TWMS J. Pure Appl. Math., V.6, N.2, 2015, pp.165-179

ON UNIFORM TOPOLOGY AND ITS APPLICATIONS

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Abstract. This paper is a review and an extended version of the report made by A.A. Borubaev at the V World Congress of the Turkic world. It contains results on some classes of uniform spaces and uniformly continuous mappings and absolutes, generalizations of metrics, normed, uniform unitary spaces, topological and uniform groups, its completions, its spectral characterizations.

Keywords: uniform space, uniformly continuous mapping, multiscalar product, multimetrics, multinorm, topological group, uniform group, completion.

AMS Subject Classification: 54C05, 54C10, 54C30, 54C35, 54C45, 54C50.

1. Introduction

The main notions relevant to the uniform topology gradually revealed in the theory of Real Analysis. Historically the first notions in the theory of uniform spaces clearly can be considered the notion of what subsequently was called as "Cauchy sequence" (1827) and the notion of uniformly continuous function which appeared in the last half of the XIX's century. French mathematician M. Frechet devised [20] (1906) a notion of "metric space" which is a special kind of "uniform space". The theory of metric spaces was deeply developed by the German mathematician F. Hausdorff [24] (1914) and especially due to the fundamental papers of the Polish mathematician S. Banach [6] (1920). With the notion of nonmetrizable spaces appeared idea of creating some natural structure expressing the idea of uniformity and in the first turn the notion of completeness and uniformly continuous function and constructing research instrument to generalize the metric approach. This situation influenced the French mathematician A. Weil [45] (1937) to create the theory of uniform spaces. This theory was presented by three significant classes, more exactly, by the classes of complete, metric and compact (precompact) spaces.

The theory of uniform spaces in present time has become logically justified, far advanced due to the fundamental papers of A. Weil, N. Bourbaki, U.M. Smirnov, V.A. Efremovich, A.A. Borubaev, J. Isbell, B.A. Pasynkov, P. Samuel, V.V. Fedorčuk, Z. Frolik, V. Kulpa, E.V. Shepin, A.P. Šostak, A.A. Chekeev and others.

Despite of independent character of uniform space it is closely connected with the theory of topological spaces and between them there is a deep analogy. So the problem of defining and research of uniform analogies of most important classes of topological spaces and continuous mappings has turned to become not only of current interest but produces fine instruments to study topological spaces itself. The first problem appeared in finding uniform analogue for paracompactness. American mathematician M.D. Rice was the first to determine uniformly

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Manuscript received February 2015.

paracompact spaces. But unfortunately this class does not include even the class of metric spaces. This is a big gap for so constructed class of uniform paracompact spaces. Then Czech mathematician Z. Frolik [21], American mathematician J. Isbell, Russian (Soviet) mathematicians B.A. Pasynkov [35], A. Aparina [5] and Kyrgyz mathematician B.E. Kanetov [27] offered different definitions for uniform paracompactness. Analysis of the relationships of all kinds of paracompact spaces has been performed by B.E. Kanetov. We offered the most successful definition of uniform paracompactness. First of all this class is the widest one, secondly all main properties of paracompact spaces are carried to this class.

2. Main results

Definition 2.1. [8]. A uniform space $(X,\mathcal{U})$ is said to be *uniformly paracompact*, if for any additive open covering γ of the space $(X,\mathcal{U})$ there exists such sequence $\alpha_n \subset \mathcal{U}$ that the following condition holds: for each point $x \in X$ there exists such number $n \in \mathbb{N}$ and $\Gamma \in \gamma$ that $\gamma_n(x) \subset \Gamma$.

Each uniformly paracompact space is paracompact and each paracompact spaces with universal uniformity becomes uniformly paracompact space. The class of uniformly paracompact spaces contains the class of metric spaces.

Theorem 2.1. [8]. *A uniform space $(X,\mathcal{U})$ is uniformly paracompact, if for any additive open covering ω of $(X,\mathcal{U})$ there is a uniformly continuous $\omega-$mapping f of $(X,\mathcal{U})$ onto some metrizable uniform space $(Y_\omega, \mathcal{V}_\omega)$.*

Theorem 2.2. [8]. *A uniform space $(X,\mathcal{U})$ is uniformly paracompact if and only if for a compactification bX and each compact $K \subset bX \backslash X$ there exists a sequence $\alpha_n \subset \mathcal{U}$ to meet the following condition: for each point $x \in X$ there exist such number $n = n(x)$, that $[\alpha_n(x)]_{bX} \cap K = \emptyset$.*

These criteria demonstrate the adequacy of the definition of uniform paracompactness. Similarly are defined another significant classes of uniform spaces: uniformly strongly paracompact spaces, uniformly Lindelöff, uniformly weekly paracompact, uniformly connected, uniformly pluming, uniformly Čech complete spaces and others. One of the central topics in general topology is the one closely related to the different kinds of extensions of topological spaces.

P.S. Aleksandroff [1] and M. Stone [42] in their fundamental papers pointed out that one of the interest and difficult problems of general topology is the study of all extensions of the given topological space.

P.S. Aleksandroff posed the problem to classify compact extensions. The brilliant solution to the problem was given by Yu.M. Smirnoff [39] who showed the one-to-one relationship between proximity structures and compact extensions on given Tychonoff space. Systemizing general problems B. Banaschewski [7] set the following problem. The problem was to show a common way for constructing extensions with given beforehand properties.

After the classical works of Yu.M. Smirnoff on the agenda was the construction of extensions likewise compactness, and in first turn, construction of all paracompact extensions. Bulgarian mathematicians D. Doichinoff [18] and Zaitsev [46] studied paracompact extensions what is more the first one used so called "supertopologies" and the second one used projection spectra. But the general method for constructing of all extensions likewise compactness by means of uniform structures has been offered by us.

Definition 2.2. [8]. Let $(X,\mathcal{U})$ *be uniform space. The uniformity $\mathcal{U}$ is said to be preparacompact, if any covering $\gamma \in \mathcal{U}$* of X *provided $\gamma \cap \mathcal{F} = \emptyset$ for each minimal Cauchy filter $\mathcal{F}$* of $(X,\mathcal{U})$.

A uniformity $\mathcal{U}$ is called *strongly preparacompact* (*Lindelöff*), if $\mathcal{U}$ is preparacompact with the base consisting of starry finite (countable) coverings.

Theorem 2.3. *Let X be Tychonoff space. Then there is a one-to-one relationship between all paracompact (strongly paracompact, Lindelöff) extensions of the space X and all preparacompact (strongly preparacompact, Lindelöff) uniform structures of X.*

Taking into account that for each proximity of space X adequately corresponds some precompact uniformity of the space X then one can say that the Theorem 2.5 is a natural continuation just previously mentioned results of Yu.M.Smirnov. General approach developed by us allows to construct other significant extensions likewise compactness such, for example, as pluming paracompact, Čech-complete paracompact, locally compact paracompact extensions.

No doubt that the theory of uniform spaces can develop successfully only in unity with the theory of their uniformly continuous mappings. Uniformly factor, uniformly open mappings are defined in natural way and their theories have been developed successfully. Difficulties appeared when defining uniform analogue for perfect mappings. In this connection many interesting results connected with perfect mappings did not get its development in uniform topology and in the first turn attempts of constructing an absolute of a uniform space failed. But completely unexpectedly in by form and content way is defined uniformly perfect mapping.

Definition 2.3. [9]. *A uniformly continuous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is called a uniformly precompact, if for any covering $\alpha \in \mathcal{U}$ there exist a covering $\beta \in \mathcal{V}$ and a finite covering $\gamma \in \mathcal{U}$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$.*

Uniformly precompact and perfect (in usual sense) mapping will be called as *uniformly perfect mapping*.

A uniformly continuous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ will be called *complete* [10], if any Cauchy filter $\mathcal{F}$ from $(X,\mathcal{U})$ converges provided $f\mathcal{F}$ converges in $(Y,\mathcal{V})$.

The following theorem demonstrates the accuracy of such defined notion of uniformly perfect mapping.

Theorem 2.4. [10]. *Uniformly continuous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly perfect if and only if the mapping f is uniformly precompact and complete.*

The accuracy and correctness of the definition of uniformly perfect mappings are confirmed by the category characteristic.

Let us consider the following square in the category $Unif$.

$$\begin{array}{ccc} (X,\mathcal{U}) & \xrightarrow{i_X} & (sX, s\mathcal{U}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle s(f)}\ (\star) \\ (Y,\mathcal{V}) & \xrightarrow[i_Y]{} & (sY, s\mathcal{V}), \end{array}$$

where $(sX, s\mathcal{U})$ and $(sY, s\mathcal{V})$ are Samuel compact extensions of uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ respectively, $s(f)$ is an extension mapping of f, i_X and i_Y are uniformly continuous canonical injections.

Theorem 2.5. [9]. *A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly perfect if and only if the square $(\star)$ is pullback in the category $Unif$.*

A uniform space $(X,\mathcal{U})$ is said to be *strongly uniformly $\tau-$pluming* (*strongly uniformly $\tau-$quasi-pluming*, respectively) [11], if there exists pseudouniformity $\mathcal{V} \subseteq \mathcal{U}$ to meet the following conditions:

(1) $w(\mathcal{V}) \leq \tau$;
(2) $\cap\{\alpha(x) : \alpha \in \mathcal{V}\} = K_x$ is compact (countably compact, respectively) for any $x \in X$;
(3) A system $\{\alpha(K_x) : \alpha \in \mathcal{V}\}$ is a neighborhoods fundamental system of K_x in topological space (X, τ_u);
(4) $\mathcal{U} = sup\{\mathcal{V}, \mathcal{U}_\beta\}$.

Assume that pseudouniform space $(X, \mathcal{V})$ is complete and $\mathcal{U} = sup\{\mathcal{V}, \mathcal{U}_\beta\}$, where $\mathcal{U}_\beta$ is a maximal precompact uniformity contained in $\mathcal{U}$ then the uniform space $(X, \mathcal{U})$ will be called *strongly uniformly* $\tau-$*Čech complete.*

Theorem 2.6. [11]. *Strongly uniformly* $\tau-$*pluming spaces (strongly uniformly Čech complete spaces) and only they are mapped onto (complete)* $\tau-$*metric spaces by means of uniformly perfect mapping.*

The role of uniformly perfect mappings to some extent is revealed by the following theorem.

Theorem 2.7. [8]. *Let* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *be uniformly perfect mapping "onto". Then the following properties of uniform spaces are direct and inverse invariants:*

(1) *completeness and index of completeness* $\leq \tau$*;*
(2) *precompactness and compactness;*
(3) $\tau-$*boundedness;*
(4) *uniform local compactness;*
(5) *uniform* $\tau-$*paracompactness;*
(6) *strongly uniform* $\tau-$*plumings;*
(7) *strongly uniform* $\tau-$*Čech-completeness.*

The theory of projective objects which arose in homological algebra entered into topology, more precisely into compact spaces due to the American mathematician A.M.Gleason [22]. But its great development under convenient and already classic name the theory of absolutes took place in the papers of V.I.Ponomarev [36].

Uniformly perfect mapping allowed to the first author to define an absolute of uniform spaces.

Definition 2.4. A uniform space $(\dot{X}, \dot{\mathcal{U}})$ is said to be an absolute of a uniform space $(X, \mathcal{U})$ provided the following conditions hold*:*

(A1) *There exists a uniformly perfect irreducible mapping* h *of the space* $(\dot{X}, \dot{\mathcal{U}})$ *onto the space* $(X, \mathcal{U})$*.*
(A2) *Any uniformly perfect irreducible mapping* g *of a uniform space* $(Z, \mathcal{W})$ *onto the uniform space* $(\dot{\mathcal{X}}, \dot{\mathcal{U}})$ *is a uniform isomorphism.*

Theorem 2.8. [10]. *Any uniform space has the unique absolute. Moreover, an absolute of any uniform space is an extremally disconnected space.*

In present time the theory of absolutes of uniform spaces has been constructed by us and far advanced. Below cited from this theory only three formulas obtained by the first author.

(1) $(s\dot{X}, s\dot{\mathcal{U}}) \cong ((sX)^{\cdot}, (s\mathcal{U})^{\cdot})$;
(2) $(\dot{\tilde{X}}, \dot{\tilde{\mathcal{U}}}) \cong ((\tilde{X})^{\cdot}, (\tilde{\mathcal{U}})^{\cdot})$;
(3) $(v\dot{X}, v\dot{\mathcal{U}}) \cong ((vX)^{\cdot}, (v\mathcal{U})^{\cdot})$,

where $(sX, s\mathcal{U})$, $(\tilde{X}, \tilde{\mathcal{U}})$, $(vX, v\mathcal{U})$ are Samuel extension, completeness, realcompact extensions of the uniform space $(X, \mathcal{U})$ respectively.

At one time S. Iliadis [25] discovered a brilliant formula $\beta\dot{X} \cong (\beta X)^{\cdot}$. In special case when we put $\mathcal{U} = \mathcal{U}_\beta$ where is $\mathcal{U}_\beta$ a maximal precompact uniformity then formulas (1) and (2) become the formulas of S.Iliadis.

Absolutes of the uniform spaces coincide exactly with the projective objects in the category of uniform spaces with respect to uniformly perfect mappings. Many wonderful results of a general topology such as factorization theorems of S. Mardesič, B.A. Pasynkov and A.V. Zarelua theorem on universal compact of weight τ and dimension n as well as E.V. Shepin's spectral theorems all they have uniform characteristics. All these results were generalized and carried to the case of uniform spaces.

Applications of uniform topology methods in functional analysis.

Let $\{\rho_\alpha : \alpha \in A\}$ *be any family of pseudometrics on* X *generating the uniformity* $\mathcal{U}$ (see [29]) *and* τ *is a cardinal of* A. *By symbol* ρ_τ *it is denoted the diagonal of the mappings* $\rho_\alpha : X \times X \to \mathbb{R}_+, \alpha \in A$, i.e. $\rho_\tau = \triangle\{\rho_\alpha : \alpha \in A\} : X \times X \to \mathbb{R}^\tau_+$.

Let (L, T) be any locally convex linear topological space T and $\{P_\beta : \beta \in B\}$ is any family of pseudonorms on L (see [40]), $\mu = |B|$. We denote as $\|\cdot\|_\mu$ the diagonal of the mappings $P_\beta : L \times L \to \mathbb{R}_+$, $\beta \in B$, i.e. $\|\cdot\|_\mu = \triangle\{P_\beta : \beta \in B\} : L \times L \to \mathbb{R}^\mu_+$.

Axiomatization of the mappings ρ_τ and $\|\cdot\|_\mu$ brings us to the concepts of $\tau-$metric and $\tau-$norm.

Definition 2.5. [14]. *Let* X *be nonempty set. A mapping* $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ *is said to be* $\tau-$ *metric or multimetric on* X *and a pair* $(X, \rho_\tau)-$ $\tau-$ *metric or multimetric space provided the following well-known axioms hold:*

(1) $\rho_\tau(x, y) = \theta$ *if* $x = y$, *where* θ *is a point from* $\mathbb{R}^\tau_+$ *with all coordinates equal to 0.*
(2) $\rho_\tau(x, y) = \rho_\tau(y, x)$ *for any* $x, y \in X$.
(3) $\rho_\tau(x, y) \leq \rho_\tau(x, z) + \rho_\tau(z, y)$ *for all* $x, y, z \in X$.

The following statement [13] demonstrates the wideness of the class of multimetric spaces and is a generalization of the classic theorem of A.Weil on metrization of uniform spaces.

A uniform space $(X,\mathcal{U})$ is $\tau-$metrizable, if $w(\mathcal{U}) = \tau$.

Definition 2.6. [14]. *A mapping* $\|\cdot\|_\tau : X \to \mathbb{R}^\tau_+$ *of a linear space (over* $\mathbb{R}$*) into* $\mathbb{R}^\tau_+$ *is called* $\tau-$ *norm or multinorm on linear space* X *and a pair* $(X, \|\cdot\|_\tau)$ *is* $\tau-$ *normed or multinormed space if the following conditions hold:*

- $\|x\|_\tau = \theta$ *if* x *is a zero element in linear space* X, *but* θ *is a point of* $\mathbb{R}^\tau_+$ *with all coordinates equal to 0;*

(2) $\|\lambda x\| = \lambda\|x\|_\tau$ *for any scalar* $\lambda \in \mathbb{R}$ *and* $x \in X$*;*
(3) $\|x + y\|_\tau \leq \|x\|_\tau + \|y\|_\tau$ *for all* $x, y \in X$.

Putting $\rho_\tau(x, y) = \|x - y\|_\tau$ in $\tau-$normed space $(X, \|\cdot\|_\tau)$ we obtain a $\tau-$metric space.

Metric and normed spaces over topological semifields were studied by the Soviet mathematicians M. Antonovskii, V.G. Boltyanskii and T.A. Sarymsakov [2].

Theorem 2.9. [14]. *Let* $\{(X_\alpha, \|\cdot\|_\tau) : \alpha \in A\}$ *be any family of* $\tau_\alpha-$*normed spaces. Then* $(X, \|\cdot\|_\tau)$ *is* $\tau-$*normed space, where* $X = \Pi\{X_\alpha : \alpha \in A\}$, $\|x\|_\tau = \{\|x\|_\tau : \alpha \in A\}$, $x = \{x_\alpha : \alpha \in A\}$, $x_\alpha \in X_\alpha$ *for each* $\alpha \in A$ *and* $\tau = sup\{\tau_\alpha : \alpha \in A\}$.

The famous Kolmogoroff's theorem [29] on norming of linear topological spaces is generalized in the following way: Linear topological space is $\tau-$normed if the zero element in X has a base consisting of $\tau-$many convex neighborhoods.

Theorem 2.10. [14]. *Multinormed (complete multinormed) spaces and only they are the limits of projective spectra consisting of normed (Banach) spaces.*

Definition 2.7. *Let X be a linear space (over field of real numbers $\mathbb{R}$). A mapping $(\cdot,\cdot)_\tau$: $X \times X \to \mathbb{R}^\tau$ is said to be $\tau-$ scalar product on linear space X if the following known axioms hold:*

(1) $(x, y)_\tau = (y, x)_\tau$ *for all* $x, y \in X$;
(2) $(\lambda x + \mu y, z)_\tau = \lambda(x, z)_\tau + \mu(y, z)_\tau$ *for any* $\lambda, \mu \in \mathbb{R}$ *and* $x, y, z \in X$;
(3) $(x, y)_\tau \geq \theta$ *for any* $x \in X$ *and* $(x, y)_\tau = \theta$, *if x is a zero element in linear space X and θ is a point in $\mathbb{R}^\tau$ with all coordinates equal to 0.*

A linear space X with τ scalar (multiscalar) product is called $\tau-$*unitary (multiunitary)* space.

Each $\tau-$unitary space $(X, (\cdot,\cdot)_\tau)$ turns to be $\tau-$normed space provided $\tau-$norm is defined as $\|x\|_\tau = \sqrt{(x, y)_\tau}$ for all $x \in X$. Here $\sqrt{(x, y)_\tau} = \{\sqrt{a_\alpha} : a_\alpha \in \mathbb{R}, \alpha \in A, |A| = \tau\}$, where $(x, y)_\tau = a \in \mathbb{R}^\tau$ and $a = \{a_\alpha : a_\alpha \geq 0, \alpha \in A\}$.

Theorem 2.11. *Completion of $\tau-$unitary space is $\tau-$unitary space again.*

Theorem 2.12. *Let $\{(X_\alpha, (\cdot,\cdot)_\tau), \alpha \in A\}$ be any family of (complete) $\tau_\alpha-$unitary spaces. Then $(X, (\cdot,\cdot)_\tau)$ is (complete) $\tau-$unitary space, where $X = \Pi\{X_\alpha : \alpha \in A\}$, $(x, y)_\tau = \{(x_\alpha, y_\alpha)_\tau : \alpha \in A\}$, $x = \{x_\alpha : \alpha \in A\}$, $y = \{y_\alpha : \alpha \in A\}$, $x_\alpha, y_\alpha \in X_\alpha$ for any $\alpha \in A$ and $\tau = \Sigma\{\tau_\alpha : \alpha \in A\}$.*

Corollary 2.13. *Product of (complete) unitary spaces is (complete) unitary space.*

Complete multiunitary space will be called *multi-Hilbert* space.

Theorem 2.14. *Multiunitary (multi-Hilbert) spaces and only they are the limits of projective spectra consisting of unitary (Hilbert) spaces.*

The following theorem is a generalization of well-known characteristics of unitary spaces in the class of normed spaces (see [19]).

Theorem 2.15. *Any $\tau-$normed space $(X, \|\cdot\|_\tau)$ is $\tau-$unitary if the following equality holds: $\|x + y\|^2_\tau + \|x - y\|^2_\tau = 2(\|x\|^2_\tau + \|y\|^2_\tau)$ for all $x, y \in X$.*

Applications of uniform topology methods in the theory of topological groups.

Traditionally on topological group $(G, \cdot, \tau)$ are considered three uniformities: left, right and two-sided ones. There are also another uniformities worth of attention.

Definition 2.8. [8]. *A triple $(G, \cdot, \mathcal{U})$ is said to be a* uniform group, *if it is both an algebraic group $(G, \cdot)$ and a uniform space satisfying to the following conditions:*

(UG1) *For any two bases of Cauchy filters F_1 and F_2 in $(G,\mathcal{U})$ the family $\{E_1 \cdot E_2 : E_1 \in F_1, E_2 \in F_2\}$ is a base of a Cauchy filter in $(G,\mathcal{U})$.*
(UG2) *For any base of a Cauchy filter F in $(G,\mathcal{U})$ the family $\{E^{-1} : E \in F\}$ is a base of a Cauchy filter in $(G,\mathcal{U})$.*

In this case the uniformity $\mathcal{U}$ is said to be a *group uniformity* of group $(G, \cdot)$.

On topological group $(G, \cdot, \tau)$ there are many group uniformities inducing topology τ, but two-sided uniformity is a maximal group uniformity.

Theorem 2.16. [8]. *Let $(G, \cdot)$ be an abstract group, $\mathcal{U}-$ be any uniformity on group G and $(\tilde{G},\tilde{\mathcal{U}})$ is a completion of the uniform space $(G,\mathcal{U})$. Then for the group operation " $\cdot$ " to be extended from topological group $(G, \cdot, \tau)$ onto topological group $(\tilde{G}, \cdot, \tau_{\tilde{u}})$ it is necessary and sufficiently for the triple $(G, \cdot, \mathcal{U})$ to be uniform group.*

The completions of topological groups with respect to maximal structure, factorization of uniform groups and inverse spectra of uniform groups were studied by A.A. Chekeev [12], [16], [17].

I.I. Guran [23] stated the problem of a continuous extendability of group operations of a topological group G over μG. The problem on characterization of topological groups, whose completion with respect to the maximal uniform structure is also a group was formulated by A.V. Arhangel'skii. In other words: *If G is a topological group, $\mathcal{U}_G$ is the maximal uniform structure on the Tychonoff space G and $(\mu G, \tilde{\mathcal{U}_G})$ is the completion of the uniform space $(G, \mathcal{U}_G)$, can the group operations $(x, y) \to x \cdot y$ and $x \mapsto x^{-1}$ be extended from the group G over μG by continuity?*

A criterion for extendability of the group operations from a group G over its Dieudonne completion μG, namely, the completion with respect to the maximal uniform structure $\mathcal{U}_G$, is given below.

Definition 2.9. [33]. *A filter F is called $\aleph_0-$ centered, if $\cap F' \neq \emptyset$ for any subfamily $F' \subset F$ of cardinality $|F'| \leq \aleph_0$.*

In other words: A uniform space in which every $\aleph_0-$ centered Cauchy filter converges is called $\aleph_0-$ complete (weakly complete in sense K.Morita [33]*).*

Definition 2.10. [16]. *A topological group G is called $\aleph_0-$ complete in Raikov sense, if the uniform space $(G, \mathcal{U}_T)$ is $\aleph_0-$complete.*

In other words, the two-sided uniform structure $\mathcal{U}_T$ [15] of the group G is an $\aleph_0-$complete uniformity.

Definition 2.11. *Any $\aleph_0-$ complete topological group $G^{\aleph_0}$ containing the group G as an everywhere dense subgroup is called an $\aleph_0-$ completion of the group G.*

Theorem 2.17. *Any topological group G has a unique, to a topological isomorphism, $\aleph_0-$completion $G_T^{\aleph_0}$ in Raikov sense.*

Theorem 2.18. *Let $\mathcal{U}$ be an arbitrary group uniformity on a group G. Then the $\aleph_0-$completion $(G^{\aleph_0}, \mathcal{U}^{\aleph_0})$ is a group, i.e. the group operations are extended from the group G over $G^{\aleph_0}$ by continuity.*

For convenience, we shall omit the sign of group operation " $\cdot$ ".

Theorem 2.19. *Let $\mathcal{U}$ be an arbitrary group uniformity on a group G. Then the $\aleph_0-$completion $(G^{\aleph_0}, \mathcal{U}^{\aleph_0})$ of the uniform space $(G, \mathcal{U})$ is homeomophically embedded in the Raikov $\aleph_0-$completion $(G_T^{\aleph_0}, \mathcal{U}_T^{\aleph_0})$.*

Corollary 2.20. *For any group uniformity $\mathcal{U}$ on group G the $\aleph_0-$completion $(G^{\aleph_0}, \mathcal{U}^{\aleph_0})$ is a subgroup of the Raikov $\aleph_0-$completion $(G_T^{\aleph_0}, \mathcal{U}_T^{\aleph_0})$.*

Remark 2.21. *For any uniform space $(G, \mathcal{U})$ the $\aleph_0-$completion $(G^{\aleph_0}, \mathcal{U}^{\aleph_0})$ can be constructed without accounting the group operations.*

Proposition 2.22. *Any uniformly continuous mapping of a uniform space $(X, \mathcal{U})$ into an $\aleph_0-$complete uniform space $(Y, \mathcal{V})$ has uniformly continuous extension $\tilde{f} : (\tilde{X}, \tilde{\mathcal{U}}) \to (Y, \mathcal{V})$ over the $\aleph_0-$completion $(\tilde{X}, \tilde{\mathcal{U}})$ of the uniform space $(X, \mathcal{U})$.*

Proposition 2.23. *For the fine uniformity $\mathcal{U}_X$ of a Tychonoff space X the $\aleph_0-$completion $(X^{\aleph_0}, \mathcal{U}_X^{\aleph_0})$ is homeomophically embedded into every $\aleph_0-$completion $(X^{\aleph_0}, \mathcal{U}^{\aleph_0})$ of the uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is any uniformity on X compatible with the topology of the space X.*

Corollary 2.24. *For the maximal uniformity* $\mathcal{U}_G$ *on a group* G *the* $\aleph_0-$*completion* $(G^{\aleph_0}, \mathcal{U}_G^{\aleph_0})$ *is homeomorphically embedded into every* $\aleph_0-$*completion* $(G^{\aleph_0}, \mathcal{U}_G^{\aleph_0})$, *where* $\mathcal{U}$ *is an arbitrary uniformity on the group* G.

For Tychonoff space X we denote by $\mathcal{U}(X)$ the set of all uniform structures compatible with the topology of the space X. For each uniformity $\mathcal{U} \in \mathcal{U}(X)$ we have the uniformly continuous mapping $1 : (X, \mathcal{U}_X) \to (X, \mathcal{U})$ and its natural uniformly continuous extensions $\tilde{1}$: $(X^{\aleph_0}, \mathcal{U}_X^{\aleph_0}) \to (X^{\aleph_0}, \mathcal{U}^{\aleph_0})$ over the $\aleph_0-$completions. As we noted above (see Proposition 2.34.), the $\aleph_0-$completion $(X^{\aleph_0}, \mathcal{U}_X^{\aleph_0})$ of the uniform space $(X, \mathcal{U}_X)$ coincides with the Dieudonne completion μX of the Tychonoff space X, i.e. it is identical with the uniform space $(\tilde{X}, \tilde{\mathcal{U}}_X)$.

For two uniformities $\mathcal{U}, \mathcal{V} \in \mathcal{U}(X)$ such that $\mathcal{U} \subset \mathcal{V}$ we have the uniformly continuous mapping $1_{\mathcal{U}}^{\mathcal{V}} : (X, \mathcal{V}) \to (X, \mathcal{V})$ admitting, uniformly continuous extension $1_{\mathcal{U}}^{\mathcal{V}} : (X^{\aleph_0}, \mathcal{V}^{\aleph_0}) \to (X^{\aleph_0}, \mathcal{U}^{\aleph_0})$ over the $\aleph_0-$completions. For uniformities $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{U}(X)$ such that $\mathcal{U} \subset \mathcal{V} \subset \mathcal{W}$ we have a natural composition $1_{\mathcal{U}}^{\mathcal{V}} = 1_{\mathcal{W}}^{\mathcal{V}} \circ 1_{\mathcal{U}}^{\mathcal{W}}$ of mappings and accordingly $\tilde{1}_{\mathcal{U}}^{\mathcal{V}} = \tilde{1}_{\mathcal{W}}^{\mathcal{V}} \circ \tilde{1}_{\mathcal{U}}^{\mathcal{W}}$. Thus, the inverse system $S = \{X^{\aleph_0}, \mathcal{U}^{\aleph_0}), \tilde{1}_{\mathcal{U}}^{\mathcal{V}}, \mathcal{U} \subset \mathcal{V}, \mathcal{U}, \mathcal{V} \in \mathcal{U}(X)\}$ is formed.

Proposition 2.25. $(\mu X, \tilde{\mathcal{U}}_X) = \varprojlim S$.

Corollary 2.26. *For maximal uniformity* $\mathcal{U}_G$ on group G *the* $\aleph_0-$*completion* $(G^{\aleph_0}, \mathcal{U}_G^{\aleph_0})$ *is homeomophic embedded to each* $\aleph_0-$*completion* $(G^{\aleph_0}, \mathcal{U}^{\aleph_0})$ *for an arbitrary uniformity* $\mathcal{U}$ *on group* G.

Theorem 2.27. *For a topological group* G *the next conditions are equivalent:*

(1) *The completion* $(\mu G, \tilde{\mathcal{U}}_G)$ *is a uniform group.*
(2) *The set* $J(G)$ *of all group uniformities forms a confinal part of the set* $\mathcal{U}(G)$ *of all uniformities.*

Definition 2.12. *A topological group* G *is called* $\mathbf{M}-$*factorizable if for any continuous mapping* $f : G \to \mathbf{M}$ *of the topological group* G *into a metric space* M *there exist a metric group* H, *a continuous epimorphism* $h : G \to H$ *and a continuous mapping* $g : H \to M$ *such that* $f = g \circ h$.

Taking into account the definition of $\mathbf{R}-$factorizable topological groups introduced by M.G. Tkachenko [43], we obtain the next

Proposition 2.28. *If a topological group* G *is* $\aleph_0-$ *bounded and* $\mathbf{M}-$ *factorizable then it is* $\mathbf{R}-$ *factorizable.*

Definition 2.13. *Let* $\mathbf{K}$ *be some class of topological groups. A topological group* G *is called* $\mathbf{K}-$ *factorizable, if for any continuous mapping* $f : G \to \mathbf{M}$ *of the topological group* G *into a metric space* M *there exist a continuous epimorphism* $h : G \to H_h$ *of* G *onto a topological group* $H_h \in \mathbf{K}$ *and a continuous mapping* $h : H_h \to M$ *such that* $f = g \circ h$.

Evidently, if $\mathbf{K}$ is the class of all metric groups, then $\mathbf{K}-$ factorization coincides with $\mathbf{M}-$ factorization.

Theorem 2.29. *Let* G *be a* $\mathbf{D}-$*factorizable group. Then* $G \in \mathbf{M}$.

Corollary 2.30. *Let* G *be a* $M-$*factorizable topological group. Then* $G \in \mathbf{M}$.

Corollary 2.31. *Let* G *be an* $M-$*factorizable topological group. Then the uniformity* $\mathcal{U}_G$ *is a group uniformity.*

The factorization for continuous homomorphisms of $R-$factorizable groups was obtained by M. Tkachenko [43]. Naturally the similar problem arises in the class of uniform groups.

So, it is necessary to find a class M of uniform groups such that any uniformly continuous homomorphism $f:(G,\mathcal{U})\to(H,\mathcal{V})$, where $(G,\mathcal{U})\in M$, is factorizable on uniform weight and uniform dimension dim.

Definition 2.14. *A uniform group $(G,\mathcal{U})$ is said to be uniformly $\mathbf{M}-$ factorizable, if for any uniformly continuous mapping $f:(G,\mathcal{U})\to M$ of the uniform group $(G,\mathcal{U})$ into a metric space M there are a metrizable group $(H,\mathcal{V})$, a uniformly continuous epimorphism $h:(G,\mathcal{U})\to H,\mathcal{V})$ and a uniformly continuous mapping $g:(H,\mathcal{V})\to M$ such that $f=g\circ h$.*

Definition 2.15. *A uniform group $(G,\mathcal{U})$ is said to be uniformly $\mathbf{R}-$factorizable, if for any uniformly continuous function $f:(G,\mathcal{V})\to\mathbf{R}$ there are an $\aleph_0-$bounded metrizable group $(H,\mathcal{V})$, a uniformly continuous epimorphism $h:(G,\mathcal{U})\to(H,\mathcal{V})$ and a uniformly continuous mapping $g:(H,\mathcal{V})\to M$ such that $f=g\circ h$.*

Proposition 2.32. *Let a topological group G be $\mathbf{M}-(\mathbf{R})-$factorizable. Then the uniform group $(G,\mathcal{U}_G)$ where $\mathcal{U}_G$ is the maximal uniformity, its uniformly $\mathbf{M}-(\mathbf{R})-$factorizable.*

Theorem 2.33. *For a uniform space $(X,\mathcal{U})$ the following conditions are equivalent:*

(1) *$dim\mathcal{U}\le n$.*
(2) *For any uniformly continuous mapping $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ of a uniform space $(X,\mathcal{U})$ into a uniform space $(Y,\mathcal{V})$ with weight $w(\mathcal{V})\le\aleph_0$ there are a uniform space $(Z,\mathcal{W})$ with weight $w(\mathcal{W})\le\aleph_0$, a surjective uniformly continuous mapping $h:(X,\mathcal{U})\to(Z,\mathcal{W})$ and a uniformly continuous mapping $g:(Z,\mathcal{W})\to(Y,\mathcal{V})$ such that $f=g\circ h$ and $dim\mathcal{W}\le n$*

Remark 2.34. *If in the conditions of Theorem 2.48. to assume that the uniform space $(X,\mathcal{U})$ is $\aleph_0-$bounded, and to replace the uniform weight w by the double uniform weight dw* [31] *in item (2), then it can be obtained one more theorem, in which the implication* (1) $\Rightarrow$ (2) *follows from Kulpa's factorization theorem* [31] *and the implication* (2) $\Rightarrow$ (1) *can be proved by analogy.*

Theorem 2.35. *For an $\aleph_0-$bounded uniform space $(X,\mathcal{U})$ the following conditions are equivalent:*

(1) *$dim\mathcal{U}\le n$.*
(2) *For any uniformly continuous mapping $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ of the uniform space $(X,\mathcal{U})$ to a uniform space $(Y,\mathcal{V})$ of double weight $dw(\mathcal{V})\le\aleph_0$ there exist a uniform space $(Z,\mathcal{W})$ of double weight $dw(\mathcal{W})\le\aleph_0$, a surjective uniformly continuous mapping $h:(X,\mathcal{U})\to(Z,\mathcal{W})$ and a uniformly continuous mapping $g:(Z,\mathcal{W})\to(Y,\mathcal{V})$ such that $f=g\circ h$ and $dim\mathcal{W}\le n$.*

Let uniformly continuous mappings $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ and $g:(X,\mathcal{U})\to(Z,\mathcal{W})$ be given. By analogy with [43], we will write $g\prec f$, if there is a uniformly continuous mapping $h:(g(X),\mathcal{W}|_{g(X)})\to(Y,\mathcal{V})$ such that $f=h\circ g$.

Proposition 2.36. *Let $(X,\mathcal{U})$ be a uniform space and for each $i\in\mathbb{N}$ uniformly continuous mapping $f_i:(X,\mathcal{U})\to(Y_i,\mathcal{V}_i)$ be given, where $f_{i+1}\prec f_i$, $w(\mathcal{V}_i)\le\aleph_0$ and $dim\mathcal{V}_i\le n$ for all $i\in\mathbb{N}$. If $f=\triangle\{f_i:i\in\mathbb{N}\}$, $Y=f(X)\subset\prod\{Y_i:i\in\mathbb{N}\}$ and $\mathcal{V}=Y\wedge\prod\{\mathcal{V}_i:i\in\{\mathbb{N}\}$, then $dim\mathcal{V}\le n$.*

Remark 2.37. *In the framework of the weight $w(\mathcal{V}_i), i\in\mathbb{N}$, may be replaced by double weight dw. In that case the statement holds too.*

Proposition 2.38. *Let $f : (G,\mathcal{U}) \to (X,\mathcal{W})$ be a uniformly continuous mapping of an $M-$factorizable uniform group $(X,\mathcal{U})$ into a uniform space $(X,\mathcal{W})$ of weight $w(\mathcal{W}) < \aleph_0$. Then there are a uniform group $(H,\mathcal{V})$ of weight $w(\mathcal{W}) \leq \aleph_0$, a uniformly continuous epimorphism $h : (G,\mathcal{U}) \to (H,\mathcal{V})$ and a uniformly continuous mapping $g : (H,\mathcal{V}) \to (X,\mathcal{W})$ such that $f = g \circ h$ and $dim\mathcal{V} \leq dim\mathcal{U}$.*

Theorem 2.39. *Let any uniformly continuous homeomorphic image of a uniform group $(G,\mathcal{U})$ be uniformly $M-$factorizable and let $h : (G,\mathcal{U}) \to (H,\mathcal{V})$ be a uniformly continuous epimorphism of the uniform group $(G,\mathcal{U})$ onto a uniform group $(H,\mathcal{V})$. Then there exist a uniform group $(G^*,\mathcal{U}^*)$ and uniformly continuous epimorphism $h_1 : (G^*,\mathcal{U}^*) \to (H,\mathcal{V})$ and $h_2 : (G,\mathcal{U}) \to (G^*,\mathcal{U}^*)$ such that $g = h_1 \circ h_2$, and $w(\mathcal{U}^*) \leq w(\mathcal{V})$.*

The inverse spectra method takes up an important place in general topology. By means of inverse spectra an approximation of various properties of spaces by properties of "simpler" construction spaces. It is to note that the method of inverse spectra takes up the central place in investigations of compacts with non-countable weight, where the E.V. Shchepin spectral theorem [41] plays an important role.

Spectral characteristics for various spaces were found by B.A. Pasynkov [34]. The spectral theorem was proven by Kulpa [31] for uniform spaces and was done by A.A. Borubaev [8] for uniformly continuous mappings.

Conditions of expandability of uniform groups into inverse group spectra are given below.

Let M be a directed set and each $a \in M$ define any uniform group $(G_a,\mathcal{U}_a)$. For each indexes $a, b \in M$, such that $a \leq b$, a uniformly continuous homomorphism $f_a^b : (G_b,\mathcal{U}_b) \to (G_a,\mathcal{U}_a)$ is defined, and $f_c^b = f_c^a \circ f_a^b$ for all indexes $a, b, c \in M$ such that $c \leq a \leq b$ and $f_a^a = 1_G$ for all $a \in M$.

Definition 2.16. *A family $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ is said to be an inverse system of uniform groups $(G_a,\mathcal{U}_a)$, $a \in M$ with connecting uniformly continuous homomorphisms f_a^b.*

Definition 2.17. *An element $\{g_a : a \in M\}$ of the product $\prod\{G_a : a \in M\}$ is said to be a thread of the inverse spectrum of uniform groups $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$, if $f_a^b(g_b) = g_a$ for all indexes $a, b \in M$, such that $a \leq b$.*

Definition 2.18. *The subspace of the uniform space $\prod\{(G_a,\mathcal{U}_a) : a \in M\}$, consisting of all threads of the inverse spectrum of uniform groups S, is said to be a limit of inverse spectrum of uniform groups $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ and is denoted as $\varprojlim S$ or $\varprojlim\{(G_a,\mathcal{U}_a), f_a^b, M\}$.*

Proposition 2.40. *The limit of inverse spectrum of uniform groups $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ is a closed uniform subgroup of group product of uniform groups $\prod\{(G_a,\mathcal{U}_a) : a \in M\}$.*

Proposition 2.41. *The family of all coverings $\pi_a^{-1}(\alpha_a)$ where α_a is a uniform covering of a group uniformity $\mathcal{U}_a$, and $a \in M'$, where M' is confinal to the set M, is a base of the group uniformity $\mathcal{U}$ of the limit of the inverse spectrum of uniform groups $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$.*

Lemma 2.42. *Let $\{\varphi, f_{a'}\}$ be a mapping of an inverse group spectrum $S = \{G_a, f_a^b, M\}$ to an inverse group spectrum $S' = \{H_{a'}, f_{a'}^{b'}, M'\}$. If all homomorphisms $f_{a'}$ are monomorphic, then the limit homomorphism $f = \varprojlim\{\varphi, f_{a'}\}$ is monomorphic too. Also, if all homomorphisms $f_{a'}$ are epimorphic, then the limit homomorphism $f = \varprojlim\{\varphi, f_{a'}\}$ is an epimorphism.*

Proposition 2.43. *Let $\{\varphi, f_{a'}\}$ be a mapping of an inverse spectrum of uniform groups $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ to an inverse spectrum of uniform groups $S' = \{(H_{a'},\mathcal{U}_{a'}), f_{a'}^{b'}, M'\}$. If all* *homomorphisms $f_{a'}$ are uniform group isomorphisms, then the limit mapping $f = \{\varphi, f_{a'}\}$ is also uniform a group isomorphism.*

Corollary 2.44. *Let $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ be an inverse spectrum of uniform groups and M' be confinal to the set M. The mapping which is the restriction of all threads of the group $(G,\mathcal{U}) = \varprojlim S$ onto M',is a uniform group isomorphism of the uniform group $(G,\mathcal{U})$ onto the uniform group $(G',\mathcal{U}') = \varprojlim S'$, where $S' = \{(G_{a'},\mathcal{U}_{a'}), f_{a'}^{b'}, M'\}$.*

Corollary 2.45. *Let $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ be an inverse spectrum of uniform groups and there exists the greatest element $a_0 \in M$ in a directed set M. Then the uniform group $(G,\mathcal{U}) = \varprojlim S$ is uniformly isomorphic to the uniform group $(G_{a_0},\mathcal{U}_{a_0})$.*

Let $(G,\mathcal{U})$ be a uniform group, $\{H_a : a \in M\}$ be a decreasing family of normal subgroups of the group G. Consider the family $\{(G/H_a,\mathcal{U}/H_a) : a \in M\}$ of uniform factor-groups and denote $G_a = G/H_a$ and $\mathcal{U}_a = \mathcal{U}/H_a$. For $a \le b$ we have $H_a \supseteq H_b$, it defines the natural mapping $f_a^b : G_b \to G_a$, transforming each class T of the group G_b to a class TH_a in a group G_a. It is easy to see that the mapping f_a^b with $a \le b$ is uniformly continuous homomorphism of the uniform group $(G_b,\mathcal{U}_b)$ onto the group $(G_a,\mathcal{U}_a)$. By such a way the inverse spectrum $S = \{(G_a,\mathcal{U}_a), f_a^b, M\}$ of uniform groups is defined. Denote $(\tilde{G},\tilde{\mathcal{U}}) = \varprojlim S$. For each $a \in M$ the canonical uniformly continuous homomorphism $f_a : (G,\mathcal{U}) \to (G_a,\mathcal{U}_a)$ is defined. Then for each point $x \in G$ the mapping $i = \triangle f_a(x) = \{f_a(x) : a \in M\}$ implements a uniformly continuous homomorphic embedding of the uniform group $(G,\mathcal{U})$ to the uniform group $(\tilde{G},\tilde{\mathcal{U}})$ under the condition that for any uniform covering $a \in \mathcal{U}$ there exists such index $a \in M$ that the partition $\{xH_a : x \in G\}$ is refined to the covering α.

Actually, let $x, y \in G$ and $x \neq y$. Then there exist such uniform covering $a \in \mathcal{U}$ and $A \in \alpha$ that x and y do not belong to A simultaneously, then $f_a(x) = xH$ and $f_a(y) = yH$ do not belong to $f_A(x) = AH$ simultaneously. Thus, the homomorphism i is injective and implements an algebraic isomorphism of the group G onto the group $i(G)$. Let us prove that i is a uniform homeomorphism.

Let $\tilde{a} \in \tilde{\mathcal{U}}$ be an arbitrary uniform covering of the group $\tilde{G}$. Then Lemma 2.60. implies $\alpha_a = f_a(a) \in \mathcal{U}_a$ for all $a \in M$. Further, Proposition 2.59. implies that for each $a \in M$ the covering $\pi_a^{-1}(\alpha_a) = \pi_a^{-1}(f(\alpha_a))$ is uniform, i.e. $\pi_a^{-1}(\alpha_a) \in \mathcal{U}$, where $\pi_a : (\tilde{G},\tilde{\mathcal{U}}) \to (G_a,\mathcal{U}_a)$. Thus, we have the following commutative diagram: This implies that $f_a = \pi_a \circ i$ or $i(a) = \pi_a^{-1}(f(\alpha_a))$.

$$\begin{tikzcd}
(G,\mathcal{U}) \arrow[rr, "i"] \arrow[dr, "f_a"'] & & (\tilde{G},\tilde{\mathcal{U}}) \\
& (G_a,\mathcal{U}_a) \arrow[ur, "\pi_a"'] &
\end{tikzcd}$$

Thus, for all open uniform coverings $\alpha \in \mathcal{U}$ $i(a)$ is a uniform covering of $\tilde{\mathcal{U}}|_{i(G)}$.

Let $\mathcal{U} \subset \tilde{G}$ be an arbitrary open set in the group $\tilde{G}$. Then there exist such index $a \in M$ and non-empty open set $\mathcal{U}_a \subset G_a$ that $\pi_a^{-1}(\mathcal{U}_a) \subset \mathcal{U}$, and, consequently, $i^{-1}(\mathcal{U}) \supset f_a^{-1}(\mathcal{U}_a)$. But f_a is surjective, hence, $i^{-1}(\mathcal{U})$ is not empty, i.e. $i(G) \cap \mathcal{U} \neq \emptyset$. Thus it is proven that $i(G)$ is an everywhere dense subgroup of the group $\tilde{G}$.

Now suppose that for each $a \in M$ the normal subgroup H_a is complete with respect to the uniformity $\mathcal{U}|_{H_a}$ and $\{T_a : a \in M\} = \dot{x}$ is an arbitrary element of the group $\tilde{G}$. Being obtained from H_a by means of a transfer, T_a is a complete subspace of the uniform space $(G,\mathcal{U})$. Thus, for each uniform covering $a \in \mathcal{U}$ there exists such index $c \in M$ that the partition $\{xH_c : x \in G\}$

is refined to α. It means that $T_c \subset A$ for some $A \in \alpha$. Consequently, the set of $T_c \in \dot{x}$ belonging to T_a forms a base of a Cauchy filter which converges to any point $x \in T_a \subset G$ in the uniform space $(T_a, \mathcal{U}|_{T_a})$. Therefore, the uniformly isomorphic embedding turns out to be surjective in this case. Thus the following theorem is above proved

Theorem 2.46. *Let $(G,\mathcal{U})$ be a uniform group, $\{H_a : a \in M\}$ be a decreasing family of normal divisors of the group G, fulfilling the following condition:*

$(\star\star)$ For each index $a \in M$, H_a is closed in $(G, \tau_{\mathcal{U}})$ and for each uniform covering $\alpha \in \mathcal{U}$ there exists such index $a \in M$, that the partition $\{xH_a : x \in G\}$ is refined to α.

Then the uniform group $(G,\mathcal{U})$ is uniformly isomorphically embedded into the group $(\tilde{G},\tilde{\mathcal{U}})$. Also, if $(H_a, \mathcal{U}|_{H_a})$ is a complete uniform space for some $a \in M$ that the uniform groups $(G,\mathcal{U})$ and $(\tilde{G},\tilde{\mathcal{U}})$ are uniformly isomorphically.

Corollary 2.47. *If the condition $(\star\star)$ holds and all uniform groups $(G_a,\mathcal{U}_a)$ are complete, then the uniform group $(G,\mathcal{U})$ has the completion, identified with the uniform group $(\tilde{G},\tilde{\mathcal{U}})$.*

Actually, the uniform group $(\tilde{G},\tilde{\mathcal{U}})$ is complete as a closed uniform subspace of the product $\prod\{(G_a,\mathcal{U}_a) : a \in M\}$ of complete uniform spaces $(G_a,\mathcal{U}_a)$. According to the uniform group $(G,\mathcal{U})$ is uniformly isomorphically and everywhere dense embedded into the uniform group $(\tilde{G},\tilde{\mathcal{U}})$, hence, its completion is the uniformly isomorphically uniform group $(\tilde{G},\tilde{\mathcal{U}})$.

Some spectral characteristics of $\tau-$balanced and $\tau-$bounded topological groups will be given below. Their two-sided uniformities are group, therefore they realize Theorem 2.64., and they can be derived from it. But we are interested in a direct construction of them.

$\aleph_0-$balanced groups, named as *groups with a quasi-invariant base*, were firstly introduced by G.A. Kats [28], and $\tau-$bounded groups were done by I.I. Guran [23].

Subgroups of products of topological groups of character $\leq \tau$ and of weight $\leq \tau$ are said to be *$\tau-$balanced* and *$\tau-$bounded* correspondingly. There are "inner" definitions of $\tau-$balanced and $\tau-$bounded topological groups.

Definition 2.19. Let V be a neighborhood of the unit of a topological group G. A system $\{V_a : a \in A\}$ of neighborhoods of the unit is said to be *quasi-invariant base* with respect to the neighborhood V, if for each $g \in G$ there exists such index $a \in A$ that $g^{-1}V_a g \subseteq V$.

A topological group G, is said to be *$\tau-$balanced*, if each neighborhood of its unit has a quasi-invariant base of cardinality $\leq \tau$.

Definition 2.20. [23]. A topological group G is said to be *$\tau-$bounded*, if for each neighborhood of the unit V there exists such set $M_V \subset G, |M_V| \leq \tau$, that $M_V \cdot V = G$.

A filter F is said to be *$\tau-$centered*, if the intersection of its each subfamily of cardinality $\leq \tau$ is nonempty.

Definition 2.21. A $\tau-$centered Cauchy filter, with respect to two-sided uniformity of a topological group G, is said to be *Cauchy $\tau-$filter*, and a topological group, in which all Cauchy $\tau-$filters converges, is said to be *$\tau-$complete*.

The following theorem gives the characteristics of closed subgroups of products of groups of character $\leq \tau$.

Theorem 2.48. *For a topological group G the following conditions are equivalent:*

(1) *A group G is $\tau-$balanced and $\tau-$complete;*

(2) *A group G is the limit of the inverse spectrum $S = \{G_a, f_a^b, M\}$, where $\chi(G_a) \leq \tau$ for all $a \in M$, f_a^b are continuous homomorphisms and M is a $\tau-$complete index set;*
(3) *A group G is closely and topological isomorphically embedded into the product $\prod\{G_a : a \in M\}$, where $\chi(G_a) \leq \tau$ for all $a \in M$.*

Corollary 2.49. *For a topological group G the following conditions are equivalent:*

(1) *A group G is $\aleph_0-$balanced and $\aleph_0-$complete;*
(2) *A group G is the limit of an inverse spectrum $S = \{G_a, f_a^b, M\}$,where is a metrizable group for all $a \in M$, f_a^b are continuous homomorphisms and M is a $\aleph_0-$complete index set;*
(3) *A group G is closed and topologically isomorphically is embedded into a product $\prod\{G_a : a \in M\}$, where G_a is a metrizable topological group for all $a \in M$.*

Corollary 2.50. *$\aleph_0-$completion of $\aleph_0-$balanced topological group G is the limit of the inverse spectrum $S = \{G_a, f_a^b, M\}$, where G_a is a metrizable group for all $a \in M$, f_a^b are continuous homomorphisms and M is a $\aleph_0-$complete index set.*

Corollary 2.51. *For a balanced group G with $\chi(G) > \tau$, where τ is an infinite cardinal, the following conditions are equivalent:*

(1) *A group G* is *$\tau-$complete;*
(2) *A group G is the limit of an inverse spectrum $S = \{G_a, f_a^b, M\}$, where G_a is a balanced topological group, $\chi(G_a) \leq \tau$ for all $a \in M$, f_a^b, is a continuous homomorphisms and M is a $\tau-$complete index set;*
(3) *A group G is closed and is topologically isomorphically embedded into the product $\prod\{G_a : a \in M\}$, where G_a is a balanced topological group, $\chi(G_a) \leq \tau$ for all $a \in M$.*

Theorem 2.52. *For a topological group G the following condition are equivalent:*

(1) *A group G is $\tau-$bounded and is $\tau-$complete;*
(2) *A group G is the limit of an inverse spectrum $S = \{G_a, f_a^b, M\}$, where $w(G_a) \leq \tau$ for all $a \in M$, f_a^b, are continuous homomorphisms and M is a $\tau-$complete index set;*
(3) *A group G is closed and is topologically isomorphically embedded into the product $\prod\{G_a : a \in M\}$, where $w(G_a) \leq \tau$ for all $a \in M$.*

Corollary 2.53. *For a topological group G the following conditions are equivalent:*

(1) *A group G is $\aleph_0-$bounded and is $\aleph_0-$complete;*
(2) *A group G is the limit of an inverse spectrum $S = \{G_a, f_a^b, M\}$, where G_a is a separable metrizable group for all $a \in M$, f_a^b are continuous homomorphisms and M is a $\aleph_0-$complete index set;*
(3) *A group G is closed and is topologically isomorphically embedded into the product $\prod\{G_a : a \in M\}$, where G_a is a separable metrizable group for all $a \in M$.*

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