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NECESSARY CONDITIONS OF OPTIMALITY FOR THE QUASI-SINGULAR RELATIVE TO THE COMPONENT CONTROLS IN THE GOURSAT-DARBOUX SYSTEMS

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Abstract. The definition of the control, which is quasi-singular relative to the component, is introduced, and on its base new necessary optimality conditions for such controls are obtained in the processes described the Goursat-Darboux systems.

Keywords: necessary optimality conditions, quasi-singular control, Goursat-Darboux system, maximum principle.

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1. Introduction

Investigation of a wide class of applied problems on the optimality, as well as, the processes of sorption and desorption of gases, drying processes and etc. leads to different optimization problems describing by the systems of nonlinear second order hyperbolic equations with Goursat boundary conditions [3,11,13,14,15]. That is why the optimal control problems for the Goursat-Darboux systems are intensively studied and the first order necessary conditions of optimality are obtained in the form of Pontryagin's maximum principle and differential maximum principle [1,3,9,11,14,15]. However, the cases are met when the first order necessary optimality conditions are degenerated. Such cases, following to L.I. Rozonoeru [10], are called singular. As is known [4], singular in the sense of Pontryagin's maximum principle controls are also quasi-singular. The converse is generally not true, i.e. quasi-singular control may not be singular in the terms of the maximum principle. In addition, the necessary conditions for the optimality of the quasi singular controls also allow one to obtain additional information on the controls which are not singular in the sense of the maximum principle.

In the present work the definition of the control, which is quasi-singular relative to the component, is introduced and on its base new scheme is proposed to derive the necessary optimality conditions of such controls. By introducing a series of needle variations new multi-point necessary optimality conditions are obtained for the quasi-singular relative to components controls in the systems described by the system of hyperbolic equations of second order with Goursat-Darboux conditions. The obtained results in some cases, allow one to identify the non-optimality as those controls that satisfy the Pontryagin's maximum condition without degeneration.

2. Problem formulation

Let in the domain $D = \{(t,x) : t \in T = [t_0, t_1], x \in X = [x_0, x_1]\}$ the controlled process be described by the system of the hyperbolic equations

$$z_{tx} = f(t, x, z, z_t, z_x, u),\ (t,x) \in D \tag{1}$$

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with conditions

$$
\begin{aligned}
&z(t,x_0)=\varphi_1(t),\ t\in T,\ \ z(t_0,x)=\varphi_2(x),\ x\in X,\\
&\varphi_1(t_0)=\varphi_2(x_0).
\end{aligned}
\tag{2}
$$

Here $z(t,x)$ is a state vector, $u(t,x)$ is a vector of controls, $f(t,x,z,z_t,z_x,u)$ is a given n-dimensional vector-function which is continuous over the set of variables together with partial derivatives relative to p $(p=(z,z_t,z_x))$ up to second order, $\varphi_1(t)$, $\varphi_2(x)$ are n-dimensional vector functions continuously differentiable on T, X respectively. As a set of admissible controls we take the set of piece-wise r-dimensional functions $u(t,x)$ taking values from given nonempty bounded set $U\subset R^r$

$$
u(t,x)\in U\subset R^r,\ \ (t,x)\in D. \tag{3}
$$

It is supposed that to each admissible control $u(t,x)$ corresponds the only absolutely continuous solution $z(t,x)$ (in the sense of [2,3,11,15]) of the problem (1), (2), defined in D. The problem consists of the minimization of the functional

$$
S(u)=\varphi(z(T_1,X_1),...,z(T_k,X_k)), \tag{4}
$$

defined on the solutions of the system (1),(2) generated by admissible controls , where $\varphi(z_1,...,z_k)$ is a given scalar function twice continuously differentiable over the set of variables, $(T_i,X_i)\in D$, $i=\overline{1,k}$ are given points, moreover $t_0<T_1<T_2<...<T_k\le t_1$, $x_0<X_1<X_2<...<X_k\le x_1$.

The problem of minimization of the functional (4) subject to the conditions (1)-(3) we call the problem (1)-(4), the solution of this problem – optimal control, and corresponding process $(u(t,x),z(t,x))$ – optimal process.

3. Definition of the quasi-singular relative to component control

Let $(u(t,x),z(t,x))$ be fixed admissible process in the problem (1)-(4). It is known that (see, for instance, [1,3,9,11,14,15]), for the optimality of the admissible process $(u(t,x),z(t,x))$ in the problem (1)-(4) it is necessary that the Pontryagin's maximum principle

$$
\Delta_u H(t,x)\equiv H(t,x,p(t,x),u,\psi(t,x))-H(t,x,p(t,x),u(t,x),\psi(t,x))\le 0, \tag{5}
$$

$$
\forall u\in U,\ \ (t,x)\in[t_0,t_1)\times[x_0,x_1).
$$

be satisfied. Here $H(t,x,p,u,\psi)=\psi' f(t,x,p,u)$, $\psi(t,x)$ is a vector function of the adjoint variables, defined by the relation

$$
\psi(t,x)=-\sum_{i=1}^{k}\lambda'(T_i,X_i;t,x)\partial\varphi(z(T_1,X_1),...,z(T_k,X_k))/\partial z_i, \tag{6}
$$

where $(')$ means transpose, and the matrix function $\lambda(t,x;\tau,s)$ is a solution of the following integral equation

$$
\lambda(t,x;\tau,s)=E+\int_\tau^t\int_s^x\lambda(t,x;\xi,\eta)f_z(\xi,\eta)d\xi d\eta+
$$

$$
+\int_\tau^t\lambda(t,x;\xi,s)f_{z_x}(\xi,s)d\xi+\int_s^x\lambda(t,x;\tau,\eta)f_{z_t}(\tau,\eta)d\eta, \tag{7}
$$

E is unit $n\times n$ matrix, $\lambda(t,x;\tau,s)=0$ by $t<\tau$ or $x<s$.

It is also known that if the function $f(t,x,p,u)$ is differentiable relative to u, and the set U is convex, then as follows from the Pontryagin's maximum principle (5),

$$
H'_u(t,x)\,(u-u(t,x))\le 0,\ \forall\,u\in U,\ (t,x)\in[t_0,t_1)\times[x_0,x_1) \tag{8}
$$

satisfied along the optimal process $(u(t,x),z(t,x))$.

Note that there is a possibility of degeneracy of the conditions (8). Such cases are called quasi-singular [6-8].

Definition 3.1. *The admissible control* $u(t,x)$ *is called quasi-singular if there exists a set* $U_0 \subset U$ *such that the condition*

$$H'_u(t,x)\,(u - u(t,x)) \equiv 0, \quad (t,x) \in [t_0,t_1) \times [x_0,x_1) \tag{9}$$

is identically satisfied for $u \in U_0$.

As follows from this definition, for the quasi-singular control the differential maximum principle (8) loses its mean, and consequently, becomes non effective. Note that necessary optimality conditions for the quasi-singular controls in the problem (1)-(4) were in [6-8]. However, those conditions are not applicable to the cases, when the right hand side of the system is not differentiable relative to some of the components of the control parameters and when the set U is not convex. Let us consider the
Example 1. Let it needs to minimize the functional

$$S(u) = z_2(1,1)$$

subject to

$$\begin{array}{l} z_{1tx} = u_1, \;\; z_{2tx} = |u_2| - u_1 z_{1t} + u_1^4, \;\; (t,x) \in [0,1] \times [0,1] = D, \\ z_i(t,0) = 0, \;\; t \in [0,1], \;\; z_i(0,x) = 0, \;\; x \in [0,1], \;\; i = 1,2, \\ U = \{(u_1,u_2) : u_1 \in [0,1], \;\; u_2 = 0, \pm 1\}. \end{array} \tag{10}$$

It is easy to show that the control $u_1(t,x) \equiv 0$, $u_2(t,x) \equiv 0$ together with the solution $z_1(t,x) \equiv 0$, $z_2(t,x) \equiv 0$ of the system (10) and corresponding solution $\psi_1(t,x) \equiv 0$, $\psi_2(t,x) \equiv -1$of the adjoint system satisfies the Pontryagin's maximum principle

$$\Delta_u H = -|u_2| - u_1^4 \le 0, \quad \forall (u_1,u_2)' \in U,$$

and consequently, the control $u_1(t,x) = 0$, $u_2(t,x) = 0$ may be optimal one. No one may show that this control is not optimal. Since the right hand side of the system (10) is not differentiable relative to u_2, and the set U is not convex, the results of the works [6-8] cannot be applied to this example. Our aim is to derive necessary optimality conditions for the case when $f(t,x,p,u)$ is not differentiable relative to some of control parameters, and the set U is not generally convex. For this as in [16] we present the control function $u = (v,w)'$, where v is r_0-dimensional vector, $w - r_1$-dimensional vector, $(0 \le r_0 \le r, \; r_1 = r - r_0)$. Then the process $(u(t,x), z(t,x))$ also may be presented in the form $(v(t,x), w(t,x), z(t,x))$. Further we suppose that $f(t,x,p,u,w)$ is continuous relative to the set of variables, twice continuously differentiable relative to p, v, and projection of the section U for each w on r_0-dimensional space is a convex set.

Definition 3.2. *The admissible control* $(v(t,x), w(t,x))'$ *is called quasi-singular relative to the component* $v(t,x)$, *if there exists a set* $U_0 \subset U$, *such that the condition*

$$H'_v(t,x)\,(v - v(t,x)) \equiv 0, \quad (t,x) \in [t_0,t_1) \times [x_0,x_1) \tag{11}$$

is identically satisfied for $(v, w(t,x))' \in U_0$, *where* $U_0 | (v(t,x), \omega(t,x))' \ne \oslash$, $(t,x) \in D$.

It is clear that the quasi-singular in the terms of the definition 1 control is aslo quasi-singular relative to v, but the converse is generally not true. The necessity of investigation of the quasi singular relative to the components control is stimulated by the fact that in some cases optimality conditions for such controls allow one to identificate the non-optimality as those controls, which leaves the Pontryagin's maximum principle among suspicious for optimality.

4. Necessary optimality conditions

Using the convexity of the projection of the set U for each w on the r_0-dimensional space, by means of linear variation of the control it is proved

Theorem 4.1. *For the optimality of the quasi-singular relative to the component $v(t,x)$ control $(v(t,x), w(t,x))'$ in the problem (1)-(4) it is necessary the fulfillment of relation*

$$2\int_{x_0}^{x_1}\left(\int_{x}^{x_1}(\mathrm{v}_1(s)-v(\theta,s))'H_{vz_t}(\theta,s)\lambda(\theta,s;\theta,x)ds\right)f_v(\theta,x)(\mathrm{v}_1(x)-v(\theta,x))\,dx+$$

$$+\int_{x_0}^{x_1}\int_{x_0}^{x_1}(\mathrm{v}_1(\tau)-v(\theta,\tau))'f_v'(\theta,\tau)M_1(\theta,\tau,s)f_v(\theta,s)(\mathrm{v}_1(s)-v(\theta,s)d\tau ds+ \tag{12}$$

$$+\int_{x_0}^{x_1}(\mathrm{v}_1(x)-v(\theta,x))'H_{vv}(\theta,x)(\mathrm{v}_1(x)-v(\theta,x))dx\le 0$$

for each $\theta\in[t_0,t_1)$, $(\mathrm{v}_1(x), w(\theta,x))'\in KC_r(X,U_0)$,

$$2\int_{t_0}^{t_1}\left(\int_{t}^{t_1}(\mathrm{v}_2(s)-v(s,\sigma))'H_{vz_x}(s,\sigma)\lambda(s,\sigma;t,\sigma)ds\right)f_v(t,\sigma)(\mathrm{v}_2(t)-v(t,\sigma))\,dt+$$

$$+\int_{t_0}^{t_1}\int_{t_0}^{t_1}(\mathrm{v}_2(\tau)-v(\tau,\sigma))'f_v'(\tau,\sigma)M_2(\sigma,\tau,s)f_v(s,\sigma)(\mathrm{v}_2(s)-v(s,\sigma))d\tau ds+ \tag{13}$$

$$+\int_{t_0}^{t_1}(\mathrm{v}_2(t)-v(t,\sigma))'H_{vv}(t,\sigma)(\mathrm{v}_2(t)-v(t,\sigma))dt\le 0$$

for each $\sigma\in[x_0,x_1)$, $(\mathrm{v}_2(t), w(t,\sigma))'\in KC_r(T,U_0)$, where

$$M_1(t,\tau,s)=\int_{x_0}^{x_1}\lambda'(t,x;t,\tau)H_{z_tz_t}(t,x)\lambda(t,x;t,s)dx,$$

$$M_2(x,\tau,s)=\int_{t_0}^{t_1}\lambda'(t,x;\tau,x)H_{z_xz_x}(t,x)\lambda(t,x;s,x)dt.$$

The inequality (12), (13) is general integral necessary condition for the quasi-singular relative to the component $v(t,x)$ control, but its checking requires complicated calculations. Therefore the problem of derivation more simple conditions from (12), (13) arises. In particular it is true

Theorem 4.2. *For the optimality of the quasi-singular relative to the component $v(t,x)$ control $(v(t,x), w(t,x))'$ in the problem (1)-(4) it is necessary the fulfillment of the inequality*

$$(\mathrm{v}-v(t,x))'H_{vv}(t,x)(\mathrm{v}-v(t,x))\le 0 \tag{14}$$

for each $(\mathrm{v}, w(t,x))'\in U_0$, $(t,x)\in[t_0,t_1)\times[x_0,x_1)$.

To prove (14) it is enough to define $\mathrm{v}_1(x)$in (12) in the form

$$\mathrm{v}_1(x) = \begin{cases} \mathrm{v}, \;\; x \in [\sigma, \sigma+\varepsilon), \\ v(\theta, x), \;\; x \in [\sigma, \sigma+\varepsilon), \end{cases}$$

where $\sigma \in [x_0, x_1)$, $(\mathrm{v}, w(\theta, x))' \in U_0$, $(t, x) \in [t_0, t_1) \times [x_0, x_1)$, and $\varepsilon > 0$ is enough small.

It is not excluded the possibility of degeneration necessary optimality condition (14). We therefore introduce

Definition 4.1. *Quasi-singular relative to the component $v(t,x)$ control $(v(t,x),\ w(t,x))'$ we call strong quasi-singular relative to $v(t,x)$ control, if there exists a set $U_c \subset U_0$, such that the condition*

$$(\mathrm{v} - v(t,x))' H_{vv}(t,x)(\mathrm{v} - v(t,x)) \equiv 0, (t,x) \in [t_0, t_1) \times [x_0, x_1).$$

is satisfied identically for all $(v, w(t,x))' \in U_c$, where $U_c | (v(t,x), w(t,x))' \neq 0$, $(t,x) \in D$.

Using (12),(13) by the help of needle variations is proved

Theorem 4.3. *For the optimality of the strong quasi-singular relative to the component $v(t,x)$ control $(v(t,x), w(t,x))'$ in the problem* (1)-(4) *it is necessary the fulfillment of the relation for any natural m*

$$\sum_{i=1}^{m} l_i (\mathrm{v}_i - v(t,\sigma_i))' H_{vz_t}(t,\sigma_i) \Big[l_i f_v(t,\sigma_i)(\mathrm{v}_i - v(t,\sigma_i)) +$$
$$+2\sum_{j=1}^{i-1} l_j \lambda(t,\sigma_i; t,\sigma_j) f_v(t,\sigma_j)(\mathrm{v}_j - v(t,\sigma_j)) \Big] + \tag{15}$$
$$+\sum_{i,j=1}^{m} l_i l_j (\mathrm{v}_i - v(t,\sigma_i))' f'_{\mathrm{v}}(t,\sigma_i) M_1(t,\sigma_i,\sigma_j)(\mathrm{v}_j - v(t,\sigma_j)) \leq 0$$

for all $l_i \geq 0$, $(\mathrm{v}_i, w(t,\sigma_i))' \in U_c$, $(t,\sigma_i) \in [t_0, t_1) \times [x_0, x_1)$, $i = \overline{1,m}$ $(x_0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \leq \sigma_m \leq x_1)$,

$$\sum_{i=1}^{m} l_i (\mathrm{v}_i - v(\theta_i,x))' H_{vz_x}(\theta_i,x) \Big[l_i f_v(\theta_i,x)(\mathrm{v}_i - v(\theta_i,x)) +$$
$$+2\sum_{j=1}^{i-1} l_j \lambda(\theta_i,x; \theta_j,x) f_v(\theta_j,x)(\mathrm{v}_j - v(\theta_j,x)) \Big] + \tag{16}$$
$$+\sum_{i,j=1}^{m} l_i l_j (\mathrm{v}_i - v(\theta_i,x))' f'_v(\theta_i,x) M_2(x,\theta_i,\theta_j) f_v(\theta_j,x)(\mathrm{v}_j - v(\theta_j,x)) \leq 0$$

for all

$$l_i \geq 0,\ (\mathrm{v}_i, w(\theta_i,x))' \in U_c,\ (\theta_i,x) \in [t_0,t_1) \times [x_0,x_1),$$
$$i = \overline{1,m}\ (t_0 \leq \theta_1 \leq \theta_2 \leq ... \leq \theta_m \leq t_1).$$

From the conditions (15), (16) different more suitable for checking necessary optimality conditions may be obtained for the strong quasi-singular relative to the component $v(t,x)$ control $(v(t,x), w(t,x))'$. For example, by $m = 1$ it holds true

Theorem 4.4. *If $(v(t,x), w(t,x))'$ is strong quasi-singular relative to the component $v(t,x)$ optimal control in the problem (1)-(4), then along the process $(v(t,x), w(t,x), z(t,x))$ take place*

$$\begin{aligned} &a(t,x;\mathrm{v}) \equiv (\mathrm{v}-v(t,x))' H_{vz_t}(t,x) f_v(t,x)(\mathrm{v}-v(t,x))+ \\ &+(\mathrm{v}-v(t,x))' f_v'(t,x) M_1(t,x,x) f_v(t,x)(\mathrm{v}-v(t,x)) \le 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &b(t,x;\mathrm{v}) \equiv (\mathrm{v}-v(t,x))' H_{vz_x}(t,x) f_v(t,x)(\mathrm{v}-v(t,x))+ \\ &+(\mathrm{v}-v(t,x))' f_v'(t,x) M_2(x,t,t) f_v(t,x)(\mathrm{v}-v(t,x)) \le 0, \end{aligned} \tag{18}$$

for all $(\mathrm{v}, w(t,x))' \in U_c$, $(t,x) \in [t_0,t_1) \times [x_0,x_1)$.

For the illustration of the efficiency of the condition (17),(18) let us consider Example 1. In this case the control $(u_1,u_2)' = (0,0)'$ is strong quasi-singular relative to the first component u_1 and along this the condition (17) takes the form $v_1^2 \le 0$, for all $(v_1,0)' \in U$, i.e. this control cannot be optimal.

The immediate consequence of the Theorem 4 is also

Theorem 4.5. *Among the strong quasi-singular relative to the component $v(t,x)$ of the process $(v(t,x),\ w(t,x),\ z(t,x))$ the following inequalities are satisfied*

$$\begin{gathered} a(t,\sigma_1;\mathrm{v}_1) \le 0, \quad a(t,\sigma_2;\mathrm{v}_2) \le 0, \\ (\mathrm{v}_2 - v(t,\sigma_2))' H_{vz_t}(t,\sigma_2) \lambda(t,\sigma_2;t,\sigma_1)(\mathrm{v}_1 - v(t,\sigma_1))+ \\ +(\mathrm{v}_2 - v(t,\sigma_2))' f_v'(t,\sigma_2) M_1(t,\sigma_2,\sigma_1) f_v(t,\sigma_1)(\mathrm{v}_1 - v(t,\sigma_1)) \le \\ \le \sqrt{a(t,\sigma_1;\mathrm{v}_1)\cdot a(t,\sigma_2;\mathrm{v}_2)} \end{gathered} \tag{19}$$

for all $(\mathrm{v}_1, w(t,\sigma_1))', (\mathrm{v}_2, w(t,\sigma_2))' \in U_c$, $(t,\sigma_1),(t,\sigma_2) \in [t_0,t_1)\times[x_0,x_1)$,

$$(x_0 \le \sigma_1 \le \sigma_2 < x_1),$$

$$\begin{gathered} b(\theta_1,x;\mathrm{v}_1) \le 0, \ b(\theta_2,x;\mathrm{v}_2) \le 0, \\ (\mathrm{v}_2 - v(\theta_2,x))' H_{vz_x}(\theta_2,x) \lambda(\theta_2,x;\theta_1,x)(\mathrm{v}_1 - v(\theta_1,x))+ \\ +(\mathrm{v}_2 - v(\theta_2,x))' f_v'(\theta_2,x) M_2(x,\theta_2,\theta_1) f_v(\theta_1,x)(\mathrm{v}_1 - v(\theta_1,x)) \le \\ \le \sqrt{b(\theta_1,x;\mathrm{v}_1)\cdot b(\theta_2,x;\mathrm{v}_2)} \end{gathered} \tag{20}$$

for all $(\mathrm{v}_1, w(\theta_1,x))', (\mathrm{v}_2, w(\theta_2,x))' \in U_c$, $(\theta_1,x),(\theta_2,x) \in [t_0,t_1)\times[x_0,x_1)$, $(t_0 \le \theta_1 \le \theta_2 < t_1)$.

The proof of this theorem follows from the condition of nonpositivity quadratic polynomial obtained from the formula (15), (16) by $m=2$.

Example 2. Consider the problem

$$\begin{aligned} &S(u) = z_3(1,1) \to \min \\ &z_{1tx} = u_1, z_{1tx} = z_{1t} + z_{2t},\ z_{3tx} = |u_2| - u_1 z_{2t} + u_1^4,\ (t,x) \in [0,1]\times[0,1], \\ &z_i(t,0) = 0,\ \ t \in [0,1], z_i(0,x) = 0, x \in [0,1], i = 1,2,3, \\ &U = \{(u_1,u_2)' : u_1 \in [0,1], u_2 = 0; \pm\tfrac{1}{2}; \pm 1\}. \end{aligned}$$

Let us study the optimality of the admissible control $(u_1,u_2)' = (0,0)'$. On this control we have $\psi(t,x) = (0,0,-1)'$, $z_i(t,x) = 0,\ i = 1,2,3,\ (t,x) \in D$. Pontryagin's maximum principle is satisfied:

$$\Delta_u H(t,x) = -|u_2| - u_1^4 \le 0,\ \ (u_1,u_2) \in U,$$

i.e. the control $(0,0)'$ claims to be optimal. Since the right hand side of the system is not differentiable with respect to u_2 and the set U is not convex the differential maximum principle (8) cannot be applied here. But as we see the right hand side of the system is differentiable with respect to u and the set U with respect to u_1 is convex.

Since $H_{u_1}(t,x)=0,\ H_{u_1u_1}(t,x)=0$, then $(0,0)'$ is strong convex relative to the component u_1 control. Among this control we have

$$\begin{aligned}&H_{u_1z_t}(t,x)=(0,1,0)',\ f_{u_1}(t,x)=(1,0,0)',\ H_{u_1z_x}(t,x)=(0,0,0)',\\ &M_2(x,t,t)=0,\ M_1(t,x,x)=0,\ \ a(t,x;u_1)\equiv 0,\ \ b(t,x;u_1)\equiv 0,\\ &\lambda(t,x;\tau,s)=\begin{pmatrix}1 & 0 & 0\\ e^{x-s}-1 & e^{x-s} & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

Therefore the optimality condition (17), (18) leaves the control $(0,0)'$ among the conterdence for optimality, and the condition (19) has a form

$$(e^{x-s}-1)v_1v_2\le 0,$$

which disturbed for all $v_1>0,\ v_2>0,\ x>s$. That is why the control $(0,0)'$ is not optimal.

Note that the optimality condition (15), (16) by $m>2$ is stronger than the conditions (19), (20). Note that, the optimal control problem in the processes described by the ordinary differential equations with two groups of control functions from various classes is considered in [5].

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