TWMS J. Pure Appl. Math. V.6, N.2, 2015, pp.213-223

COMMON FIXED POINT THEOREM FOR SIX MAPPINGS ON FUZZY METRIC SPACES

BHAVANA DESHPANDE[1], ROHIT PATHAK[2]

Abstract. In this paper we extend the result of Turkoglu et al [29] and prove a common fixed point theorem for compatible maps of type (α) on fuzzy metric spaces. We also give an example to validate our result.

Keywords: coincidence point, common fixed point, compatible maps, weak commutativity of type (KB).

AMS Subject Classification: 54H25, 47H10.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [31]. Since then, many authors have tried to use this concept in topology and analysis and developed the theory of fuzzy sets and applications. Especially, Deng [7], Erceg [8], Kaleva & Seikkala [17], Kramosil & Michalek [18] have introduced the concept of fuzzy metric spaces in different ways. Grabiec [11] followed Kramosil & Michalek [18] and obtained the fuzzy version of Banach's fixed point theorem. Many authors have studied the fixed point theory in fuzzy metric spaces. The most interesting references are [2], [5], [9], [11], [12], [19], [23]-[25].

Sessa [22] generalized the concept of commutativity and introduced weak commutativity of mappings. Further, more generalized commutativity called compatibility was introduced by Jungck [13]. Mishra et al [19] introduced the concept of compatibility in fuzzy metric spaces and obtained common fixed point theorems for compatible maps.

Jungck et al [15] introduced the concept of compatible maps of type (A) in metric spaces and proved common fixed point theorems in metric spaces. Cho [6] introduced the notion of compatible maps of type (α) in fuzzy metric spaces.

Many generalizations of metric spaces have appeared (see [16], [30]). Several others ([1], [3], [4], [20]) studied common fixed point theorems in various spaces under different conditions.

In this paper, we extend the result of Turkoglu et al [29] and prove a common fixed point theorem for compatible maps of type (α) on fuzzy metric spaces. We also give an example to validate our result.

2. Preliminaries

Definition 2.1. [21] *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is called a continuous* t*-norm if* $([0,1], *)$ *is an Abelian topological monoid with the unit* 1 *such that* $a*b \le c*d$ *whenever* $a \le c$ *and* $b \le d$ *for all* $a,b,c,d \in [0,1]$. *Examples of* t*-norms are* $a*b = ab$ *and* $a*b = \min\{a,b\}$.

[1]Department of Mathematics, Govt. Arts and Science P.G. College, Ratlam, M.P., India
[2]Institute of Engineering and Technology, DAVV, Indore, M.P., India
e-mail: bhavnadeshpande@yahoo.com, rohitpathakres@yahoo.in
Manuscript received June 2014.

Definition 2.2. [10] *The 3-tuple $(X, M, *)$ is called a fuzzy metric space (Shortly FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*
for all $x, y, z \in X$ and $t, s > 0$,
(fm-1) $M(x, y, t) > 0$,
(fm-2) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$,
(fm-3) $M(x, y, t) = M(y, x, t)$,
(fm-4) $M(x, y, t) * M(y, z, s) \leq M(x, y, t + s)$,
(fm-5) $M(x, y, .) : X^2 \times [0, \infty) \to [0, 1]$ *is continuous.*

Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞ and we can find some topological properties and examples of fuzzy metric spaces in [10].

Lemma 2.1. [11] *For all $x, y \in X$, $M(x, y, .)$ is nondecreasing.*

Definition 2.3. [11] *Let $(X, M, *)$ be a FM-Space:*
1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ i.e. $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} M(x_n, y, t) = 1$ for all $t > 0$.
2. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$$

for all $t > 0$ and $p > 0$.
3. A FM-space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.1. *Since $*$ is continuous, it follows from (fm-4) that the limit of sequence in FM-space is uniquely determined.*

Throughout this paper $(X, M, *)$ will denote the fuzzy metric space with the following condition:
(fm-6) $\lim_{n\to\infty} M(x_n, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Lemma 2.2. [6] *Let $\{y_n\}$ be a sequence in an FM-space $(X, M, *)$ with $t * t > t$ for all $t \in [0, 1]$. If there exists a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, 3....$, then $\{y_n\}$ is a Cauchy sequence in X.

3. Compatible maps of type (α)

In this section, we give the concept of compatible maps of type (α) in FM-spaces and some properties of these maps.

Definition 3.1. [19] *Let A and B be maps from an FM-space $(X, M, *)$ into itself. The maps A and B are said to be compatible if*

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

Definition 3.2. [6] *Let A and B be maps from an FM-space $(X, M, *)$ into itself. The maps A and B are said to be compatible of type (α) if*

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

Remark 3.1. *In* [14], [15] *we can find the equivalent formulations of Definitions 4 and 5 and their examples in metric spaces. Such maps are independent of each other and more general then commuting and weakly commuting maps (*[13], [22]*).*

Proposition 3.1. [6] *Let $(X, M, *)$ be an FM-space with $t * t \geq t$ for all $t \in [0, 1]$ and A, B be continuous maps from X into itself. Then A and B are compatible if and only if they are compatible of type (α).*

Proposition 3.2. [10] *Let $(X, M, *)$ be an FM-space with $t * t \geq t$ for all $t \in [0, 1]$ and A, B be continuous maps from X into itself. If A and B are compatible of type (α) and $Az = Bz$ for some $z \in X$, then*

$$ABz = BBz = BAz = AAz.$$

Proposition 3.3. [6] *Let $(X, M, *)$ be an FM-space with $t * t \geq t$ for all $t \in [0, 1]$ and A, B be compatible maps of type (α) from X into itself. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Then we have the following:*

(i) $\lim_{n\to\infty} Bx_n = Az$ *if A is continuous at z,*

(ii) $ABz = BAz$ *and $Az = Bz$ if A and B are continuous at z.*

Example 3.1. Let $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define

$$M(x, y, t) = \frac{t}{d(x, y) + t}$$

for all $x, y \in X$. Clearly $(X, M, *)$ is a fuzzy metric space where $*$ is defined by $a * b = ab$.

Define $A, B : X \to X$ by

$$Ax = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & x \geq 1 \end{cases}, \quad Bx = \begin{cases} 2 - x^2 & 0 \leq x < 1 \\ 2 & x \geq 1 \end{cases}$$

Clearly A and B are discontinuous at $x = 1$. Consider the sequence $\{x_n\}$ in X defined by $x_n = 1 - \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 1 \in X.$$

Also

$$ABx_n \to 2, BBx_n \to 2 \text{ as } n \to \infty$$

and

$$\begin{aligned} BAx_n &= 1 - \frac{4}{n^2} - \frac{1}{n^4} + \frac{4}{n} + \frac{2}{n^3} - \frac{2}{n^2}, \\ AAx_n &= 1 + \frac{4}{n^2} + \frac{1}{n^4} - \frac{4}{n} - \frac{4}{n^3} + \frac{2}{n^2}. \end{aligned}$$

Then

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) \neq 1$$

but

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1$$

as $n \to \infty$. Thus A and B are compatible of type (α) but they are not compatible.

Example 3.2. Let $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define

$$M(x, y, t) = \frac{t}{d(x, y) + t}$$

for all $x, y \in X$. Clearly $(X, M, *)$ is a fuzzy metric space where $*$ is defined by $a * b = ab$. Define $A, B : X \to X$ by

$$Ax = \begin{cases} 1 + x & 0 \leq x < 1 \\ x & x \geq 1 \end{cases}, Bx = \begin{cases} 1 - x & 0 \leq x < 1 \\ 2x & x \geq 1 \end{cases}.$$

Clearly A and B are discontinuous at $x = 1$. Consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 1 \in X.$$

Further

$$\begin{aligned} ABx_n &= 2 - \frac{1}{n}, BAx_n = 2 + \frac{2}{n}, \\ AAx_n &= 1 + \frac{1}{n}, BBx_n = \frac{1}{n}. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} M(ABx_n, BAx_n, t) &= 1, \\ \lim_{n\to\infty} M(ABx_n, BBx_n, t) &\neq 1 \end{aligned}$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) \neq 1$$

as $n \to \infty$. Thus A and B are compatible but they are not compatible of type (α).

4. Main results

Theorem 4.1. *Let $(X, M, *)$ be a complete FM-space with $t * t \geq t$ for all $t \in [0, 1]$ and let A, B, P, Q, S and T be maps from X into itself such that-*

(i) $P(ST)(X) \subseteq AB(ST)(X), Q(AB)(X) \subseteq AB(ST)(X)$,

(ii) there exists a constant $k \in [0, 1)$ such that

$$\begin{aligned} & M^2(Px, Qy, kt) * [M(ABx, Px, kt)M(STy, Qy, kt)] \\ & *M^2(STy, Qy, kt) \geq \\ \geq \ & [pM(ABx, Px, t) + qM(ABx, STy, t)]M(ABx, Qy, 2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$, where $0 < p, q < 1$ such that $p + q = 1$,

(iii) A, B, S and T are continuous,

(iv) $AB = BA, ST = TS, PB = BP, TQ = QT, AB(ST) = ST(AB)$,

(v) the pairs (P, AB) and (Q, ST) are compatible of type (α).

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. By (i) we can construct a sequence $\{x_n\}$ in X as follows-

$$\begin{aligned} P(ST)x_{2n} &= AB(ST)x_{2n+1}, \\ Q(AB)x_{2n+1} &= AB(ST)x_{2n+2}, \end{aligned}$$

$n = 0, 1, 2, 3.....$

Let $z_n = AB(ST)x_n$, then by (ii),

$$\begin{aligned} & M^2(P(ST)x_{2n}, Q(AB)x_{2n+1}, kt) * \\ & [M(AB(ST)x_{2n}, P(ST)x_{2n}, kt)M(ST(AB)x_{2n+1}, Q(AB)x_{2n+1}, kt)] * \\ & M^2(ST(AB)x_{2n+1}, Q(AB)x_{2n+1}, kt) \geq \\ \geq\ & [pM(AB(ST)x_{2n}, P(ST)x_{2n}, t) + qM(AB(ST)x_{2n}, ST(AB)x_{2n+1}, t)] \\ & M(AB(ST)x_{2n}, Q(AB)x_{2n+1}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M^2(AB(ST)x_{2n+1}, AB(ST)x_{2n+2}, kt) * \\ & [M(z_{2n}, AB(ST)x_{2n+1}, kt)M(z_{2n+1}, AB(ST)x_{2n+2}, kt)] * \\ & M^2(z_{2n+1}, AB(ST)x_{2n+2}, kt) \geq \\ \geq\ & [pM(z_{2n}, AB(ST)x_{2n+1}, t) + qM(z_{2n}, z_{2n+1}, t)]M(z_{2n}, AB(ST)x_{2n+2}, 2kt), \end{aligned}$$

then

$$\begin{aligned} & M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)M(z_{2n+1}, z_{2n+2}, kt)] * \\ & M^2(z_{2n+1}, z_{2n+2}, kt) \geq \\ \geq\ & [pM(z_{2n}, z_{2n+1}, t) + qM(z_{2n}, z_{2n+1}, t)]M(z_{2n}, z_{2n+2}, 2kt), \end{aligned}$$

so

$$\begin{aligned} & M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)M(z_{2n+1}, z_{2n+2}, kt)] \geq \\ \geq\ & (p+q)M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M^2(z_{2n+1}, z_{2n+2}, kt)[M(z_{2n}, z_{2n+1}, kt) * M(z_{2n+1}, z_{2n+2}, kt)] \geq \\ \geq\ & (p+q)M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \geq \\ \geq\ & (p+q)M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt). \end{aligned}$$

Then it follows that

$$M(z_{2n+1}, z_{2n+2}, kt) \geq M(z_{2n}, z_{2n+1}, t),$$

for $0 < k < 1$ and for all $t > 0$.

Similarly, we also have

$$M(z_{2n+1}, z_{2n+3}, kt) \geq M(z_{2n+1}, z_{2n+2}, t),$$

for $0 < k < 1$ and for all $t > 0$.

In general,

$$M(z_{m+1}, z_{m+2}, kt) \geq M(z_m, z_{m+1}, t), 2n+1 = m = 1, 2, 3.....$$

for $0 < k < 1$ and for all $t > 0$.

Therefore by Lemma 2, $\{z_n\}$ is a Cauchy sequence in X. Since $(X, M, *)$ is complete, $\{z_n\}$ converges to a point $x \in X$ and since

$$\{P(ST)x_{2n}\} \text{ and } \{Q(AB)x_{2n+1}\}$$

are subsequences of $\{z_n\}, P(ST)x_{2n} \to z$ and $Q(AB)x_{2n+1} \to z$ as $n \to \infty$.

Let $y_n = STx_n$ and $w_n = ABx_n$ for $n = 1, 2, 3...$, then we have

$$Py_{2n} \to z, ABy_{2n} \to z, STw_{2n+1} \to z \text{ and } Qw_{2n+1} \to z$$

as $n \to \infty$.

Since the pairs (P, AB) and (Q, ST) are compatible of type (α), we have as $n \to \infty$

$$\begin{aligned} M(P(AB)y_{2n}, AB(AB)y_{2n}, t) &\to 1, \\ M((AB)Py_{2n}, PPy_{2n}, t) &\to 1, \\ M((ST)Qw_{2n+1}, QQy_{2n+1}, t) &\to 1, \\ M(Q(ST)w_{2n+1}, ST(ST)w_{2n+1}, t) &\to 1. \end{aligned}$$

Moreover, by the continuity of A, B, S and T and Proposition 3, we have

$$\begin{aligned} Q(ST)w_{2n+1} &\to STz, ST(ST)w_{2n+1} \to STz, \\ P(AB)y_{2n} &\to ABy_{2n}, AB(AB)y_{2n} \to ABy_{2n} \end{aligned}$$

as $n \to \infty$. Now taking $x = y_{2n}$ and $y = STw_{2n+1}$ in (ii), we have

$$\begin{aligned} & M^2(Py_{2n}, Q(ST)w_{2n+1}, kt) * \\ & [M(ABy_{2n}, Py_{2n}, kt)M(ST(ST)w_{2n+1}, Q(ST)w_{2n+1}, kt)] * \\ & M^2(ST(ST)w_{2n+1}, Q(ST)w_{2n+1}, kt) \geq \\ \geq \; & [pM(ABy_{2n}, Py_{2n}, t) + qM(ABy_{2n}, ST(ST)w_{2n+1}, t)] \\ & M(ABy_{2n}, Q(ST)w_{2n+1}, 2kt). \end{aligned}$$

This implies as $n \to \infty$

$$\begin{aligned} & M^2(z, STz, kt) * [M(z, z, kt)M(STz, STz, kt)] * \\ & M^2(STz, STz, kt) \geq \\ \geq \; & [pM(z, z, t) + qM(z, STz, t)]M(z, STz, 2kt). \end{aligned}$$

Then it follows that

$$M^2(z, STz, kt) \geq [p + qM(z, STz, t)]M(z, STz, 2kt)$$

and since $M(x, y, .)$ is non decreasing for all $x, y \in X$, we have

$$M(z, STz, 2kt)M(z, STz, t) \geq [p + qM(z, STz, t)]M(z, STz, 2kt).$$

Thus

$$\begin{aligned} M(z, STz, t) &\geq p + qM(z, STz, t) \\ &\Rightarrow M(z, STz, t) \geq \frac{p}{1-q} = 1 \end{aligned}$$

for all $t > 0$.

So $z = STz$. Similarly $z = ABz$.

Now taking $x = y_{2n}$ and $y = z$ in (ii), we have

$$\begin{aligned} & M^2(Py_{2n}, Qz, kt) * [M(ABy_{2n}, Py_{2n}, kt)M(STz, Qz, kt)] * \\ & M^2(STz, Qz, kt) \geq \\ \geq \quad & [pM(ABy_{2n}, Py_{2n}, t) + qM(ABy_{2n}, STz, t)]M(ABy_{2n}, Qz, 2kt). \end{aligned}$$

This implies as $n \to \infty$

$$M^2(z, Qz, kt) * M(z, Qz, kt) \geq (p+q)M(z, Qz, 2kt),$$

so

$$M(z, Qz, kt)[M(z, Qz, kt) * 1] \geq M(z, Qz, 2kt)$$

and since $M(x, y, .)$ is non decreasing for all $x, y \in X$, we have

$$M(z, Qz, 2kt)M(z, Qz, t) \geq M(z, Qz, 2kt).$$

Then it follows that $M(z, Qz, t) = 1$ for all $t > 0$. So $z = Qz$. Similarly we have $z = Pz$.

Now we show $Bz = z$ and $Tz = z$. Taking $x = Bz$ and $y = z$ in (ii), we get

$$\begin{aligned} & M^2(P(Bz), Qz, kt) * [M(AB(Bz), P(Bz), kt)M(STz, Qz, kt)] * \\ & M^2(STz, Qz, kt) \geq \\ \geq \quad & [pM(AB(Bz), P(Bz), t) + qM(AB(Bz), STz, t)]M(AB(Bz), Qz, 2kt), \end{aligned}$$

which gives

$$\begin{aligned} & M^2(Bz, z, kt) * [M(Bz, Bz, kt)M(z, z, kt)] * M^2(z, z, kt) \geq \\ \geq \quad & [pM(Bz, Bz, t) + qM(Bz, z, t)]M(Bz, z, 2kt) \\ & \Rightarrow M^2(Bz, z, kt) \geq [p + qM(Bz, z, t)]M(Bz, z, 2kt) \end{aligned}$$

and since $M(x, y, .)$ is non decreasing for all $x, y \in X$, we have

$$M(Bz, z, 2kt)M(Bz, z, t) \geq [p + qM(Bz, z, t)]M(Bz, z, 2kt).$$

Thus

$$\begin{aligned} M(Bz, z, t) \quad & \geq \quad p + qM(Bz, z, t) \\ & \Rightarrow \quad M(Bz, z, t) \geq \frac{p}{1-q} = 1 \end{aligned}$$

for all $t > 0$.

So $Bz = z$.Similarly we have $Tz = z$.

Since $z = ABz$, therefore $Az = z$ and since $Tz = z$ therefore $Sz = z$. By combining the above results, we have

$$Az = Bz = Sz = Tz = Pz = Qz = z,$$

that is, z is the common fixed point of A, B, P, Q, S and T.

To prove uniqueness, let $v \neq z$ be another fixed point of A, B, P, Q, S and T. Then using (ii),

$$\begin{aligned} & M^2(Pz, Qv, kt) * [M(ABz, Pz, kt)M(STv, Qv, kt)] * \\ & M^2(STv, Qv, kt) \geq \\ \geq \quad & [pM(ABz, Pz, t) + qM(ABz, STv, t)]M(ABz, Qv, 2kt) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & M^2(z, v, kt) * [M(z, z, kt)M(v, v, kt)] * \\ & M^2(v, v, kt) \geq \\ \geq \quad & [pM(z, z, t) + qM(z, v, t)]M(z, v, 2kt) \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad & M^2(z,v,kt) * [M(z,z,kt)M(v,v,kt)] * \\ & M^2(v,v,kt) \geq \\ \geq\quad & [pM(z,z,t) + qM(z,v,t)]M(z,v,2kt) \end{aligned}$$

$$\Rightarrow M^2(z,v,kt) \geq [p + qM(z,v,t)]M(z,v,2kt)$$

and since $M(x,y,.)$ is non decreasing for all $x, y \in X$, we have

$$M(z,v,kt)M(z,v,2kt) \geq [p + qM(z,v,t)]M(z,v,2kt).$$

Thus it follows that $M(z,v,t) \geq \dfrac{p}{1-q} = 1$ for all $t > 0$.

So $v = z$. Hence A, B, P, Q, S and T have a unique common fixed point. □

Remark 4.1. *The above theorem extends the result of Turkoglu et al* [29].

If we put $B = T = I$, (the identity map on X) in the Theorem 4.1, we have the following result due to Turkoglu et al [29].

Corollary 4.1. *Let $(X, M, *)$ be a complete FM-space with $t * t \geq t$ for all $t \in [0,1]$ and let A, P, S and Q be maps from X into itself such that-*
(i) $PS(X) \subseteq AS(X), QA(X) \subseteq AS(X)$,
(ii) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} & M^2(Px,Qy,kt) * [M(Ax,Px,kt)M(Sy,Qy,kt)] \\ & *M^2(Sy,Qy,kt) \geq \\ \geq\quad & [pM(Ax,Px,t) + qM(Ax,Sy,t)]M(Ax,Qy,2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$, where $0 < p, q < 1$ such that $p + q = 1$,
(iii) A and S are continuous,
(iv) $AS = SA$,
(v) the pairs (P, A) and (Q, S) are compatible of type (α).
Then A, P, S and Q have a unique common fixed point in X.

If we put $A = S$, $B = T$ and $P = Q$ in the Theorem 4.1, we have the following:

Corollary 4.2. *Let $(X, M, *)$ be a complete FM-space with $t * t \geq t$ for all $t \in [0,1]$ and let A, B and P be maps from X into itself such that-*
(i) $P(AB)(X) \subseteq AB$,
(ii) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} & M^2(Px,Py,kt) * [M(ABx,Px,kt)M(ABy,Py,kt)] \\ & *M^2(ABy,Py,kt) \geq \\ \geq\quad & [pM(ABx,Px,t) + qM(ABx,ABy,t)]M(ABx,Py,2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$, where $0 < p, q < 1$ such that $p + q = 1$,
(iii) A and B are continuous,
(iv) $AB = BA, PB = BP$,
(v) the pair (P, AB) is compatible of type (α).
Then A, B and P have a unique common fixed point in X.

The following example illustrates our main theorem.

Example 4.1. Let $X = [-1, 1]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define

$$M(x, y, t) = \frac{t}{d(x, y) + t}$$

for all $x, y \in X$. Clearly $(X, M, *)$ is a complete fuzzy metric space where $*$ is defined by $a * b = ab$. Let A, B, P, Q, S and T be maps from X into itself defined as

$$Ax = \frac{x}{2}, Bx = \frac{x}{8}, \ Sx = \frac{x}{3}, Tx = \frac{x}{5}, \ Px = \frac{x}{16}, Qx = \frac{x}{15}.$$

Then

$$P(ST)(X) = \left[\frac{-1}{240}, \frac{1}{240}\right] \subseteq AB(ST)(X) = \left[\frac{-1}{240}, \frac{1}{240}\right]$$

and

$$Q(AB)(X) = \left[\frac{-1}{240}, \frac{1}{240}\right] \subseteq AB(ST)(X) = \left[\frac{-1}{240}, \frac{1}{240}\right].$$

Thus (i) is satisfied. Also (iii) and (iv) are satisfied. Now define a sequence $\{x_n\}$ in X such that $x_n = \dfrac{n}{n+1}$. Then

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = \frac{1}{16},$$

$$\lim_{n\to\infty} M(P(AB)x_n, AB(AB)x_n, t) = 1$$

and

$$\lim_{n\to\infty} M((AB)Px_n, PPx_n, t) = 1.$$

Thus the pair (P, AB) is compatible of type (α). Similarly,

$$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} STx_n = \frac{1}{15},$$

$$\lim_{n\to\infty} M(Q(ST)x_n, ST(ST)x_n, t) = 1$$

and

$$\lim_{n\to\infty} M((ST)Qx_n, QQx_n, t) = 1.$$

Therefore the pair (Q, ST) is also compatible of type (α). For $p = \frac{7}{8}, q = \frac{1}{8}, \ k = \frac{1}{4}$ we can see that the condition (ii) is satisfied. Hence all the conditions of our main Theorem 1 are satisfied and the unique common fixed point is $x = 0$.

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Bhavana Deshpande is a Professor of Mathematics in Department of Mathematics of Govt. P. G. Arts and Science College, Ratlam (MP), India. She has 30 years of teaching experience of teaching undergraduate and postgraduate students of Mathematics. Her research areas are fixed point theory and fuzzy set theory. She has more than 63 research papers on fixed point theorems published in referred International Journals.

Rohit Pathak is a Sr. Lecturer of Mathematics in Department of Applied Sciences in Institute of Engineering and Technology, DAVV, Indore, MP, India. He has 12 years of teaching experience of teaching undergraduate and postgraduate students of Mathematics and Management. His research area is fixed point theory and its applications. Currently he is a teaching engineering mathematics to students.