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COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. An analytic function f defined on the open unit disk $U = \{z : |z| < 1\}$ is bi-univalent if the function f and its inverse f^{-1} are univalent in U. Inspired by the recent work of Hamidi et al. [8], we propose to investigate the coefficient estimates for a general class of analytic and bi-univalent functions. Also, we obtain estimates on the coefficients $|a_2|, |a_3|$ and $|a_n|$ for functions in this class. Some earlier results are shown to be special cases of our results.

Keywords: analytic and univalent functions, bi-univalent functions, Salagean derivative.

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1. Introduction

Let A denotes the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of the form (1) which are also univalent in U and let P be the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ that are analytic in U and satisfy the condition $Re\,(p(z)) > 0$ in U. By the Caratheodory's lemma (e.g., see [7]) we have $|p_n| \leq 2$.

Let $f \in A$. We define the differential operator D^k, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, \ldots\}$, by (see [12])

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= Df(z) = zf'(z); \\ &\vdots \\ D^k f(z) &= D^1\left(D^{k-1} f(z)\right). \end{aligned}$$

For $k \in \mathbb{N}_0$, $0 \leq \beta < 1$, $\lambda \geq 0$, we introduce the subclass $Q(k, \lambda, \beta)$ of S of functions of the form (1) satisfying the condition

$$Re\left\{\frac{(1-\lambda) D^k f(z) + \lambda D^{k+1} f(z)}{z}\right\} > \beta, \quad z \in U, \tag{2}$$

where D^k stands for Salagean derivative introduced by Salagean [12] .For $f \in A$, the class $Q(k, \lambda, \beta) \subset S$ and was first defined and investigated by Porwal and Darus [11].

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It is easy to see that $Q(k,\ \lambda_1,\ \beta) \subset Q(k,\ \lambda_2,\ \beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1$, $0 \leq \beta < 1$, $Q(k,\ \lambda,\ \beta) \subset Q(k,\ 1,\ \beta) = \left\{ f \in A : Re \dfrac{D^{k+1} f(z)}{z} > \beta;\ 0 \leq \beta < 1 \right\}$.

It is well known that every $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z\ ,\ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w\ ,\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see [10]). But the interest on the bounds for the coefficients of classes of bi-univalent functions picked up by the publications of Brannan and Taha [6], and Srivastava et al. [13]. Not much is known about behavior of higher order coefficients of classes of bi-univalent functions, as Ali et al. [5]. Motivating with their work, we let $f \in Q(k,\ \lambda,\ \beta)$ and $g = f^{-1} \in Q(k,\ \lambda,\ \beta)$ and use the Faber polynomial coefficient expansion to provide bounds for the general coefficients $|a_n|$ of such functions with a given gap series [8].

2. Main results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)\, w^n, \tag{3}$$

where

$$\begin{aligned} K_{n-1}^{-n} \;=\; & \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!\,(n-3)!} a_2^{n-3} a_3 + \\ & + \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 + \\ & + \frac{(-n)!}{[2(-n+2)]!\,(n-5)!} a_2^{n-5} \left[a_5 + (-n+2)\, a_3^2\right] + \\ & + \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5)\, a_3 a_4\right] + \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{4}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{5}$$

In general, for any $p \in \mathbb{N}$, an expansion of K_n^p is as, [3],

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!3!}E_n^3 + ... + \frac{p!}{(p-n)!n!}E_n^n, \tag{6}$$

where $E_n^p = E_n^p(a_2, a_3, ...)$ and by [1],

$$E_n^m(a_1, a_2, ..., a_n) = \sum_{m=1}^{\infty} \frac{m!\,(a_1)^{\mu_1} ... (a_n)^{\mu_n}}{\mu_1!...\mu_n!}, \tag{7}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ ...\ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ ...\ + n\mu_n &= n. \end{aligned} \tag{8}$$

Evidently, $E_n^n(a_1, a_2, ..., a_n) = a_1^n$, [2].

Theorem 2.1. *For* $0 \leq \beta < 1$, $\lambda \geq 1$ *and* $k \in \mathbb{N}_0$, *let* $f \in Q(k, \lambda, \beta)$ *and* $g \in Q(k, \lambda, \beta)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2(1-\beta)}{n^k[1+(n-1)\lambda]}; \qquad n \geq 4 \tag{9}$$

Proof. For analytic functions f of the form (1) we have

$$\frac{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)}{z} = 1 + \sum_{n=2}^{\infty} n^k [1+(n-1)\lambda] a_n z^{n-1}, \tag{10}$$

and for its inverse map, $g = f^{-1}$, we have

$$\begin{aligned} \frac{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)}{w} &= 1 + \sum_{n=2}^{\infty} n^k [1+(n-1)\lambda] b_n w^{n-1} = \\ &= 1 + \sum_{n=2}^{\infty} n^k [1+(n-1)\lambda] \times \\ &\quad \times \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^{n-1}. \end{aligned} \tag{11}$$

On the other hand, since $f \in Q(k, \lambda, \beta)$ and $g = f^{-1} \in Q(k, \lambda, \beta)$ by definition, there exist two positive real part functions $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum\limits_{n=1}^{\infty} d_n w^n$ where $Re p(z) > 0$ and $Re q(w) > 0$ in P so that

$$\frac{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)}{z} = 1 + (1-\beta)\sum_{n=1}^{\infty} K_n^1(c_1, c_2, ..., c_n) z^n, \tag{12}$$

$$\frac{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)}{w} = 1 + (1-\beta)\sum_{n=1}^{\infty} K_n^1(d_1, d_2, ..., d_n) w^n. \tag{13}$$

Comparing the corresponding coefficients of (10) and (12) yields

$$n^k\left[1+(n-1)\lambda\right]a_n=(1-\beta)\,K_{n-1}^{1}\left(c_1,c_2,...,c_{n-1}\right), \tag{14}$$

and similarly, from (11) and (13) we obtain

$$\frac{1}{n}n^k\left[1+(n-1)\lambda\right]K_{n-1}^{-n}\left(b_0,b_1,...,b_n\right)=(1-\beta)\,K_{n-1}^{1}\left(d_1,d_2,...,d_{n-1}\right). \tag{15}$$

Note that for $a_m=0\ ;\ 2\leq m\leq n-1$ we have $b_n=-a_n$ and so

$$\begin{aligned} n^k\left[1+(n-1)\lambda\right]a_n &= (1-\beta)\,c_{n-1} \\ -n^k\left[1+(n-1)\lambda\right]a_n &= (1-\beta)\,d_{n-1} \end{aligned} \tag{16}$$

Now taking the absolute values of either of the above two equations and applying the Caratheodory's lemma, we obtain

$$|a_n|\leq\frac{(1-\beta)\,|c_{n-1}|}{|n^k\left[1+(n-1)\lambda\right]|}=\frac{(1-\beta)\,|d_{n-1}|}{|n^k\left[1+(n-1)\lambda\right]|}\leq\frac{2\,(1-\beta)}{n^k\left[1+(n-1)\lambda\right]}. \tag{17}$$

$\square$

Theorem 2.2. *For* $0\leq\beta<1,\ 1\leq\lambda\leq1+\sqrt{2}$, *let* $f\in Q\,(k,\ \lambda,\ \beta)$ *and* $g\in Q\,(k,\ \lambda,\ \beta)$. *Then*

$$\begin{aligned}
(i)\qquad |a_2| &\leq \begin{cases} \sqrt{\dfrac{2\,(1-\beta)}{3^k\,(1+2\lambda)}}, & 0\leq\beta<\frac{3^k(1+2\lambda)-2^{2k-1}(1+2\lambda)^2}{3^k(1+2\lambda)}; \\ \dfrac{2\,(1-\beta)}{2^k\,(1+2\lambda)}, & \frac{3^k(1+2\lambda)-2^{2k-1}(1+2\lambda)^2}{3^k(1+2\lambda)}\leq\beta<1. \end{cases} \\
(ii)\qquad |a_3| &\leq \frac{2\,(1-\beta)}{3^k\,(1+2\lambda)}. \\
(iii)\qquad \left|a_3-2a_2^2\right| &\leq \frac{2\,(1-\beta)}{3^k\,(1+2\lambda)}.
\end{aligned} \tag{18}$$

Proof. Replacing n by 2 and 3 in (14) and (15), respectively, we find that

$$2^k\,(1+\lambda)\,a_2=(1-\beta)\,c_1, \tag{19}$$

$$3^k\,(1+2\lambda)\,a_3=(1-\beta)\,c_2, \tag{20}$$

$$-2^k\,(1+\lambda)\,a_2=(1-\beta)\,d_1, \tag{21}$$

Dividing (19) or (21), by $2^k\,(1+\lambda)$, taking their absolute values, and applying the Caratheodory's lemma, we obtain

$$|a_2|\leq\frac{(1-\beta)\,|c_1|}{2^k\,(1+\lambda)}=\frac{(1-\beta)\,|d_1|}{2^k\,(1+\lambda)}\leq\frac{2\,(1-\beta)}{2^k\,(1+\lambda)}. \tag{22}$$

Adding (20) to (22) implies

$$2.3^k\,(1+2\lambda)\,a_2^2=(1-\beta)\,(c_2+d_2) \tag{23}$$

or, equivalently,

$$a_2^2=\frac{(1-\beta)\,(c_2+d_2)}{2.3^k\,(1+2\lambda)}. \tag{24}$$

An application of Caratheodory's lemma followed by taking the square roots yields

$$|a_3|=\frac{(1-\beta)\,|c_2|}{3^k\,(1+2\lambda)}\leq\frac{2\,(1-\beta)}{3^k\,(1+2\lambda)}. \tag{25}$$

In the following, dividing (22) by $(1+2\lambda)$, taking the absolute values of both sides, and applying the Caratheodory's lemma, we obtain

$$\left|a_3 - 2a_2^2\right| = \frac{(1-\beta)\left|d_2\right|}{3^k\left(1+2\lambda\right)} \le \frac{(1-\beta)}{3^k\left(1+2\lambda\right)}. \tag{26}$$

□

Remark 2.1. *If we put $k=0$ in Theorem 1, we obtain the corresponding results due to Jahangiri and Hamidi* [9].

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