{3}MD^*(F_1,u_1)$$

$$= 4(b-a)\omega_{[a,b]}(KF_1,h) + 2nhMD^*(F_1,u_1)$$

$$= 4(b-a)M\omega_{[a,b]}(F_1,h) + 4(b-a)\|F_1\|\,\omega_s(K,h) + (b-a)MD^*(F_1,u_1).$$

So,

$$D^*(F_2,u_2) \le 4(b-a)M\omega_{[a,b]}(F_1,h) + 4(b-a)\|F_1\|\,\omega_s(K,h) + (b-a)MD^*(F_1,u_1).$$

By induction, for $m \ge 3$, we get

$$D^*(F_m,u_m) \le 4(b-a)M\omega_{[a,b]}(F_{m-1},h) + 4(b-a)\|F_{m-1}\|\,\omega_s(K,h)$$

$$+(b-a)MD^*(F_{m-1},u_{m-1}),$$

$$D^*(F_{m-1},u_{m-1}) \le 4(b-a)M\omega_{[a,b]}(F_{m-2},h) + 4(b-a)\|F_{m-2}\|\,\omega_s(K,h)$$

$$+(b-a)MD^*(F_{m-2},u_{m-2}),$$

$$\vdots$$

$$D^*(F_1,u_1) \le 4(b-a)M\omega_{[a,b]}(F_0,h) + 4(b-a)\|F_0\|\,\omega_s(K,h).$$

Now, we obtain

$$D^*(F_m,u_m) \le 4(b-a)M\omega_{[a,b]}(F_{m-1},h) + 4(b-a)\|F_{m-1}\|\,\omega_s(K,h)$$

$$+4(b-a)^2M^2\omega_{[a,b]}(F_{m-2},h) + 4(b-a)^2M\|F_{m-2}\|\,\omega_s(K,h)$$

$$+(b-a)^2M^2D^*(F_{m-2},u_{m-2})$$

$$= 4(b-a)M\omega_{[a,b]}(F_{m-1},h) + 4(b-a)\|F_{m-1}\|\,\omega_s(K,h)$$

$$+4(b-a)^2M^2\omega_{[a,b]}(F_{m-2},h) + 4(b-a)^2M\|F_{m-2}\|\,\omega_s(K,h)$$

$$+(b-a)^2M^2\Bigg(4(b-a)M\omega_{[a,b]}(F_{m-3},h) + 4(b-a)\|F_{m-3}\|\,\omega_s(K,h)$$

$$+ (b-a)MD^*(F_{m-3},u_{m-3})\Bigg)$$

$$= 4(b-a)M\omega_{[a,b]}(F_{m-1},h) + 4(b-a)\|F_{m-1}\|\,\omega_s(K,h)$$

$$+4(b-a)^2M^2\omega_{[a,b]}(F_{m-2},h) + 4(b-a)^2M\|F_{m-2}\|\,\omega_s(K,h)$$

$$+(b-a)^2M^2\Bigg(4(b-a)M\omega_{[a,b]}(F_{m-3},h) + 4(b-a)\|F_{m-3}\|\,\omega_s(K,h)$$

$$+ (b-a)MD^*(F_{m-3},u_{m-3})\Bigg)$$

$$= 4(b-a)M\omega_{[a,b]}(F_{m-1},h) + 4(b-a)\|F_{m-1}\|\,\omega_s(K,h)$$

$$+4(b-a)^2M^2\omega_{[a,b]}(F_{m-2},h)+4(b-a)^2M\left\|F_{m-2}\right\|\omega_s(K,h)$$

$$+4(b-a)^3M^3\omega_{[a,b]}(F_{m-3},h)+4(b-a)^3M^2\left\|F_{m-3}\right\|\omega_s(K,h)$$

$$+(b-a)^3M^3D^*(F_{m-3},u_{m-3}).$$

Clearly, we can obtain the relation between the modulus of continuity of F_m and f for t_1, $t_2 \in [a,b]$ such that $|t_1-t_2| \leq h$, as follows

$$D(F_m(t_1),F_m(t_2)) = D\Bigg(f(t_1)\oplus(FR)\int_a^b K(s,t_1)\odot F_{m-1}(s)ds,$$

$$f(t_2)\oplus(FR)\int_a^b K(s,t_2)\odot F_{m-1}(s)ds\Bigg) \leq D(f(t_1),f(t_2))$$

$$+\int_a^b D(K(s,t_1)\odot F_{m-1}(s),K(s,t_2)\odot F_{m-1}(s))ds$$

$$\leq \omega_{[a,b]}(f,h)+\int_a^b |K(s,t_1)-K(s,t_2)|\,D(F_{m-1}(s),\tilde{0})ds$$

$$=\omega_{[a,b]}(f,h)+(b-a)\left\|F_{m-1}\right\|\omega_t(K,h),$$

hence

$$\omega_{[a,b]}(F_m,h) \leq \omega_{[a,b]}(f,h)+(b-a)\left\|F_{m-1}\right\|\omega_t(K,h). \tag{30}$$

By backward substitution of the above inequality into Eq. (31), we have

$$D^*(F_m,u_m) \leq 4(b-a)M\omega_{[a,b]}(f,h)\Big(1+(b-a)M+\cdots+(b-a)^{m-1}M^{m-1}\Big)$$

$$+\,4(b-a)^2M\omega_t(K,h)\Big(\left\|F_{m-2}\right\|+(b-a)M\left\|F_{m-3}\right\|$$

$$+\,(b-a)^2M^2\left\|F_{m-4}\right\|+\cdots+(b-a)^{m-2}M^{m-2}\left\|F_0\right\|\Big)$$

$$+\,4(b-a)\omega_s(K,h)\Big(\left\|F_{m-1}\right\|+(b-a)M\left\|F_{m-2}\right\|$$

$$+\,(b-a)^2M^2\left\|F_{m-3}\right\|+\cdots+(b-a)^{m-1}M^{m-1}\left\|F_0\right\|\Big)$$

$$\leq 4C\omega_{[a,b]}(f,h)\Big(1+C+C^2+\cdots+C^{m-1}\Big)$$

$$+\,4C(b-a)\omega_t(K,h)\Big(\left\|F_{m-2}\right\|+C\left\|F_{m-3}\right\|+\cdots+C^{m-2}\left\|F_0\right\|\Big)$$

$$+\,4(b-a)\omega_s(K,h)\Big(\left\|F_{m-1}\right\|+C\left\|F_{m-2}\right\|+\cdots+C^{m-1}\left\|F_0\right\|\Big).$$

So,

$$D^*(F_m, u_m) \leq \frac{4C}{1-C}\omega_{[a,b]}(f,h)$$
$$+4C(b-a)\omega_t(K,h)\bigg(\|F_{m-2}\| + C\|F_{m-3}\| + \cdots + C^{m-2}\|F_0\| \bigg)$$
$$+4(b-a)\omega_s(K,h)\bigg(\|F_{m-1}\| + C\|F_{m-2}\| + \cdots + C^{m-1}\|F_0\| \bigg).$$

Now, by using (12), we conclude that

$$D(F_m(t), F_{m-1}(t)) = D(f(t) \oplus (FR)\int_a^b K(s,t) \odot F_{m-1}(s)ds,$$
$$f(t) \oplus (FR)\int_a^b K(s,t) \odot F_{m-2}(s)ds)$$
$$\leq (FR)\int_a^b |K(s,t)|\, D(F_{m-1}(s), F_{m-2}(s))ds$$
$$\leq (b-a)MD^*(F_{m-1}, F_{m-2}) \leq \bigg((b-a)M\bigg)^{m-1} D^*(F_1, F_0).$$

Consequently,

$$D(F_m(t), F_0(t)) \leq D(F_m(t), F_{m-1}(t)) + D(F_{m-1}(t), F_{m-2}(t)) + \cdots + D(F_1, F_0).$$

Taking supremum for $t \in [a,b]$ from above inequality, we conclude that

$$D^*(F_m, F_0) \leq \bigg(\bigg((b-a)M\bigg)^{m-1} + \bigg((b-a)M\bigg)^{m-2} + \cdots + 1\bigg)D^*(F_1, F_0)$$
$$= \bigg(C^{m-1} + C^{m-2} + \cdots + 1\bigg)D^*(F_1, F_0) \leq \frac{1}{1-C}D^*(F_1, F_0).$$

It is obvious that

$$D(F_1(t), F_0(t)) = D(f(t) \oplus (FR)\int_a^b K(s,t) \odot F_0(s)ds, f(t))$$
$$\leq (FR)\int_a^b |K(s,t)|\, D(f(s), \tilde{0})ds \leq (b-a)M\|f\| = C\|f\|,$$

and

$$D(F_m(t), \tilde{0}) \leq D(F_m(t), F_0(t)) + D(F_0(t), \tilde{0})$$
$$\leq \frac{1}{1-C}D^*(F_1, F_0) + \|f\| = \frac{1}{1-C}(b-a)M\|f\| + \|f\|.$$

So, we have:

$$D^*(F_m, \tilde{0}) \leq \frac{1}{1-C}\|f\|.$$

Hence,

$$
\begin{aligned}
D^*(F_m,u_m) &\le \frac{4C}{1-C}\omega_{[a,b]}(f,h) \\
&\quad + 4C(b-a)\omega_t(K,h)\Big(\frac{1}{1-C}\|f\| + \frac{C}{1-C}\|f\| + \cdots + \frac{C^{m-2}}{1-C}\|f\|\Big) \\
&\quad + 4(b-a)\omega_s(K,h)\Big(\frac{1}{1-C}\|f\| + \frac{C}{1-C}\|f\| + \cdots + \frac{C^{m-1}}{1-C}\|f\|\Big) \\
&= \frac{4C}{1-C}\omega_{[a,b]}(f,h) + \frac{4C}{1-C}(b-a)\|f\|\omega_t(K,h)\Big(1+C+\cdots+C^{m-2}\Big) \\
&\quad + \frac{4}{1-C}(b-a)\|f\|\omega_s(K,h)\Big(1+C+\cdots+C^{m-1}\Big) \\
&\le \frac{4C}{1-C}\Big(\omega_{[a,b]}(f,h) + \frac{C}{(1-C)M}\|f\|\omega_t(K,h) + \frac{1}{(1-C)M}\|f\|\omega_s(K,h)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
D^*(F^*,u_m) &\le D^*(F^*,F_m) + D^*(F_m,u_m) \\
&\le \frac{C^{m+1}}{1-C}M_0 + \frac{4C}{1-C}\Big(\omega_{[a,b]}(f,h) + \frac{C}{(1-C)M}\|f\|\omega_t(K,h) \\
&\quad + \frac{1}{(1-C)M}\|f\|\omega_s(K,h)\Big).
\end{aligned}
$$

□

Since $C<1$, it follows that $\lim_{m\to\infty} C^{m+1}=0$. In addition,

$$\lim_{h\to 0}\omega_{[a,b]}(f,h)=0,\ \ \lim_{h\to 0}\omega_s(K,h)=0,\ \ \lim_{h\to 0}\omega_t(K,h)=0.$$

So, $\lim_{h\to 0, m\to\infty} D^*(F^*,u_m)=0$ that shows the convergence of the proposed method.

5.1. **Stopping criterion.** It is critical to recognize that practical features are other than theoretical aspects. So, we can give the optimal values of m and h.

Suppose that the function f is Lipscitzian. By using Definition 2.6, we have:

$$\omega(f,h) = \sup_{x,y\in[a,b]}\{|f(x)-f(y)|;\ \ |x-y|\le h\} \le L|x-y| \le Lh.$$

Also, by using (27), (28), (29) and (30), we conclude that

$$D^*(F^*,u_m) \le \frac{4C}{1-C}\Big(Lh + \frac{C}{(1-C)M}\|f\|(L_1h+L_2h)\Big).$$

Proposition 5.1 ([9])**.** *Let $f:[a,b]\to\mathbb{R}_F$ be a Lipschitz function. Then*

$$
\begin{aligned}
&D\Big((FR)\int_a^b f(t)dt, (\alpha-a)\odot f(u)\oplus(\beta-\alpha)\odot f(v)\oplus(b-\beta)\odot f(w)\Big) \\
&\qquad\qquad \le 2L\max\Big((\alpha-a)^2,(v-\alpha)^2,(\beta-v)^2,(b-\beta)^2\Big),
\end{aligned}
$$

for any $\alpha,\beta \in [a,b]$, and $u \in [a,\alpha]$, $v \in [\alpha,\beta]$, $w \in [\beta,b]$.
Considering $u=a$, $v=\frac{a+b}{2}$ and $w=b$, $\alpha=\frac{5a+b}{6}$ and $\beta=\frac{a+5b}{6}$, we obtain the fuzzy Simpson quadrature formula

$$D\left((FR)\int_a^b f(t)dt, \frac{b-a}{6} \odot \left[f(a) \oplus 4f\left(\frac{a+b}{2}\right) \oplus f(b)\right]\right) \leq L.\frac{2(b-a)^2}{9}.$$

Clearly, we can extend the above formula for Simpson quadrature rule for uniform partitions

$$\Delta : a = t_0 < t_1 < \cdots < t_{2n-1} < t_{2n} = b,$$

with $h = \frac{b-a}{2n}$, in the following Corollary:

Corollary 5.1. *For uniform partitions, the following Simpson inequality holds:*

$$D\left((FR)\int_a^b f(t)dt, \sum_{i=1}^{n} \frac{(t_{2i}-t_{2i-2})}{6} \odot [f(t_{2i-2}) \oplus 4f(t_{2i-1}) + f(t_{2i})]\right) \leq L.\frac{2(b-a)^2}{9n}. \qquad (31)$$

Proof. As in [10], we have

$$\begin{aligned}
&D\left((FR)\int_a^b f(t)dt, \sum_{i=1}^{n} \frac{(t_{2i}-t_{2i-2})}{6} \odot [f(t_{2i-2}) \oplus 4f(t_{2i-1}) + f(t_{2i})]\right)\\
&= D\left(\sum_{i=1}^{n}(FR)\int_{t_{2i-2}}^{t_{2i}} f(t)dt, \sum_{i=1}^{n} \frac{(t_{2i}-t_{2i-2})}{6} \odot [f(t_{2i-2}) \oplus 4f(t_{2i-1}) + f(t_{2i})]\right)\\
&\leq \sum_{i=1}^{n} D\left((FR)\int_{t_{2i-2}}^{t_{2i}} f(t), \frac{(t_{2i}-t_{2i-2})}{6} \odot [f(t_{2i-2}) \oplus 4f(t_{2i-1}) + f(t_{2i})]\right)\\
&\leq \sum_{i=1}^{n} L.\frac{2(t_{2i}-t_{2i-2})^2}{9} = L.\frac{2(b-a)^2}{9n}.
\end{aligned}$$

□

Theorem 5.2. *Suppose that:*

1. *there is $L' > 0$ such that $|K(\alpha,t) - K(\beta,t)| \leq L' |\alpha - \beta|$, $\forall\, t,\alpha,\beta \in [a,b]$;*
2. *there is $L > 0$ such that $|f(\alpha) - f(\beta)| \leq L |\alpha - \beta|$, $\forall\, \alpha,\beta \in [a,b]$,*
   and under the assumptions of Theorem 3.3, we have:

$$D(F^*(t), u_m(t)) \leq \frac{C^{m+1}}{1-C}M_0 + L''.\frac{2(b-a)^2}{9n(1-C)}, \qquad (32)$$

and

$$m > \left[\log_C^{\frac{(1-C)\epsilon}{2M_0}}\right] - 1, \;\; n > L''.\frac{4(b-a)^n}{9\epsilon(1-C)},$$

where

$$L'' = L' \|f\| + ML.$$

Proof. Since, the function $K \odot f$ is Lipscitzian with upper bound $L'' = L' \|f\| + ML$,

$$\begin{aligned}
&D(K(s,t) \odot f(s), K(\alpha,t) \odot f(\alpha)) \\
&\quad \le D(K(s,t) \odot f(s), K(\alpha,t) \odot f(s)) + D(K(\alpha,t) \odot f(s), K(\alpha,t) \odot f(\alpha)) \\
&\quad \le |K(s,t) - K(\alpha,t)| \, D(f(s), \tilde{0}) \\
&\quad + |K(\alpha,t)| \, D(f(s), f(\alpha)) \le L' |s-\alpha| \, \|f\| + ML \, |s-\alpha| \\
&\quad = |s-\alpha| \, (L' \|f\| + ML) = L'' |s-\alpha| \, , \qquad \forall \, s, \alpha \in [a,b],
\end{aligned}$$

from (12) and (26), we have

$$\begin{aligned}
&D\Big(F_1(t), u_1(t)\Big) = D\Big(f(t) \oplus \int_a^b K(s,t) \odot F_0(s) ds, \\
&\quad f(t) \oplus \sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot u_0(t_{2j-2}) \oplus 4K(s,t) \odot u_0(t_{2j-1}) \oplus K(s,t) \odot u_0(t_{2j})]\Big) \\
&= D\Big(\int_a^b K(s,t) \odot f(s) ds, \sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot f(t_{2j-2}) \oplus 4K(s,t) \odot f(t_{2j-1}) \\
&\quad \oplus K(s,t) \odot f(t_{2j})]\Big) \le L''.\frac{2(b-a)^2}{9n}.
\end{aligned}$$

Now,

$$\begin{aligned}
&D\Big(F_2(t), u_2(t)\Big) = D\Big(f(t) \oplus \int_a^b K(s,t) \odot F_1(s) ds, \\
&\quad f(t) \oplus \sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot u_1(t_{2j-2}) \oplus 4K(s,t) \odot u_1(t_{2j-1}) \oplus K(s,t) \odot u_1(t_{2j})]\Big) \\
&\le D\Big(\int_a^b K(s,t) \odot F_1(s) ds, \sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot F_1(t_{2j-2}) \oplus 4K(s,t) \odot F_1(t_{2j-1}) \\
&\quad \oplus K(s,t) \odot F_1(t_{2j})]\Big) + D\Big(\sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot F_1(t_{2j-2}) \oplus 4K(s,t) \odot F_1(t_{2j-1}) \\
&\quad \oplus K(s,t) \odot F_1(t_{2j})], \sum_{i=1}^{n} \frac{b-a}{6n} \odot [K(s,t) \odot u_1(t_{2j-2}) \oplus 4K(s,t) \odot u_1(t_{2j-1}) \\
&\quad \oplus K(s,t) \odot u_1(t_{2j})]\Big) \\
&\le L''.\frac{2(b-a)^2}{9n} + \sum_{i=1}^{n} \frac{b-a}{6n} \, |K(s,t)| \, [D(F_1(t_{2j-2}), u_1(t_{2j-2})) \\
&\quad + 4D(F_1(t_{2j-1}), u_1(t_{2j-1})) + D(F_1(t_{2j}), u_1(t_{2j}))] \\
&\le L''.\frac{2(b-a)^2}{9n}(1 + (b-a)M).
\end{aligned}$$

So, by induction for $m > 2$, we conclude that

$$D(F_m(t), u_m(t)) \le L''.\frac{2(b-a)^2}{9n}[1 + (b-a)M + \cdots + (b-a)^{m-1}M^{m-1}].$$

Suppose that $C = (b-a)M < 1$, we get

$$\frac{1-(b-a)^m M^m}{1-(b-a)M} < \frac{1}{1-(b-a)M} = \frac{1}{1-C},$$

we conclude

$$D(F_m(t), u_m(t)) \leq L''.\frac{2(b-a)^2}{9n(1-C)}.$$

Finally

$$\begin{aligned} D(F^*(t), u_m(t)) &\leq D(F^*(t), F_m(t)) + D(F_m(t), u_m(t)) \\ &\leq \frac{C^{m+1}}{1-C} M_0 + L''.\frac{2(b-a)^2}{9n(1-C)}. \end{aligned}$$

For given $\epsilon > 0$ the numbers $n,\ m \in \mathbb{N}$ will be determined as

$$m > \left[\log_C^{\frac{(1-C)\epsilon}{2M_0}} \right] - 1,$$

and

$$n > L''.\frac{4(b-a)^n}{9\epsilon(1-C)}.$$

$\square$

6. Numerical stability analysis

In this section, we prove the numerical stability analysis for the presented method. As [37], we consider new starting approximation $y(t) = Y_0(t)$ such that $\exists\ \epsilon > 0$ for which $D(F_0(t), Y_0(t)) < \epsilon,\ \forall\ t \in [a, b]$. The acquired sequence of successive approximations is

$$Y_m(t) = y(t) \oplus (FR) \int_a^b K(s,t) \odot Y_{m-1}(s) ds,$$

and using the identical iterative method, the terms of produced sequence are

$$\begin{aligned} v_0(t) &= Y_0(t) = y(t), \\ v_m(t) &= y(t) \oplus \sum_{i=1}^{n} \frac{h}{3} \odot \Big(K(s_{2i-2}, t) \odot v_{m-1}(s_{2i-2}) \\ &\quad \oplus 4K(s_{2i-1}, t) \odot v_{m-1}(s_{2i-1}) \oplus K(s_{2i}, t) \odot v_{m-1}(s_{2i}) \Big). \quad m \geq 1 \end{aligned}$$

Definition 6.1. *The algorithm of the iterative method applied to the LFFIE* (1) *is said to be numerically stable with respect to the choice of the first iteration iff there exist four constants* $k_1, k_2, k_3, k_4 > 0$ *which are independent by* $h = \frac{b-a}{2n}$ *such that*

$$D^*(u_m, v_m) \leq k_1 \epsilon + k_2 \Big(\omega_{[a,b]}(f, h) + \omega_{[a,b]}(v, h) \Big) + k_3 \omega_t(K, h) + k_4 \omega_s(K, h), \tag{33}$$

where

$$\begin{aligned} k_1 &= \frac{1}{1-C},\ k_2 = \frac{4C}{1-C}, \\ k_3 &= \frac{4C^2}{(1-C)^2 M}(\|f\| + \|v\|),\ k_4 = \frac{4C}{(1-C)^2 M}(\|f\| + \|v\|). \end{aligned}$$

Theorem 6.1. *By considering Theorem 5.1, the proposed method (25) is numerically stable in respect of the first iteration.*

Proof. At first, we obtain that

$$\begin{aligned}D(u_m(t), v_m(t)) &\leq D(u_m(t), F_m(t)) + D(F_m(t), Y_m(t)) + D(Y_m(t), v_m(t))\\ &\leq \frac{4C}{1-C}\bigg(\omega_{[a,b]}(f,h) + \frac{C}{(1-C)M}\,\|f\|\,\omega_t(K,h) + \frac{1}{(1-C)M}\,\|f\|\,\omega_s(K,h)\bigg)\\ &\quad + D(F_m(t), Y_m(t)) + \frac{4C}{(1-C)}\bigg(\omega_{[a,b]}(v,h) + \frac{C}{(1-C)M}\,\|v\|\,\omega_t(K,h)\\ &\quad + \frac{1}{(1-C)M}\,\|v\|\,\omega_s(K,h)\bigg).\end{aligned}$$

However,

$$\begin{aligned}&D(F_m(t), Y_m(t))\\ &= D\bigg(f(t) \oplus (FR)\int_a^b K(s,t)\odot F_{m-1}(s)ds,\, y(t)\oplus (FR)\int_a^b K(s,t)\odot Y_{m-1}(s)ds\bigg)\\ &\leq D(f(t), y(t)) + D\bigg((FR)\int_a^b K(s,t)\odot F_{m-1}(s)ds,\, (FR)\int_a^b K(s,t)\odot Y_{m-1}(s)ds\bigg)\\ &\leq \epsilon + (FR)\int_a^b |K(s,t)|\, D(F_{m-1}(s), Y_{m-1}(s))ds.\end{aligned}$$

We conclude that

$$D^*(F_m, Y_m) \leq \epsilon + \int_a^b MD^*(F_{m-1}, Y_{m-1})ds = \epsilon + CD^*(F_{m-1}, Y_{m-1}),$$

and thus

$$\begin{aligned}D^*(F_m, Y_m) &\leq \epsilon + CD^*(F_{m-1}, Y_{m-1})\\ D^*(F_{m-1}, Y_{m-1}) &\leq \epsilon + CD^*(F_{m-2}, Y_{m-2})\\ &\vdots\\ D^*(F_1, Y_1) &\leq \epsilon + CD^*(F_0, Y_0).\end{aligned}$$

So,

$$\begin{aligned}D^*(F_m, Y_m) &\leq \epsilon + C\bigg(\epsilon + CD^*(F_{m-2}, Y_{m-2})\bigg)\\ &\leq \epsilon + C\epsilon + C^2\bigg(\epsilon + CD^*(F_{m-3}, Y_{m-3})\bigg)\\ &\vdots\\ &\leq \epsilon + C\epsilon + C^2\epsilon + C^3\epsilon + \cdots + C^m D^*(F_0, Y_0)\\ &\leq \epsilon\bigg(1 + C + C^2 + C^3 + \cdots + C^m\bigg) \leq \frac{\epsilon}{1-C}.\end{aligned}$$

Therefore,

$$\begin{aligned} D^*(u_m, v_m) \leq & \frac{4C}{1-C}\Bigg((\omega_{[a,b]}(f,h) + \omega_{[a,b]}(v,h)) \\ & + \frac{C}{(1-C)M}\omega_t(K,h)(\|f\| + \|v\|) + \frac{1}{(1-C)M}\omega_s(K,h)(\|f\| + \|v\|)\Bigg) + \frac{\epsilon}{1-C} \\ = & k_1\epsilon + k_2\Big(\omega_{[a,b]}(f,h) + \omega_{[a,b]}(v,h)\Big) + k_3\omega_t(K,h) + k_4\omega_s(K,h), \end{aligned}$$

where

$$k_1 = \frac{1}{1-C},\ k_2 = \frac{4C}{1-C},\ k_3 = \frac{4C^2}{(1-C)^2M}(\|f\| + \|v\|),\ k_4 = \frac{4C}{(1-C)^2M}(\|f\| + \|v\|).$$

□

7. Numerical experiments

Finally, in this section, we solve some examples using the proposed method and compare results with the method of [16].

Consider the following LFFIE

$$F(t) = f(t) \oplus (FR)\int_0^1 K(s,t) \odot F(s)ds,$$

where

$$\underline{f}(t,r) = \frac{-1}{52}r(5 - 52t + 2t^2),$$
$$\overline{f}(t,r) = \frac{1}{52}(r-2)(5 - 52t + 2t^2),$$

and

$$K(s,t) = \frac{s^2 + t^2 + 2}{13},\ \ 0 \leq s,t \leq 1 \text{ and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(t,r) = rt,\ \ \overline{u}(t,r) = (2-r)t.$$

To compare the results with the results of [16], see Table 1.

Table 1. The accuracy on the level sets for Example 1 by using the method [16] and the proposed method in $t = 0.5$ for $h = \frac{1}{10}$ and $m = 10$.

	The method of [16]		Proposed method	
$r - level$	$\lvert\underline{F}^* - \underline{u}_m\rvert,$	$\overline{F}^* - \overline{u}_m$	$\lvert\underline{F}^* - \underline{u}_m\rvert,$	$\overline{F}^* - \overline{u}_m$
0.00	0.	4.81157×10^{-4}	0.	3.14809×10^{-8}
0.25	6.01446×10^{-5}	4.21012×10^{-4}	3.93512×10^{-9}	2.75458×10^{-8}
0.50	1.20289×10^{-4}	3.60868×10^{-4}	7.87024×10^{-9}	2.36107×10^{-8}
0.75	1.80434×10^{-4}	3.00723×10^{-4}	1.18054×10^{-8}	1.96756×10^{-8}
1.00	2.40578×10^{-4}	2.40578×10^{-4}	1.57405×10^{-8}	1.57405×10^{-8}

Consider the following LFFIE

$$F(t) = f(t) \oplus (FR)\int_0^1 K(s,t) \odot F(s)ds,$$

where

$$\underline{f}(t,r) = \frac{2}{3}(2+r)t,$$
$$\overline{f}(t,r) = \frac{-2}{3}(r-4)t,$$

and

$$K(s,t) = st, \ \ 0 \leq s,t \leq 1 \ \text{ and } \ \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(t,r) = (2+r)t, \ \ \overline{u}(t,r) = (4-r)t.$$

In Table 2, we compared the results of the proposed method with the results of [16].

Table 2. The accuracy on the level sets for Example 2 by using the method of [16] and the proposed method in $t = 0.5$ for $h = \frac{1}{10}$ and $m = 10$.

	The method of [16]		Proposed method	
$r - level$	$\lvert \underline{F}^* - \underline{u}_m \rvert,$	$\overline{F}^* - \overline{u}_m$	$\lvert \underline{F}^* - \underline{u}_m \rvert,$	$\overline{F}^* - \overline{u}_m$
0.00	2.50029×10^{-3}	5.00057×10^{-3}	5.64503×10^{-6}	1.12901×10^{-5}
0.25	2.81282×10^{-3}	4.68804×10^{-3}	6.35066×10^{-6}	1.05844×10^{-5}
0.50	3.12536×10^{-3}	4.3755×10^{-3}	7.05629×10^{-6}	9.8788×10^{-6}
0.75	3.4379×10^{-3}	4.06297×10^{-3}	7.76192×10^{-6}	9.17317×10^{-6}
1.00	3.75043×10^{-3}	3.75043×10^{-3}	8.46754×10^{-6}	8.46754×10^{-6}

Consider the following LFFIE

$$F(t) = f(t) \oplus (FR) \int_0^{\frac{1}{2}} K(s,t) \odot F(s) ds,$$

where

$$\underline{f}(t,r) = r(t + \cos(t)) - \cos\left(\frac{1}{2} + t\right) - \frac{1}{2}\sin\left(\frac{1}{2} + t\right),$$
$$\overline{f}(t,r) = \frac{1}{2}(-2+r)\Bigg(-2t - 2\cos t + 2\cos\left(\frac{1}{2} + t\right) + \sin\left(\frac{1}{2} + t\right)\Bigg),$$

and

$$K(s,t) = \cos(s+t), \ \ 0 \leq s,t \leq \frac{1}{2} \ \text{ and } \ \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(t,r) = rt, \ \ \overline{u}(t,r) = (2-r)t.$$

For more details, see Table 3.

Table 3. The accuracy on the level sets for Example 3 by using the method of [16] and the proposed method in $t = 0.25$ for $h = \frac{1}{20}$ and $m = 20$.

	The method of [16]		Proposed method	
$r - level$	$\lvert \underline{F}^* - \underline{u}_m \rvert,$	$\lvert \overline{F}^* - \overline{u}_m \rvert$	$\lvert \underline{F}^* - \underline{u}_m \rvert,$	$\lvert \overline{F}^* - \overline{u}_m \rvert$
0.00	0.	4.20233×10^{-4}	0.	1.16667×10^{-7}
0.25	5.25291×10^{-5}	3.67704×10^{-4}	1.45834×10^{-8}	1.02084×10^{-7}
0.50	1.05058×10^{-4}	3.15175×10^{-4}	2.91668×10^{-8}	8.75004×10^{-8}
0.75	1.57587×10^{-4}	2.62646×10^{-4}	4.37502×10^{-8}	7.2917×10^{-8}
1.00	2.10117×10^{-4}	2.10117×10^{-4}	5.83336×10^{-8}	5.83336×10^{-8}

8. Conclusions

First, we reviewed some of the numerical methods for solving linear and nonlinear FFIEs which had been done by several authors. Then, by using iterative method and Simpson quadrature rule, we proposed a new approach to solve linear FFIE. Also, in two theorems, we presented convergence analysis and the numerical stability of the proposed numerical method. The numerical results show that the proposed method can be a suitable method for solving linear FFIEs. Clearly, the proposed method can be applied to solve nonlinear FFIEs.

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