TWMS J. Pure Appl. Math. V.6, N.2, 2015, pp.202-212

ON THE THEORY OF INFINITE SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS*

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Abstract. The work provides an overview of the author's papers, which formed the basis of the new theory of general infinite systems of linear algebraic equations. In these papers the author extended the Gaussian elimination and Cramer's rule to infinite systems. The special particular solution, so-called strictly particular solution, of inhomogeneous infinite systems was received. This strictly particular solution is written as a series. The divergence of this series demonstrates incompatibility of the original system. The necessary and sufficient conditions for the existence of nontrivial solutions of the homogeneous infinite systems are considered.

Keywords: infinite systems, rank and decrement, infinite matrices and determinants, strictly particular solution.

AMS Subject Classification: 15A06, 15A15

1. Introduction

The theory of infinite systems of linear algebraic equations with an infinite number of the unknowns started to be developed since the late 19th century with joint efforts of such great mathematicians as Poincare, Fredholm, Hilbert, Riesz. Although previously it was Fourier (1807), who tried to solve one particular infinite system in order to expand the function with some properties in a trigonometric series. Infinite systems attracted these great mathematicians with absolutely different positions; while Fredholm and Hilbert sparked interest in connection with the solution of integral equations, Poincare and G. Hill were actually interested in terms of solutions of ordinary differential equations; as for Fourier, he applied them to the expansion of a trigonometric series with a view to solve the boundary value problems of mathematical physics. Naturally, such a wide range of applications of infinite systems could not be generate interest of mathematicians around the world. However, being extremely complex on the one hand, this theory, on the other hand, has a very rich content and a wide range of applications in many fields of mathematics. All this made mathematicians go the way of research of particular classes of infinite systems. For more than one hundred years of infinite systems study extensive literature has been accumulated. Until now, however, the theory of infinite systems has not completely formed yet. Currently almost a dozen of infinite systems classes have been completely studied:

* This work was supported by the Russian Ministry of Education and Science within the base part of the state task to North-Eastern Federal University (project no. 3047)

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Manuscript received June 2014.

normal systems, regular and completely regular systems, multiplicative systems, with difference-index and others. Recently, the author has discovered and described in detail a new class of infinite systems, called the class of periodic infinite systems [9]. The theory of periodic infinite systems enabled the study of general infinite systems and allowed to move on from the crisis point in recent years. In the author's recent monograph [10] classes of infinite systems have been studied systematically since their emergence as independent theory.

Basic information, concepts and definitions of infinite systems and matrices, determinants sufficiently are available from the sources [9, 10, 28, 19, 20, 7, 27].

In this paper we highlight the main aspects of the general theory of infinite systems proposed and promoted by us [9, 10, 14, 8, 12, 13, 15, 16, 11]. The extensive literature on infinite systems is reviewed in monograph [10]. This overview allow to argue that the theory of infinite systems had been developed only for a narrow class at the time when the monograph [10] was published (there are 201 bibliographies). In particular, it is pointed out, for example, in works [29, 2, 22, 1, 5, 23]. This tendency is also observed at the present time [30, 4, 25, 3, 24, 6].

It must be emphasized that one chapter of monograph [10] is devoted to research on the infinite systems with difference-index (index of coefficients of the system is the difference between i and j; also it is called "discrete Wiener-Hopf equations" [21, 18]). These systems are the simplest type of periodic systems and they are completely investigated in monograph [9]. We note that even homogeneous infinite systems with difference-index cannot be studied by the methods of work similar to [21, 18] not to mention a general infinite systems. The research of inhomogeneous and homogeneous infinite systems is based on fundamentally different approaches. Furthermore the convergence of approximate methods (such as method of successive approximations or projection method) for solving a inhomogeneous infinite systems does not guarantee a real solution of these systems. The approximate solutions can be the real solutions of these systems if only the systems are consistent. Otherwise it is not possible to check whether these solutions satisfy the infinite systems (in report [26] there are some examples which point out this statement).

Let the infinite system of linear algebraic equations with an infinite number of unknowns [10, 20] be given by:

$$\left.\begin{array}{r} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n + ... = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n + ... = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,n}x_n + ... = b_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{1}$$

where $a_{i,k}$ are known coefficients, b_i are the constant terms and x_k – unknown from some field F.

Set of numerical values of the variables x_1, x_2, ... is called *a solution of system* (1) if, after substituting these values in the left-hand side of (1) we obtain a convergent series, and all these equalities are satisfied.

In the case of the solvability, the infinite system is called *consistent*, otherwise – *inconsistent*.

In the future, the system (1) will be called simply infinite system (1).

Under *the infinite matrix* we consider the table of coefficients of an infinite system (1):

$$A = (a_{i,j}) = A(a_{i,j})_1^{\infty} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & \dots \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & \dots \\ . & . & \dots & . & \dots \\ . & . & \dots & . & \dots \\ . & . & \dots & . & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & \dots \\ . & . & \dots & . & \dots \end{pmatrix}, \tag{2}$$

which is called *the (general) coefficient matrix* of the system (1), and matrix

$$\overline{A} = \begin{pmatrix} b_1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} & \dots \\ b_2 & a_{2,1} & a_{2,2} & \dots & a_{2,n} & \dots \\ . & . & . & \dots & . & \dots \\ . & . & . & \dots & . & \dots \\ . & . & . & \dots & . & \dots \\ b_n & a_{n,1} & a_{n,2} & \dots & a_{n,n} & \dots \\ . & . & . & \dots & . & \dots \end{pmatrix}, \tag{3}$$

– *the augmented matrix* of the system (1). Then, an infinite system (1) is equivalent to a matrix equation of the form

$$AX = B, \tag{1'}$$

where the X is a column vector of unknowns and B is a column vector of constant terms of (1).

Having highlighted the elements contained in the first n columns and the first n rows of the matrix A, we may obtain the determinant of n-th order – $D_n = |A_n|$. Value of this determinant, obviously depends on n, i.e., on the order of the obtained determinant. And the $D_n = |A_n|$ is called *the main* determinant of n-th order generated by the matrix A.

If the value of the main determinant tends to certain limit $|A|$ as its order n increases without limit, then we say that there is *an infinite determinant* of coefficient matrix A, and that $|A|$ is the value of this determinant [19, 20, 27].

We do not do any restrictive assumptions on the coefficients and the constant terms of system (1). In all known modern studies [10, 20] it is assumed that constant terms are limited altogether, i.e. $|b_j| \le K > 0$. In addition, for the coefficients of system there is the weakest restrictive that is assumed, is concluded in the following: $\sum\limits_{i=1}^{\infty} |d_{i,j}| < \infty$, $a_{i,i} = 1 + d_{i,i}$, $a_{i,j} = d_{i,j}$. Systems, for which these assumptions are not fulfilled, a priori are excluded as subject of investigation, and this ultimately led to critical situation. There is an example of such system:

$$\sum_{p=0}^{\infty} \frac{(2j+2p)!}{(2p)!} x_{j+p} = b^j, \;\; j = \overline{0,\infty}, \quad b = const > 0. \tag{4}$$

It is obvious that in each equation of (4) the sum of the absolute value of the coefficient is infinite

$$\sum_{p=0}^{\infty} |a_{j,j+p}| = \sum_{p=0}^{\infty} \frac{(2j+2p)!}{(2p)!} = \infty, \;\; \forall \; j = 0, 1, 2, ...,$$

besides, the constant terms b^j for $b > 1$ are not bounded. Nevertheless, this system has a solution. Moreover, we were able to find a particular solution of the inhomogeneous system (4) and the fundamental solution of homogeneous ($b = 0$) system (4) [9, 10]:

$$x_i^{(k)} = \frac{b^i}{(2i)!\mathrm{ch}(\sqrt{\mathrm{b}})} + \frac{(-1)^i \pi^{2i} (2k+1)^{2i} x_0}{(2i)! 2^{2i}}, \quad i, k = 0, 1, 2, \dots, \tag{5}$$

and hence we have found out its general solution.

It should be emphasized, that the systems of type (4) appeared in solving of quite real boundary value problems of mathematical physics by the boundary method similar to work [8], i.e., they are not some kind of abstract systems.

Several examples of this type are given in [9, 10, 14, 8]. In these papers we solved some boundary value problems of mathematical physics by use of boundary method [8].

Thus, in our investigations we will avoid any restrictive assumptions on the coefficients and the constant terms of system (1) in advance. Only some necessary algebraic assumptions may be possible.

Our the main task is to extend the Gaussian elimination to infinite systems (1).

2. Gaussian elimination

It is known [19, 17], that the infinite matrix A has finite rank only in exceptional cases, moreover, the infinite matrix can have finite rank if and only if its determinant equals zero, and that is not always so.

Let the infinite determinant $|A|$ of system (1) is nonzero, then, obviously, the infinite matrix A has infinite rank.

Using the theory of Gauss method for finite systems, as set out in the monograph of F.R. Gantmakher [17], we obtained the theorem that extends the Gaussian elimination to infinite systems (1) [12]:

Theorem 2.1. *Every matrix $A(a_{i,k})_1^\infty$ of infinite rank, which has the sequence of principal minors, that are non-zero, i.e., $D_k \neq 0$ $(k = 1, 2, ..., \infty)$ can be represented as a product of triangular matrix B by Gaussian infinite matrix C $(c_{i,i} \neq 0)$:*

$$A = BC = \begin{pmatrix} b_{1,1} & 0 & \dots & 0 & \dots \\ b_{2,1} & b_{2,2} & \dots & 0 & \dots \\ . & . & \dots & . & \dots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} & \dots \\ . & . & \dots & . & \dots \end{pmatrix} \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} & \dots \\ 0 & c_{2,2} & \dots & c_{2,n} & \dots \\ . & . & \dots & . & \dots \\ 0 & 0 & \dots & c_{n,n} & \dots \\ . & . & \dots & . & \dots \end{pmatrix}. \tag{6}$$

In this

$$b_{1,1}c_{1,1} = D_1, \quad b_{2,2}c_{2,2} = \frac{D_2}{D_1}, \dots, \quad b_{n,n}c_{n,n} = \frac{D_n}{D_{n-1}} \dots, \tag{7}$$

$$b_{j,k} = b_{k,k} \frac{A\begin{pmatrix} 1\,2 \dots k-1\,j \\ 1\,2 \dots k-1\,k \end{pmatrix}}{A\begin{pmatrix} 1\,2 \dots k \\ 1\,2 \dots k \end{pmatrix}}, \quad c_{j,k} = c_{k,k} \frac{A\begin{pmatrix} 1\,2 \dots k-1\,k \\ 1\,2 \dots k-1\,j \end{pmatrix}}{A\begin{pmatrix} 1\,2 \dots k \\ 1\,2 \dots k \end{pmatrix}} \tag{8}$$

$$(j = k, k+1, ..., \infty; \quad k = 1, 2, ..., \infty),$$

where

$$A\begin{pmatrix} i_1 i_2 \dots i_p \\ k_1 k_2 \dots k_p \end{pmatrix} = \begin{vmatrix} a_{i_1,k_1} & a_{i_1,k_2} & \dots & a_{i_1,k_p} \\ a_{i_2,k_1} & a_{i_2,k_2} & \dots & a_{i_2,k_p} \\ . & . & \dots & . \\ a_{i_p,k_1} & a_{i_p,k_2} & \dots & a_{i_p,k_p} \end{vmatrix}, \quad D_n = A\begin{pmatrix} 1\,2 \dots n \\ 1\,2 \dots n \end{pmatrix}, \; n = 1, 2, ..., \infty.$$

Diagonal elements of the matrices B and C can be an arbitrary numbers satisfying conditions in (7). It should be noted that all diagonal elements $b_{i,i}$, $a_{i,i}$ of matrices B and C are not equal to zero, since it is assumed that the matrix A has a nonzero determinant.

Corollary 2.1. *Elements of columns of the matrix B and rows of the matrix C are associated with the the matrix A elements by recurrence relations:*

$$b_{i,k} = \frac{a_{i,k} - \sum\limits_{j=1}^{k-1} b_{i,j} c_{j,k}}{c_{k,k}}, \quad i \geq k;\ i = 1, 2, ..., \infty;\ k = 1, 2, ..., \infty, \tag{9}$$

$$c_{i,k} = \frac{a_{i,k} - \sum\limits_{j=1}^{i-1} b_{i,j} c_{j,k}}{b_{i,i}}, \quad i \leq k;\ i = 1, 2, ..., \infty;\ k = 1, 2, ..., \infty. \tag{10}$$

In work [7] for systems (1) with a triangular matrix the following theorem is proved:

Theorem 2.2. *Infinite system (1) with a triangular matrix having all elements of the main diagonal being not equal to zero, has the unique right side inverse matrix, which is a triangular matrix with all diagonal elements are equal $\frac{1}{a_{i,i}}$ respectively.*

It should be noted that:

Note 1. Let A be the triangular matrix and $a_{i,i} \neq 0$ for all i, so by Theorem 2.2 it has a unique right side inverse matrix X. Then X is also left side inverse matrix to A, and hence it is the unique two-side inverse matrix to A.

Corollary 2.2. *If the diagonal elements $b_{i,i}\,(i = 1, 2, ...\infty)$ of the matrix B are equal to the unity, we obtain the Gaussian elimination for infinite systems.*

Proof. Since the matrix B is triangular, then according to Note 1 it has the unique two-side inverse matrix B^{-1}. Therefore, based on the matrix equation $(1')$, the following ratio is valid: $AX = BCX = F$ and $B^{-1}BCX = B^{-1}F$, where $CX = B^{-1}F$ and, besides on the basis of Theorem 2.1 the matrix C is a Gaussian matrix.

3. Method of reduction

Infinite systems work analysis [10] showed that the most effective way to solve the system (1) is a combination of the simple reduction method and successive approximations method. However, in this case the convergence of method of reduction becomes dependent on the convergence of the method of successive approximations. To eliminate such situation, one must clearly separate these processes. Such opportunity is given by the Gaussian elimination for infinite system (1). Thus, in the conditions of Theorem 2.1 instead of the general system (1) we can consider the infinite system in the Gaussian form ($a_{j,j} \neq 0$ for all j):

$$\sum_{p=0}^{\infty} a_{j,j+p} x_{j+p} = b_j, \quad j = 0, 1, 2, \ldots. \tag{11}$$

On the one hand, when we apply the simple reduction method to system (11) it becomes possible to find an exact solution of a finite truncated system of n-th order without use of the successive approximations method. On the other hand, this exact solution can be expressed by iterative method (on order of reduction n).

Before applying the simple reduction method to system (11), the following should be noted. We introduced a different interpretation of the reduction method [9, 10]. If in the reduction method for solving infinite systems of algebraic equations the number of unknowns and the number of equations remain the same in the truncated system, then we can say that reduction method *is understood in the narrow sense* (simple reduction method), and if the number of unknowns is greater than the number of equations, then we say that the method of reduction *is understood in a broad sense.* Such separation proved to be necessary as it is impossible to find a nontrivial solution of the homogeneous infinite system by simple reduction method if it exists.

Let the infinite system (11) is truncated by the reduction method in the narrow sense:

$$\sum_{p=0}^{n-j} a_{j,j+p} \overset{n}{x}_{j+p} = b_j, \quad a_{j,j} \neq 0, \quad j = \overline{0,n}. \tag{12}$$

Theorem 3.1. *Solution of the finite system (12) is the expression:*

$$\overset{n}{x}_j = B_{n-j}, \quad j = 0,1,...,n, \tag{13}$$

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=0}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p, \quad B_0 = \frac{b_n}{a_{n,n}}, \qquad j = \overline{0, n-1}. \tag{14}$$

Suppose that in (14) it is possible to pass term-by-term to the limit in the next formula

$$\lim_{n\to\infty} \sum_{p=j+1}^{n} \frac{a_{j,p}}{a_{j,j}} B_{n-p} = \sum_{p=j+1}^{\infty} \frac{a_{j,p}}{a_{j,j}} \lim_{n\to\infty} B_{n-p}. \tag{14$'$}$$

In [13] the concept of strictly particular solution of inhomogeneous infinite systems was firstly introduced.

Further we assume the next condition is hold: a) let the limit $\lim\limits_{n\to\infty} B_{n-j} = B(j)$ exists, besides not all of $B(j)$ equal 0.

Theorem 3.2. *Let the condition a) and the equality (14') hold then the limit value $B(j)$ is a particular solution of infinite system (11)*

Definition 1. *Particular solution $x_j = B(j)$ of inhomogeneous infinite Gaussian system (11) is called strictly particular solution of system (11).*

Theorem 3.3. *Under the fulfillment of the condition a), the passage to the limit in (14) is possible if and only if the set of $B(j)$, $j = 0,1,\dots$ is a strictly particular solution of infinite Gaussian system (11).*

Theorem 3.4. *The inhomogeneous infinite Gaussian system (11) is consistent if and only if the strictly particular solution of it exists.*

Only the limits $\lim\limits_{n\to\infty} B_{n-j} = B(j)$ should be found. In [15, 16] it is shown that the (14) actually defines a special determinant:

$$B_{n-j} = \begin{vmatrix} b_j & b_{j+1} & b_{j+2} & ... & b_{n-2} & b_{n-1} \\ a_{j,j+1} & 1 & 0 & ... & 0 & 0 \\ a_{j,j+2} & a_{j+1,j+2} & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ a_{j,j+k} & a_{j+1,j+k} & a_{j+2,j+k} & ... & 0 & 0 \\ . & . & . & ... & . & . \\ a_{j,n-2} & a_{j+1,n-2} & a_{j+2,n-2} & ... & 1 & 0 \\ a_{j,n-1} & a_{j+1,n-1} & a_{j+2,n-1} & ... & a_{n-2,n-1} & 1 \end{vmatrix}, \tag{15}$$

where $B_0 = b_n$. Here, without loss of generality, we puted $a_{j,j} = 1$, and in the general case, instead of b_{j+p} in (15) we mean $\frac{b_{j+p}}{a_{j+p,j+p}}$, and instead of $a_{j+p,j+k}$ we mean $\frac{a_{j+p,j+k}}{a_{j+p,j+p}}$, $p = 0, 1, ..., n-j-1$, $k = 1, 2, ..., n-j-1$.

Determinant (15) in the infinite case, if it exists, will obviously have the form

$$B(j) = \begin{vmatrix} b_j & b_{j+1} & b_{j+2} & \dots & b_{n-1} & . \\ a_{j,j+1} & 1 & 0 & \dots & 0 & . \\ a_{j,j+2} & a_{j+1,j+2} & 1 & \dots & 0 & . \\ . & . & . & \dots & . & . \\ a_{j,j+k} & a_{j+1,j+k} & a_{j+2,j+k} & \dots & 0 & . \\ . & . & . & \dots & . & . \\ a_{j,n-1} & a_{j+1,n-1} & a_{j+2,n-1} & \dots & 1 & . \\ . & . & . & \dots & . & . \end{vmatrix}, \tag{16}$$

Thus, to find the limits $\lim\limits_{n\to\infty} B_{n-j}$ we should calculate the infinite determinant (16). Deleting the first row in the determinant (16), we form sequence of principal minors of the obtained determinant, while we assume $A_0(j) = 1$, and for the rest of $n > 0$, we obtain

$$A_n(j) = \begin{vmatrix} a_{j,j+1} & 1 & \dots & 0 & 0 \\ a_{j,j+2} & a_{j+1,j+2} & \dots & 0 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ a_{j,j+n-1} & a_{j+1,j+n-1} & \dots & a_{j+n-2,j+n-1} & 1 \\ a_{j,j+n} & a_{j+1,j+n} & \dots & a_{j+n-2,j+n} & a_{j+n-1,j+n} \end{vmatrix}. \tag{17}$$

The sequence of determinants (17) is called *the characteristic sequence of Gaussian system* (11).

Theorem 3.5. *Characteristic sequence (17) is calculated by recurrent relation:*

$$A_p(j) = \sum_{k=0}^{p-1} (-1)^{p-1-k} a_{j+k,j+p} A_k(j), \quad A_0(j) = 1. \tag{18}$$

In work [16] the determinant (15) is calculated:

Theorem 3.6. *The following takes place:*

$$B_{n-j} = \sum_{p=0}^{n-j} (-1)^p A_p(j) b_{j+p}, \quad j = 0, 1, ...n\,, \tag{19}$$

where $A_p(j)$ is calculated by recurrent relation (18).

Theorem 3.7. *Let the Gaussian system (11) be consistent, then the strictly particular solution of (11) is expressed by the formula:*

$$\lim_{n\to\infty} \overset{n}{x}_j = x_j = \lim_{n\to\infty} B_{n-j} = B(j) = \sum_{p=0}^{\infty} (-1)^p A_p(j) b_{j+p}, \, , j = 0, ..., \infty. \tag{20}$$

Theorem 3.8. *The strictly particular solution of (11) is expressed by Cramer's formula:*

$$x_j = B(j) = \sum_{p=0}^{\infty} (-1)^p A_p(j) b_{j+p} = \frac{\Delta^{(j+1)}}{\Delta}, \quad j = 0, 1, ...\infty\,, \tag{21}$$

where $\Delta = |A|$ and $\Delta^{(j+1)}$ is the determinant of matrix A, where $j+1$-th column (j starts with zero) is replaced by the column of constant terms of system (11).

The numerical implementation of the formula (20) is given in [13].

It is worth mentioning the critical importance of strictly particular solution properties. This solution possesses the following properties:

1) The strictly particular solution is obtained by the reduction in the narrow sense. Thus, the existence of a strictly particular solution proves the convergence of the reduction method.

2) The strictly particular solution is unique and can be expressed by Cramer's formula. It follows that Cramer's formula for infinite system is obtained from Cramer's formula for finite truncated Gaussian system by use of the passage to the limit.

3) The strictly particular solution does not contain a nontrivial solution of the corresponding homogeneous system.

In the solution (5) of example (4), the first term on the right-hand side is a strictly particular solution of the system (4), and a linear combination of the second terms in k is the general solution of the homogeneous system. In connection with this we see the validity of the property 3) for a strictly particular solution of (4).

4) The strictly particular solution is *the principal solution* [20] of the infinite system.

Up to the present, the method of successive approximations was used to find the principal solution without linking it with the consistency of the system (1).

4. The existence of solutions of infinite systems

By Theorem 3.4 to prove the existence of solution of infinite system is sufficient to investigate the existence of its strictly particular solution.

Using characteristic determinants (17) $A_p(j)$ we generate vectors $\overline{a}_j = \{A_0(j), -A_1(j), ..., (-1)^p A_p(j), ...\}$, and using the constant terms b_j – vectors $\overline{b}_j = \{b_j, b_{j+1}, b_{j+2}, ...\}$, $j \geq 0$. Then, the Theorem 3.7 can be rewritten as follows:

Theorem 4.1. *Let the inhomogeneous Gaussian system (11) be consistent, then its strictly particular solution x_j is the scalar product of vectors $\overline{a}_j$ and $\overline{b}_j$:*

$$x_j = B(j) = (\overline{a}_j, \overline{b}_j), \quad j = 0, 1, \dots .$$

Theorem 4.2. *If the series in (20) diverges at least for one $j = j_0$, i.e. the scalar product of vectors $\overline{a}_{j_0}$ and $\overline{b}_{j_0}$ is not limited, then Gaussian system (11) is inconsistent.*

Theorem 4.3. *If vector $\overline{a}_j$ is orthogonal to vector $\overline{b}_j$ for all j, i.e.*

$$(\overline{a}_j, \overline{b}_j) = 0, \ \forall j,$$

but $b_{j_0} \neq 0$ for some $j = j_0$, then Gaussian system (11) is inconsistent.

5. The existence of nontrivial solutions of the homogeneous infinite systems

As stated above, if we solve the homogeneous Gaussian system by the reduction in the narrow sense then we get only a trivial solution. Therefore, to find the solution of homogeneous Gaussian systems, it is necessary to apply the reduction method in a broad sense.

When we solve an infinite system by the reduction in the broad sense [9, 10], an infinite system (1) is truncated to finite system in which the number of unknowns is on one more than the number of equations. Therefore, we shall consider the truncated system of type (12), i.e., a finite system of n first equations with the $(n+1)$ unknowns: $x_0, x_1, ...x_n$. For such finite systems the following results were obtained in [15].

Theorem 5.1. *Let the following finite system holds*

$$\sum_{p=0}^{n-j} a_{j,j+p}x_{j+p} = b_j, \quad a_{j,j} \neq 0, \quad j = \overline{0, n-1}. \tag{22}$$

Then unknowns x_i are expressed by x_0 as follows:

$$x_i = B_{n-i} + \frac{(-1)^{i+1}B_n}{\prod\limits_{p=1}^{i} S_{n-i+p}} + \frac{(-1)^i x_0}{\prod\limits_{p=1}^{i} S_{n-i+p}} \qquad i = \overline{1, n}, \tag{23}$$

where

$$B_j = \frac{b_{n-j}}{a_{n-j,n-j}} - \sum_{p=1}^{j-1} \frac{a_{n-j,n-p}}{a_{n-j,n-j}} B_p, \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}, \qquad j = \overline{2, n}, \tag{24}$$

and

$$S_j = \frac{a_{n-j,n-j+1}}{a_{n-j,n-j}} + \sum_{p=2}^{j} \frac{(-1)^{p+1}a_{n-j,n-j+p}}{a_{n-j,n-j}\prod_{k=1}^{p-1} S_{j-k}},$$
$$S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \qquad j = \overline{2, n}. \tag{25}$$

Here x_0 – is an arbitrary real number.

Corollary 5.1. *In the system (22) the neighboring unknowns are related to each other as follows*

$$x_i = B_{n-i} + S_{n-i}B_{n-i-1} - S_{n-i}x_{i+1}, \qquad i = \overline{0, n-1}. \tag{26}$$

It should be noted that in the case of infinite system (11) in expressions (23) and (26) the approximate values $\overset{n}{x}_i$ of unknown values x_i of (11) is taken for x_i.

Obviously, recurrence relations (24) and (25) can be rewritten respectively as:

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=1}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}, \qquad j = \overline{0, n-2}, \tag{27}$$

$$S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j} \frac{(-1)^{p+1}a_{j,j+p}}{a_{j,j}\prod\limits_{k=1}^{p-1} S_{n-j-k}}, \ S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \qquad j = \overline{0, n-2}. \tag{28}$$

It is easy to verify that the limits of expressions (14) and (27) are the same in case of their existence.

If we assume that there is a limit $\lim\limits_{n\to\infty} S_{n-j} = S(j)$ and the passage to the limit is possible in (28) as well as in (14′), then the following equality holds for each j:

$$\sum_{p=0}^{\infty} \frac{(-1)^p a_{j,j+p}}{a_{j,j}\prod\limits_{k=0}^{p-1} S(j+k)} = 0, \quad j = 0, 1, 2, \dots, \tag{29}$$

where to unify the notation, the following is adopted $\prod\limits_{k=0}^{-1} S(j+k) = 1$ for $\forall j$.

The following theorem gives conditions for the existence of nontrivial solution for general homogeneous infinite Gaussian systems.

Theorem 5.2. *The necessary and sufficient condition for the existence of nontrivial solution of the homogeneous Gaussian system (11) is the fulfillment of conditions (29) for each j. When the conditions (29) hold, the solution of the system (11) are expressions of the following form:*

$$x_i = \frac{(-1)^i x_0}{\prod_{k=0}^{i-1} S(k)}, \quad i = 1, 2, \dots, \tag{30}$$

where x_0 – an arbitrary real number, $S(k)$ satisfy the equation (29) for each j.

Note 2. If an infinite determinant $|A|$ of system (1) is not equal to zero, then all the results obtained here for Gaussian systems are valid for general systems (1).

Note 3. In the case of periodic systems, as shown in [9, 10], the conditions (29) is transformed into one condition – one characteristic equation $f(x) = 0$. Substituting zeros of function $f(x)$ in the expression (30), we obtain an independent nontrivial solutions of the homogeneous Gaussian system (11). Thus, the dimension of subspace of independent solutions of the homogeneous Gaussian system (11) coincides with the number of zeros of the function $f(x)$.

In the case of infinite determinant $|A|$ of the system (1) is zero, then we can obtain Kronecker-Capelli theorem for infinite systems. To do this we introduce the concept of decrement of infinite matrices and determinants.

Thus basis for the general theory of infinite systems of linear algebraic equations is obtained, which in combination with Kronecker-Capelli theorem constitutes the general theory of infinite systems.

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