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NUMERICAL-ANALYTICAL METHOD FOR SOLVING OF THE FIRST ORDER PARTIAL QUASI-LINEAR EQUATIONS

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Abstract. Two-dimensional first order non-linear (quasi-linear) partial differential equation which arises in the solving of the system of hyperbolic equations describing of oil production by gas lift using the sweep method, is considered. By using the method of characteristics is shown that the seeking solution can be defined from the corresponding implicit algebraic relation by the help of the fixed point method. The computational algorithm for solving of implicit algebraic relation is giving. For the particular case , when the initial function is constant the analytical solution of the partial quasi linear equation is found.

Keywords: partial differential equation, quasi linear equation, method of characteristics, implicit solution, numerical algorithms, gas lift.

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1. Introduction

For ordinary linear differential equations with constant coefficients the general solution can be determined on the basis of Euler's theory [13], where the finding of the explicit representation both of Cauchy problem [5] and the boundary problem is not difficult [1]. For the problems for the ordinary differential equations with variable coefficients, the solutions are obtained in an explicit form [5, 7, 14, 18]. It should be noted that such studies for partial differential equations are almost neglected [14].

As is known, the solution of the Cauchy problem for the second order differential equation of hyperbolic type with constant coefficients is given explicitly by d'Alembert's (in the two dimensions case), Poisson's (in the three dimensions case) and Kirchhoff's (in the case of four dimensions) formulas [6, 19]. The solution of the mixed problem for these equations is, generally, represented as the series [16, 17]. Considering the presentation in the explicit form of the solution of the problem for the partial linear differential equations with constant and variable coefficients, only narrow class of such problems are covered [15].

In this work one example of the first order quasi-linear partial differential equation, which arises in the mathematical modeling of the oil production by the gas lift method is considered [12]. Based on the method of characteristics [11, 18] the representation of the solution of Cauchy problem for this equation is given. The results are illustrated by the example arising from the specific practical problems [3, 9].

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2. Statement of the problem

Let the first order quasi-linear partial differential equation in the form

$$\frac{\partial S(x,t)}{\partial x} + FS(x,t)\frac{\partial S(x,t)}{\partial t} - 2aS(x,t) = 0, \ \ x \in (0,l), \ \ t > 0, \tag{1}$$

with the initial condition

$$S(x,0) = \varphi(x), \ \ x \in [0,l], \tag{2}$$

where F and a are real constants, $\varphi(x)$ is the known continuous real-valued function, and $S(x,t)$ is the seeking function, be given.

The solution of equation (1) will be sought in an implicit form [11]:

$$\chi(x,t,S(x,t)) = 0. \tag{3}$$

Then differentiating (3) both by x and t, we'll have:

$$\frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial S}\frac{\partial S}{\partial x} = 0, \ \ \frac{\partial \chi}{\partial t} + \frac{\partial \chi}{\partial S}\frac{\partial S}{\partial t} = 0,$$

or

$$\frac{\partial S(x,t)}{\partial x} = -\frac{\frac{\partial \chi}{\partial x}}{\frac{\partial \chi}{\partial S}}, \qquad \frac{\partial S(x,t)}{\partial t} = -\frac{\frac{\partial \chi}{\partial t}}{\frac{\partial \chi}{\partial S}}. \tag{4}$$

Substituting (4) into (1) we obtain:

$$-\frac{\frac{\partial \chi}{\partial x}}{\frac{\partial \chi}{\partial S}} + FS\frac{-\frac{\partial \chi}{\partial t}}{\frac{\partial \chi}{\partial S}} - 2aS = 0$$

or

$$\frac{\partial \chi}{\partial x} + FS\frac{\partial \chi}{\partial t} + 2aS\frac{\partial \chi}{\partial S} = 0, \tag{5}$$

which is the first order linear partial differential equation for the function $\chi((x,t, \ S(x,t))$ from (3).

3. The method of characteristics

Let us define the characteristics of the equation (5):

$$\frac{dx}{1} = \frac{dt}{FS} = \frac{dS}{2aS}$$

from which we obtain the following equations:

$$dx = \frac{dS}{2aS}$$

and

$$\frac{dt}{F} = \frac{dS}{2a},$$

or

$$\frac{dS}{S} = 2adx,$$

$$dS = \frac{2a}{F} dt.$$

After integration of the last we define the following characteristics or a functional-invariant Yerugin's solution [10]

$$\begin{cases} \ln S(x,t) - 2ax = C_1 \\ S(x,t) - \frac{2at}{F} = C_2. \end{cases} \tag{6}$$

Here C_1 and C_2 are constants with which the characteristics are defined. Taking $t = 0$ in (6) we have:

$$\begin{cases} \ln S(x,0) - 2ax = C_1, \\ S(x,0) = C_2. \end{cases}$$

Taking into account the boundary condition (2), we obtain:

$$\begin{cases} \ln \varphi(x) - 2ax = C_1, \\ \varphi(x) = C_2, \end{cases}$$

or from the second equation we define

$$x = \varphi^{-1}(C_2),$$

and substituting it into the first equation before the past, we find

$$\ln C_2 - 2a\varphi^{-1}(C_2) = C_1,$$

or

$$\varphi^{-1}(C_2) = \frac{1}{2a} \ln C_2 - \frac{C_1}{2a}$$

or

$$C_2 = \varphi\left(\frac{1}{2a} \ln C_2 - \frac{C_1}{2a}\right) .$$

Finally, taking into account (6) from the last, we have:

$$S(x,t) - \frac{2a}{F}t = \varphi\left[\frac{1}{2a} \ln\left(S(x,t) - \frac{2a}{F}t\right) - \frac{1}{2a} \ln S(x,t) + x\right]$$

or

$$S(x,t) - \frac{2a}{F}t = \varphi\left(\frac{1}{2a} \ln\left(1 - \frac{2at}{FS(x,t)}\right) + x\right). \tag{7}$$

It is easy to see that (7) satisfies the initial condition (2). Now we show that (7) satisfies the equation (1), too.

Indeed,

$$\frac{\partial S}{\partial t} = \frac{2a}{F} + \varphi'\left(\frac{1}{2a} \ln\left(1 - \frac{2at}{FS(x,t)}\right) + x\right) \frac{1}{2a} \frac{-\frac{2a}{F}\frac{S(x,t) - t\frac{\partial S(x,t)}{\partial t}}{S^2(x,t)}}{1 - \frac{2at}{FS(x,t)}} =$$
$$= \frac{2a}{F} - \varphi'\left(\frac{1}{2a} \ln\left(1 - \frac{2at}{FS(x,t)}\right) + x\right) \frac{S(x,t) - t\frac{\partial S(x,t)}{\partial t}}{(FS(x,t) - 2at)S(x,t)},$$

or

$$\frac{\partial S}{\partial t} = \frac{2a(FS^2 - 2atS) - \varphi' SF}{F^2S^2 - 2atFS - \varphi' tF}. \tag{8}$$

Similarly

$$\frac{\partial S}{\partial x} = \varphi' \left[1 + \frac{1}{2a} \frac{\frac{2at}{F} \frac{1}{S^2} \frac{\partial S}{\partial x}}{1 - \frac{2at}{FS}} \right],$$

or

$$\frac{\partial S}{\partial x} = \varphi' \left[1 + \frac{t \frac{\partial S}{\partial x}}{S(FS - 2at)} \right],$$

$$\frac{\partial S}{\partial x} = \frac{\varphi' S(FS - 2at)}{FS^2 - 2atS - \varphi' t}. \tag{9}$$

Substituting (8) and (9) into (1) we have:

$$\frac{\varphi' S(FS - 2at)}{FS^2 - 2atS - \varphi' t} + FS \frac{2a(FS^2 - 2atS) - \varphi' SF}{F^2S^2 - 2atFS - \varphi' tF} - 2aS =$$
$$= \frac{\varphi' FS^2 - \varphi' S2at + 2aFS^3 - 4a^2tS^2 - \varphi' SFS - 2aFS^3 + 4a^2tS^2 + 2aS\varphi' t}{FS^2 - 2atS - \varphi' t} = 0,$$

i.e. the implicit function $S(x, t)$, given in the form (7), is a solution of the equation (1).

4. Numerical solution of algebraic equation (7)

Depending on the complexity of the structure of the initial function $\varphi(x)$ from (2) the finding $S(x, t)$ from (7) in an explicit form is problematic. So, using the different numerical algorithms we'll give an approximate methods for the solution of the equation (7).

(1) Since (7) has the form $x = G(x)$, we construct the next iteration

$$S^{(k+1)}(x, t) = \frac{2a}{F} t + \varphi \left(\frac{1}{2a} \ln \left(1 - \frac{2at}{FS^k(x, t)} \right) + x \right).$$

Then we have

$$S^{(k+1)}(\cdot) - S^{(k)}(\cdot) = \varphi \left(\frac{1}{2a} \ln \left(1 - \frac{2at}{FS^k(\cdot)} \right) + x \right) - \varphi \left(\frac{1}{2a} \ln \left(1 - \frac{2at}{FS^{k-1}(\cdot)} \right) + x \right),$$

where $S(\cdot) = S(x, t)$.

If assume that $\psi(x, t, S(x, t))$ in the right-hand side of (7) satisfies the Lipschitz condition by the variable $S(x, t)$ with constant Lipschitz coefficient $0<L < 1$, we have

$$\left| S^{(k+1)}(\cdot) - S^{(k)}(\cdot) \right| \le L \left| S^{(k)}(\cdot) - S^{(k-1)}(\cdot) \right| \le L^k \left| S^{(1)}(\cdot) - S^{(0)}(\cdot) \right|, \tag{10}$$

or

$$S^{(k+1)}(\cdot) - S^{(0)}(\cdot) = \sum_{m=1}^{k+1} \left(S^{(m)}(\cdot) - S^{(m-1)}(\cdot) \right).$$

This after evaluation (12) has the form

$$\left| S^{(k+1)}(\cdot) - S^0(\cdot) \right| \le \left| S^1(\cdot) - S^0(\cdot) \right| \sum_{m=0}^{k} L^m = \frac{1 - L^{k+1}}{1 - L} \times$$
$$\times \left| S^{(1)}(\cdot) - S^{(0)}(\cdot) \right| < \frac{1}{1 - L} \left| S^{(1)}(\cdot) - S^{(0)}(\cdot) \right|.$$

As can be seen from (10) the series $S^{(k)}(x,t)$ converges at $k \to \infty$.

Note that for the approximate value of $S(x,t)$ we'll take $S^{(k)}(x,t)$ for which the inequality

$$\left|S^{(k+1)}(\cdot) - S^{(k)}(\cdot)\right| > \left|S^{(k)}(\cdot) - S^{(k-1)}(\cdot)\right| \tag{11}$$

holds true.

Thus, we have the following numerical-analytical algorithm for solving the problem (1), (2).

Algorithm.

1. The coefficients F and a of equation (1) are given.
2. The initial functions $\varphi(x)$ of (2) is selected such that the Lipschitz constant L was $0<L<1$.
3. The initial approximation $S^0(x,t)$ is selected.
4. $S^{(k)}(x,t)$ from the relation (10) is calculated.
5. The condition (11) is checked at the satisfaction of which $S^{(k)}(x,t)$ is taken as an approximate solution.

Note that for the solution of the equation (7) other methods can be used, for example, the method of quasi-linearization [8]

Let us illustrate the above with the following example.

Example. As in [4], let $\varphi(z) = Ce^{-\beta z}$[1] from (2) ($C$ and β are real constants [4]). Then the non-linear algebraic equation (7) takes the form

$$S(x,t) = \frac{2at}{F} + Ce^{-\beta x}\left(1 - \frac{2at}{FS(x,t)}\right)^{-\beta/2a}.$$

After some transformations we obtain

$$S(x,t) = \frac{2at}{F} + Ce^{-\frac{2a\beta x}{2a+\beta}} S^{\frac{\beta}{2a+\beta}}(x,t).$$

Then, analogically to (10), the iterative process is as follows:

$$S^{k+1}(x,t) = \frac{2at}{F} + Ce^{-\frac{2a\beta x}{2a+\beta}} S^{(k)\frac{\beta}{2a+\beta}}(x,t), \tag{12}$$

at the given initial approximation $S^{(0)}(x,t)$. In the case when the given volume of gas is constant at the beginning of the well, then $\beta = 0$ and $\varphi(z) = C$ where the iterative process (12) converges to the exact solution of the problem (1), (2)

$$S(x,t) = C + \frac{2a}{F}t$$

This coincides with the result of [2].

References

[1] Aliev, F.A., (1989), Methods for solving of applied problems of dynamical systems optimization, Baku: Elm, 320p.

[2] Aliev, F.A., Hasanov, K.K., Guliyev, A.P., (2014), Sweep method for solving a system of hyperbolic equations describing the movement in oil exploration, Proccedings of the Institute of Applied Mathematics, 3(2), pp.249-255.

[3] Aliev, F.A., Ismailov, N.A., Mukhtarova, N.S., (2015), Algorithm to determine the optimal solution of a boundary control problem, Automation and Remote Control, 76(4), pp.627-633.

[4] Aliev, N.A., Aliev, F.A., Guliyev, A.P., Ilyasov, M.H., (2013), Method of series in solving a boundary value problem for a system of hyperbolic equations arising from the production of oil, Proceedings of the Institute of Applied Mathematics, 12(2), pp.113-138.

[5] Andreev, Y.N., (1976), Control of Finite-Dimensional Linear Objects, Moscow, Science, 424p.

[1] Such choice provides $L < 1$.

[6] Babich, V.M., Kapilevich, M.B., Mikhlin, S.G., Natanson, G.I, Ryz, P.., Slobodetskiy, L.N., Smirnov, M.M., (1964), Linear Equations of Mathematical Physics, Moscow, Nauka, 368p.

[7] Bordyug, V.A., Larin, V.B., Tymoshenko, A.G., (1985), Problems of control of walking aids, Kiev, Naukova Dumka, 263p.

[8] Bellman, R., Kalaba, R., (1966), Quazilinearization and Linear Boundary Value Problems, Moscow: Mir, 153p.

[9] Camponogara, V.I., Plucenio, A., Teixeira, A.F., Compos, S.R.V., (2010), Automation system for gas-lifted oil wells: model identification control and optimization, Journal of petroleum science and engineering, 70, pp.157-167.

[10] Erugin, N.P., (1949), Functional-invariant solutions of equations of second order with two independent variables, L., Scien.Notes of Univ., series Mat., 16, pp.142-166.

[11] Goursat, E., (1933), Course of Mathematical Analysis, II(II), Differential Equations, GTTI, Moscow, Leningrad, 285p.

[12] Gurbanov, R.S., Nuriyev, N.B., Gurbanov, R.S., (2013), Technological control and optimization problems in oil production: theory and practice, Appl. Comput. Math., 12(3), pp.314-324.

[13] Hasanov, K.K, Tanriverdiyev, T.S., (2015), Optimal control problem describing by the Cauchy problem for the first Ooder linear hyperbolic system with two independent variables, TWMS J. Pure Appl. Math., 6(1), pp.100-110

[14] Kamke, E., (1971), Guide to Ordinary Differential Equations, Nauka.

[15] Mikhlin, S.G., (1977), Partial Linear Equations, Moscow, High school, 430p.

[16] Petrovsky I.G., (1961), Lectures on Partial Differential Equations, GIFML, Moscow, 400p.

[17] Sobolev, S.L., (1954), The Equations of Mathematical Physics, GITTL, Moscow, 444p.

[18] Stepanov, V.V., (1966), Course of Differential Equations, Moscow, Nauka.

[19] Vladimirov, V.S., (1981), The equations of Mathematical Physics, Moscow, Nauka, 512p.

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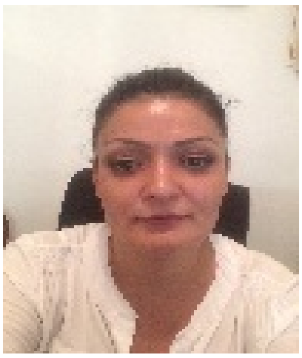

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