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COMMON COUPLED FIXED POINT THEOREM FOR GENERALIZED COMPATIBLE PAIR OF MAPPINGS UNDER GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION

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Abstract. We establish a coupled coincidence point theorem for generalized compatible pair of mappings $F, G{:}X\times X\rightarrow X$ under generalized Mizoguchi-Takahashi contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of $F{:}X\times X\rightarrow X$. An example supporting to our result has also been cited. As an application, we obtain the solution of integral equations to illustrate the usability of the obtained results. We improve, extend and generalize several known results.

Keywords: Coupled coincidence point, generalized Mizoguchi-Takahashi contraction, generalized compatibility, increasing mapping, mixed monotone mapping, commuting mapping.

AMS Subject Classification:47H10, 54H25.

1. Introduction and Preliminaries

Gnana-Bhaskar and Lakshmikantham [2] introduced the notion of coupled fixed point, mixed monotone mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces. These concepts are defined as follows.

Definition 1 [2]. Let $(X, \preccurlyeq)$ be a partially ordered set and endow the product space $X\times X$ with the following partial order:

$$(u, v)\preccurlyeq(x, y)\Leftrightarrow x\succcurlyeq u \text{ and } y\preccurlyeq v, \text{ for all } (u, v), (x, y)\in X\times X.$$

Definition 2 [2]. Let X be a set. An element $(x, y)\in X\times X$ is called a coupled fixed point of the mapping $F{:}X\times X\rightarrow X$ if $F(x, y)=x$ and $F(y, x)=y$.

Definition 3 [2]. Let $(X, \preccurlyeq)$ be a partially ordered set. Suppose $F{:}X\times X\rightarrow X$ be a given mapping. We say that F has the mixed monotone property if for all $x, y\in X$, we have

$$x_1, x_2\in X,\ x_1\preccurlyeq x_2 \text{ implies } F(x_1, y)\preccurlyeq F(x_2, y),$$

and

$$y_1, y_2\in X,\ y_1\preccurlyeq y_2 \text{ implies } F(x, y_1)\succcurlyeq F(x, y_2).$$

Lakshmikantham and Ciric [12] extended the notion of mixed monotone property to mixed g-monotone property and generalized the results of Gnana-Bhaskar and Lakshmikantham [2] by establishing the existence of coupled coincidence point results using a pair of commutative mappings.

Definition 4 [12]. Let X be a set. An element $(x, y)\in X\times X$ is called a coupled coincidence point of the mappings $F:X\times X\to X$ and $g:X\to X$ if

$$F(x, y)=g(x) \text{ and } F(y, x)=g(y).$$

Definition 5 [12]. Let X be a set. An element $(x, y)\in X\times X$ is called a common coupled fixed point of the mappings $F:X\times X\to X$ and $g:X\to X$ if

$$x=F(x, y)=g(x) \text{ and } y=F(y, x)=g(y).$$

Definition 6 [12]. Let X be a set. The mappings $F:X\times X\to X$ and $g:X\to X$ are said to be commutative if

$$g(F(x, y))=F(g(x), g(y)), \text{ for all } (x, y)\in X\times X.$$

Definition 7 [12]. Let $(X, \preccurlyeq)$ be a partially ordered set and $F:X\times X\to X$ and $g:X\to X$ are given mappings. We say that F has the mixed g-monotone property if for all x, $y\in X$, we have

$$x_1, x_2\in X,\ g(x_1)\preccurlyeq g(x_2) \text{ implies } F(x_1, y)\preccurlyeq F(x_2, y),$$

and

$$y_1, y_2\in X,\ g(y_1)\preccurlyeq g(y_2) \text{ implies } F(x, y_1)\succcurlyeq F(x, y_2).$$

Definition 8 [4]. Let X be a set. The mappings $F:X\times X\to X$ and $g:X\to X$ are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n=x,$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n=y, \text{ for some } x, y\in X.$$

As an application, these results used to study the existence and uniqueness of solution for periodic boundary value problems. Luong and Thuan [13] generalized the results of Gnana-Bhaskar and Lakshmikantham [2]. Berinde [1] extended the results of Gnana-Bhaskar and Lakshmikantham [2] and Luong and Thuan [13]. Jain et al. [11] extended and generalized the results of Berinde [1], Gnana-Bhaskar and Lakshmikantham [2], Lakshmikantham and Ciric [12] and Luong and Thuan [13].

In [14], Hussain et al. introduced a new concept of generalized compatibility of a pair of mappings defined on a product space and proved some coupled coincidence point results. Hussain et al. [14] also deduce some coupled fixed point results without mixed monotone property.

Definition 9 [14]. Let $(X, \preccurlyeq)$ be a partially ordered set and $F, G:X\times X\to X$ are two mappings. F is said to be G-increasing with respect to $\preccurlyeq$ if for all $x, y, u, v\in X$, with $G(x, y)\preccurlyeq G(u, v)$ we have $F(x, y)\preccurlyeq F(u, v)$.

Example 1 [14]. Let $X=(0, +\infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $F, G:X\times X\to X$ by $F(x, y)=\ln(x+y)$ and $G(x, y)=x+y$ for all $(x, y)\in X\times X$. Note that F is G-increasing with respect to $\leq$.

Example 2 [14]. Let $X=\mathbb{N}$ endowed with the partial order defined by $x, y\in X\times X$, $x\preccurlyeq y$ if and only if y divides x. Define the mappings $F, G:X\times X\to X$ by $F(x, y)=x^2y^2$ and $G(x, y)=xy$ for all $(x, y)\in X\times X$. Then F is G-increasing with respect to $\preccurlyeq$.

Definition 10 [14]. Let X be a set. An element $(x, y)\in X\times X$ is called a coupled coincidence point of mappings $F, G:X\times X\to X$ if $F(x, y)=G(x, y)$ and $F(y, x)=G(y, x)$.

Example 3 [14]. Let $F, G:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by $F(x, y) =xy$ and $G(x, y)=(2/3)(x+y)$ for all $(x, y)\in X\times X$. Note that $(0, 0)$, $(1, 2)$ and $(2, 1)$ are coupled coincidence points of F and G.

Definition 11 [14]. Let X be a set and $F, G:X\times X\to X$ be two mappings. We say that the pair $\{F, G\}$ is commuting if

$$F(G(x, y), G(y, x))=G(F(x, y), F(y, x)), \text{ for all } x, y\in X.$$

Definition 12 [14]. Let $(X, \preccurlyeq)$ be a partially ordered set, $F:X\times X\to X$ and $g:X\to X$ be two mappings. We say that F is g-increasing with respect to $\preccurlyeq$ if for any $x, y\in X$,

$$gx_1\preccurlyeq gx_2 \text{ implies } F(x_1, y)\preccurlyeq F(x_2, y),$$

and

$$gy_1\preccurlyeq gy_2 \text{ implies } F(x, y_1)\preccurlyeq F(x, y_2).$$

Definition 13 [14]. Let $(X, \preccurlyeq)$ be a partially ordered set, $F:X\times X\to X$ be a mapping. We say that F is increasing with respect to $\preccurlyeq$ if for any $x, y\in X$,

$$x_1\preccurlyeq x_2 \text{ implies } F(x_1, y)\preccurlyeq F(x_2, y),$$

and

$$y_1\preccurlyeq y_2 \text{ implies } F(x, y_1)\preccurlyeq F(x, y_2).$$

Definition 14 [14]. Let X be a set and $F, G:X\times X\to X$ be two mappings. We say that the pair $\{F, G\}$ is generalized compatible if

$$\lim_{n\to\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0,$$
$$\lim_{n\to\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0,$$

whenever (x_n) and (y_n) are sequences in X such that

$$\lim_{n\to\infty} G(x_n, y_n) = \lim_{n\to\infty} F(x_n, y_n)=x\in X,$$
$$\lim_{n\to\infty} G(y_n, x_n) = \lim_{n\to\infty} F(y_n, x_n)=y\in X.$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Recently Ciric et al. [3] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi condition in the setting of ordered metric spaces. Main results of Ciric et al. [3] extended and generalized the results of Gnana-Bhaskar and Lakshmikantham [2], Du [9] and Harjani et al. [10].

Very recentlySamet et al. [19] claimed that most of the coupled fixed point theorems on ordered metric spaces are consequences of well-known fixed point theorems. For more details, see [3, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18, 20, 21] and the reference therein.

In this paper, we establish a coupled coincidence point theorem for generalized compatible pair of mappings $F, G:X\times X\to X$ under generalized Mizoguchi-Takahashi contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of $F:X\times X\to X$. An example supporting to our result has also been cited. As an application, we obtain the solution of integral equations to illustrate the usability of the obtained results.

We improve, extend and generalize the results of Gnana-Bhaskar and Lakshmikantham [2], Ciric et al. [3], Du [9] and Harjani et al. [10].

2. Main results

Let Φ denote the set of all functions $\phi:[0, +\infty)\to[0, +\infty)$ satisfying
(i_ϕ) ϕ is non-decreasing,
(ii_ϕ) $\phi(t)=0\Leftrightarrow t=0$,
(iii_ϕ) $\limsup_{t\to 0+}\left(\frac{t}{\phi(t)}\right)<\infty$.
Let Ψ denote the set of all functions $\psi:[0, +\infty)\to [0, 1)$ which satisfies $\lim_{r\to t+}\psi(r)<1$ for all $t\geq 0$.

Theorem 1. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G:X\times X\to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0\in X$ with

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

Suppose that there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$\phi(d(F(x, y), F(u, v))) \tag{1}$$
$$\leq\psi\left(\phi\left(max\begin{Bmatrix}d(G(x,y),G(u,v)),\\ d(G(y,x),G(v,u))\end{Bmatrix}\right)\right)\phi\left(max\begin{Bmatrix}d(G(x,y),G(u,v)),\\ d(G(y,x),G(v,u))\end{Bmatrix}\right)$$

for all $x, y, u, v\in X$, where $G(x, y)\preccurlyeq G(u, v)$ and $G(y, x)\succcurlyeq G(v, u)$. Suppose that for any $x, y\in X$, there exist $u, v\in X$ such that

$$F(x, y)=G(u, v) \text{ and } F(y, x)=G(v, u). \tag{2}$$

Also suppose that either
(a) F is continuous or
(b) X has the following properties:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n\preccurlyeq x$, for all n,
(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x\preccurlyeq x_n$, for all n.
Then F and G have a coupled coincidence point.

Proof. By hypothesis, there exist $x_0, y_0\in X$ such that

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

From (2), we can choose $x_1, y_1\in X$ such that

$$G(x_1, y_1)=F(x_0, y_0) \text{ and } G(y_1, x_1)=F(y_0, x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$G(x_{n+1}, y_{n+1})=F(x_n, y_n) \text{ and } G(y_{n+1}, x_{n+1})=F(y_n, x_n), \text{ for all } n\geq 0. \tag{3}$$

We shall show that

$$G(x_n, y_n)\preccurlyeq G(x_{n+1}, y_{n+1}) \text{ and } G(y_n, x_n)\succcurlyeq G(y_{n+1}, x_{n+1}), \text{ for all } n\geq 0. \tag{4}$$

We shall use the mathematical induction. Let n=0, since

$$G(x_0, y_0) \preccurlyeq F(x_0, y_0)=G(x_1, y_1),$$
$$G(y_0, x_0) \succcurlyeq F(y_0, x_0)=G(y_1, x_1),$$

we have

$$G(x_0, y_0)\preccurlyeq G(x_1, y_1) \text{ and } G(y_0, x_0)\succcurlyeq G(y_1, x_1).$$

Thus (4) hold for n=0. Suppose now that (4) hold for some fixed $n\in\mathbb{N}$. Then since

$$G(x_n, y_n)\preccurlyeq G(x_{n+1}, y_{n+1}) \text{ and } G(y_n, x_n)\succcurlyeq G(y_{n+1}, x_{n+1}),$$

and as F is G-increasing with respect to $\preccurlyeq$, from (3), we have

$$G(x_{n+1}, y_{n+1})=F(x_n, y_n)\preccurlyeq F(x_{n+1}, y_{n+1})=G(x_{n+2}, y_{n+2}),$$
$$G(y_{n+1}, x_{n+1})=F(y_n, x_n)\succcurlyeq F(y_{n+1}, x_{n+1})=G(y_{n+2}, x_{n+2}).$$

Thus by the mathematical induction we conclude that (4) hold for all $n\geq 0$. Therefore

$$G(x_0, y_0)\preccurlyeq G(x_1, y_1)\preccurlyeq\ldots\preccurlyeq G(x_n, y_n)\preccurlyeq G(x_{n+1}, y_{n+1})\preccurlyeq\ldots$$

and

$$G(y_0, x_0)\succcurlyeq G(y_1, x_1)\succcurlyeq\ldots\succcurlyeq G(y_n, x_n)\succcurlyeq G(y_{n+1}, x_{n+1})\succcurlyeq\ldots$$

Now, by (1) and (i_ϕ), we have

$\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1})))$
$=\phi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)))$
$\leq\psi(\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}])\times\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$,

which, by the fact that $\psi<1$, implies

$\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1})))$
$\leq\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.

Similarly

$\phi(d(G(y_n, x_n), G(y_{n+1}, x_{n+1})))$
$\leq\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.

Combining them, we get

$\max\{\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1}))), \phi(d(G(y_n, x_n), G(y_{n+1}, x_{n+1})))\}$
$\leq\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.

Since ϕ is non-decreasing, it follows that

$$\phi[\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}]$$
$$\leq\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]. \qquad (5)$$

Now (5) shows that $\{\phi[\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}]\}$ is a non-increasing sequence. Therefore, there exists some $\delta\geq 0$ such that

$$\lim_{n\to\infty}\phi\left(max\left\{\begin{matrix}d(G(x_n, y_n), G(x_{n+1}, y_{n+1})),\\ d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\end{matrix}\right\}\right)=\delta. \qquad (6)$$

Since $\psi\in\Psi$, we have $\lim_{r\to\delta+}\psi(r)<1$ and $\psi(\delta)<1$. Then there exists $\alpha\in[0, 1)$ and $\varepsilon>0$ such that $\psi(r)\leq\alpha$ for all $r\in[\delta, \delta+\varepsilon)$. From (6), we can take $n_0\geq 0$ such that $\delta\leq\phi[\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}]\leq\delta+\varepsilon$ for all $n\geq n_0$. Then, by (1) and (i_ϕ), for all $n\geq n_0$, we have

$\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1})))$
$=\phi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)))$
$\leq\psi(\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}])$
$\times\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$
$\leq \alpha\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.

Thus, for all $n\geq n_0$, we have

$\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1})))$

$\leq\alpha\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.
Similarly, for all $n\geq n_0$, we have
$\phi(d(G(y_n, x_n), G(y_{n+1}, x_{n+1})))$
$\leq\alpha\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.
Combining them, for all $n\geq n_0$, we get
$\max\{\phi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1}))), \phi(d(G(y_n, x_n), G(y_{n+1}, x_{n+1})))\}$
$\leq\alpha\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}]$.
Since ϕ is non-decreasing, it follows that

$$\phi[\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}] \tag{7}$$
$$\leq\alpha\phi[\max\{d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), d(G(y_{n-1}, x_{n-1}), G(y_n, x_n))\}],$$

for all $n\geq n_0$. Letting $n\to\infty$ in the above inequality and using (6), we obtain that $\delta\leq\alpha\delta$. Since $\alpha\in[0, 1)$, therefore $\delta=0$. Thus

$$\lim_{n\to\infty}\phi\left(max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}\right)=0. \tag{8}$$

Since $\{\phi[\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}]\}$ is a non-increasing sequence and ϕ is non-decreasing, then $\{\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\theta\geq0$ such that

$$\lim_{n\to\infty} max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}=\theta. \tag{9}$$

Since ϕ is non-decreasing, we have

$$\phi\left(max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}\right)\geq\phi(\theta).$$

Letting $n\to\infty$ in this inequality, by using (2.8), we get $0\geq\phi(\theta)$ which, by (ii_ϕ), implies that $\theta=0$. Thus, by (2.9), we get

$$\lim_{n\to\infty} max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}=0. \tag{10}$$

Suppose that $\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}=0$, for some $n\geq0$. Then, we have $d(G(x_n, y_n), G(x_{n+1}, y_{n+1}))=0$ and $d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))=0$ which implies that $G(x_n, y_n)=G(x_{n+1}, y_{n+1})=F(x_n, y_n)$ and $G(y_n, x_n)=G(y_{n+1}, x_{n+1})=F(y_n, x_n)$, that is, (x_n, y_n) is a coupled coincidence point of F and G. Now, suppose that $\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}\neq0$, for all ≥0. Denote

$$a_n=\phi\left(max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}\right), \text{ for all } n\geq0.$$

From (7), we have

$$a_n\leq\alpha a_{n-1}, \text{ for all } n\geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n\leq\sum_{n=0}^{n_0} a_n+\sum_{n=n_0+1}^{\infty}\alpha^{n-n_0}a_{n_0}<\infty. \tag{11}$$

On the other hand, by (iii_ϕ), we have

$$\limsup_{n\to\infty}\left(\frac{max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}}{\phi\left(max\left\{\begin{matrix}d(G(x_n,y_n),G(x_{n+1},y_{n+1})),\\ d(G(y_n,x_n),G(y_{n+1},x_{n+1}))\end{matrix}\right\}\right)}\right)<\infty. \quad (12)$$

Thus, by (11) and (12), we have $\sum\max\{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))\}<\infty$. It means that $\{G(x_n, y_n)_{n\geq 0}$ and $\{G(y_n, x_n)_{n\geq 0}$ are Cauchy sequences in X. Since X is complete, therefore there exist some $x, y\in X$ such that

$$\lim_{n\to\infty}G(x_n, y_n)=\lim_{n\to\infty}F(x_n, y_n)=x, \quad (13)$$

$$\lim_{n\to\infty}G(y_n, x_n)=\lim_{n\to\infty}F(y_n, x_n)=y.$$

Since the pair {F, G} satisfies the generalized compatibility, from (13), we get

$$\lim_{n\to\infty}d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n)))=0, \quad (14)$$

and

$$\lim_{n\to\infty}d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n)))=0. \quad (15)$$

Suppose that assumption (a) holds. Then

$d(F(G(x_n, y_n), G(y_n, x_n)), G(x, y))$

$\leq d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n)))+d(G(F(x_n, y_n), F(y_n, x_n)), G(x, y))$.

Taking limit as $n\to\infty$ in the above inequality, using (13), (14) and the fact that F and G are continuous, we have

$$F(x, y)=G(x, y).$$

Similarly we can show that

$$F(y, x)=G(y, x).$$

Thus (x, y) is a coupled coincidence point of F and G.

Now, suppose that (b) holds. Now we show that (x, y) is a coupled coincidence point of F and G. By (4) and (13), we have $\{G(x_n, y_n)\}$ is a non-decreasing sequence, $G(x_n, y_n)\to x$ and $\{G(y_n, x_n)\}$ is a non-increasing sequence, $G(y_n, x_n)\to y$ as $n\to\infty$. Thus for all n, we have

$$G(x_n, y_n)\preccurlyeq x \text{ and } G(y_n, x_n)\succcurlyeq y. \quad (16)$$

Since G is continuous, by (13), (14) and (15), we have

$\lim_{n\to\infty}G(G(x_n, y_n), G(y_n, x_n))$

$=G(x, y)$

$=\lim_{n\to\infty}G(F(x_n, y_n), F(y_n, x_n))$

$$=\lim_{n\to\infty}F(G(x_n, y_n), G(y_n, x_n)) \quad (17)$$

and

$\lim_{n\to\infty}G(G(y_n, x_n), G(x_n, y_n))$

$=G(y, x)$

$=\lim_{n\to\infty}G(F(y_n, x_n), F(x_n, y_n))$

$$=\lim_{n\to\infty}F(G(y_n, x_n), G(x_n, y_n)). \quad (18)$$

Since G has the mixed monotone property, it follows from (16) that $G(G(x_n, y_n), G(y_n, x_n))\preccurlyeq G(x, y)$ and $G(G(y_n, x_n), G(x_n, y_n))\succcurlyeq G(y, x)$. Now, by using (1) and (i_ϕ), we have

$\phi(d(G(x, y), F(x, y)))$

$=\lim_{n\to\infty}\phi(d(G(F(x_n, y_n), F(y_n, x_n)), F(x, y)))$

$=\lim_{n\to\infty}\phi(d(F(G(x_n, y_n), G(y_n, x_n)), F(x, y)))$

$\leq\lim_{n\to\infty}\psi(\phi[\max\{d(G(G(x_n, y_n), G(y_n, x_n)), G(x, y)), d(G(G(y_n, x_n), G(x_n, y_n)), G(y, x))\}])\times\phi[\max\{d(G(G(x_n, y_n), G(y_n, x_n)), G(x, y)), d(G(G(y_n, x_n), G(x_n, y_n)), G(y, x))\}]$,

it follows that

$\phi(d(G(x, y), F(x, y)))$

$\leq\lim_{n\to\infty}\phi[\max\{d(G(G(x_n, y_n), G(y_n, x_n)), G(x, y)), d(G(G(y_n, x_n), G(x_n, y_n)), G(y, x))\}]$,

which, by (i_ϕ), implies

$d(G(x, y), F(x, y))$

$\leq\lim_{n\to\infty}\max\{d(G(G(x_n, y_n), G(y_n, x_n)), G(x, y)), d(G(G(y_n, x_n), G(x_n, y_n)), G(y, x))\}$.

Thus, by (17) and (18), we get

$$d(G(x, y), F(x, y)))=0.$$

Similarly, we can show that

$$d(G(y, x), F(y, x)))=0.$$

It follows that

$$G(x, y)=F(x, y) \text{ and } G(y, x)=F(y, x),$$

that is, (x, y) is a coupled coincidence point of F and G.

Corollary 1. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G:X\times X\to X$ be two commuting mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0\in X$ with

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

Suppose that the inequalities (1) and (2) hold and either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n\preccurlyeq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x\preccurlyeq x_n$, for all n.

Then F and G have a coupled coincidence point.

Now, we deduce result without g-mixed monotone property of F.

Corollary 2. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F:X\times X\to X$ and $g:X\to X$ be two mappings such that F is g-increasing with respect to $\preccurlyeq$, and there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$\phi(d(F(x, y), F(u, v)))$

$\leq\psi(\phi[\max\{d(gx, gu), d(gy, gv)\}])\phi[\max\{d(gx, gu), d(gy, gv)\}]$,

for all $x, y, u, v\in X$, where $g(x)\preccurlyeq g(u)$ and $g(y)\succcurlyeq g(v)$. Suppose that $F(X\times X)\subseteq g(X)$, g is continuous and monotone increasing with respect to $\preccurlyeq$ and the pair $\{F, g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n\preccurlyeq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x\preccurlyeq x_n$, for all n.

If there exist two elements $x_0, y_0\in X$ with

$$gx_0 \preccurlyeq F(x_0, y_0) \text{ and } gy_0 \succcurlyeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Corollary 3. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F:X\times X\to X$ and $g:X\to X$ be two mappings such that F is g-increasing with respect to $\preccurlyeq$, and there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$\phi(d(F(x, y), F(u, v)))$

$\leq\psi(\phi[\max\{d(gx, gu), d(gy, gv)\}])\phi[\max\{d(gx, gu), d(gy, gv)\}]$,

for all $x, y, u, v\in X$, where $g(x)\preccurlyeq g(u)$ and $g(y)\succcurlyeq g(v)$. Suppose that $F(X\times X)\subseteq g(X)$, g is continuous and monotone increasing with respect to $\preccurlyeq$ and the pair $\{F, g\}$ is commuting. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n\preccurlyeq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x\preccurlyeq x_n$, for all n.

If there exist two elements $x_0, y_0\in X$ with

$$gx_0 \preccurlyeq F(x_0, y_0) \text{ and } gy_0 \succcurlyeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Corollary 4. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F:X\times X\to X$ be an increasing mapping with respect to $\preccurlyeq$ and there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$\phi(d(F(x, y), F(u, v)))$

$\leq\psi(\phi[\max\{d(x, u), d(y, v)\}])\phi[\max\{d(x, u), d(y, v)\}]$,

for all $x, y, u, v\in X$, where $x\preccurlyeq u$ and $y\succcurlyeq v$. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n\preccurlyeq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x\preccurlyeq x_n$, for all n.

If there exist two elements $x_0, y_0\in X$ with

$$x_0 \preccurlyeq F(x_0, y_0) \text{ and } y_0 \succcurlyeq F(y_0, x_0).$$

Then F has a coupled fixed point.

If we put $\psi(t)=1-(\psi(t)/t)$ for all $t\geq 0$ in Theorem 1, then we get the following result:

Corollary 5. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G:X\times X\to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0\in X$ with

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

Suppose that there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$\phi(d(F(x, y), F(u, v)))$

$\leq\phi[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}]-\psi(\phi[\max\{d(G(x, y), G(u, v)),d(G(y, x), G(v, u))\}])$,

for all $x, y, u, v\in X$, where $G(x, y)\preccurlyeq G(u, v)$ and $G(y, x)\succcurlyeq G(v, u)$. Suppose that for any $x, y\in X$, there exist $u, v\in X$ such that

$$F(x, y)=G(u, v) \text{ and } F(y, x)=G(v, u).$$

Also suppose that either
(a) F is continuous or
(b) X has the following properties:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n \preccurlyeq x$, for all n,
(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x \preccurlyeq x_n$, for all n.
Then F and G have a coupled coincidence point.
If we put $\phi(t)=2t$ for all $t\geq 0$ in Theorem 1, then we get the following result:
Corollary 6. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume F, $G:X\times X\to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0\in X$ with

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

Suppose that there exists some $\psi\in\Psi$ such that
$d(F(x, y), F(u, v))$
$\leq\psi(2\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\})\times\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}$,
for all $x, y, u, v\in X$, where $G(x, y)\preccurlyeq G(u, v)$ and $G(y, x)\succcurlyeq G(v, u)$. Suppose that for any $x, y\in X$, there exist $u, v\in X$ such that

$$F(x, y)=G(u, v) \text{ and } F(y, x)=G(v, u).$$

Also suppose that either
(a) F is continuous or
(b) X has the following properties:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n \preccurlyeq x$, for all n,
(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x \preccurlyeq x_n$, for all n.
Then F and G have a coupled coincidence point.
If we put $\psi(t)=k$ where $0<k<1$, for all $t\geq 0$ in Corollary 6, then we get the following result:
Corollary 7. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete metric d on X. Assume F, $G:X\times X\to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0\in X$ with

$$G(x_0, y_0)\preccurlyeq F(x_0, y_0) \text{ and } G(y_0, x_0)\succcurlyeq F(y_0, x_0).$$

Suppose
$d(F(x, y), F(u, v))\leq k\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}$,
for all $x, y, u, v\in X$ and $0<k<1$, where $G(x, y)\preccurlyeq G(u, v)$ and $G(y, x)\succcurlyeq G(v, u)$. Suppose that for any $x, y\in X$, there exist $u, v\in X$ such that

$$F(x, y)=G(u, v) \text{ and } F(y, x)=G(v, u).$$

Also suppose that either
(a) F is continuous or
(b) X has the following properties:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in X then $x_n \preccurlyeq x$, for all n,
(ii) if a non-increasing sequence $\{x_n\}\to x$ in X then $x \preccurlyeq x_n$, for all n.
Then F and G have a coupled coincidence point.

Example 4. Suppose that X=[0, 1] be endowed with the natural ordering of real numbers $\leq$ and equipped with the usual metric $d:X\times X\to[0, +\infty)$. Then (X, d) is a complete metric space. Let F, $G:X\times X\to X$ be defined as

$$F(x, y)=\begin{cases}\frac{x^2-y^2}{4}, \text{if } x\geq y,\\ 0, \text{if } x<y,\end{cases}$$

and

$$G(x, y)=\begin{cases}x^2-y^2, \text{if } x\geq y,\\ 0, \text{if } x<y.\end{cases}$$

Define $\phi:[0, +\infty)\to[0, +\infty)$ by

$$\phi(t)=\begin{cases}\ln(t+1), \text{for } t\neq 1,\\ (3/4), \text{for } t=1,\end{cases}$$

and $\psi:[0, +\infty)\to[0, 1)$ defined by

$$\psi(t)=\phi(t)/t, \text{ for all } t\geq 0.$$

First, we shall show that F is G-increasing. Let (x, y), (u, v)$\in X\times X$ with $G(x, y)\leq G(u, v)$. We consider the following cases:

Case 1: If $x<y$, then $F(x, y)=0\leq F(u, v)$.

Case 2: If $x\geq y$ and $u\geq v$, then $G(x, y)\leq G(u, v)\Rightarrow x^2-y^2\leq u^2-v^2\Rightarrow(x^2-y^2)/4\leq(u^2-v^2)/4\Rightarrow F(x, y)\leq F(u, v)$. But if $u<v$, then $G(x, y)\leq G(u, v)\Rightarrow 0\leq x^2-y^2\leq 0\Rightarrow x^2=y^2\Rightarrow F(x, y)=0\leq F(u, v)$.

Thus F is G-increasing. Now, we prove that for any x, y$\in$X, there exist u, v$\in$X such that

$$F(x, y)=G(u, v) \text{ and } F(y, x)=G(v, u).$$

Let (x, y), (u, v)$\in X\times X$ be fixed. We consider the following cases:

Case 1: If x=y, then we have $F(x, y)=0=G(x, y)$ and $F(y, x)=0=G(y, x)$.

Case 2: If $x>y$, then we have $F(x, y)=(x^2-y^2)/4=G(x/2, y/2)$ and $F(y, x)=0=G(y/2, x/2)$.

Case 3: If $x<y$, then we have $F(x, y)=0=G(x/2, y/2)$ and $F(y, x)=(y^2-x^2)/4=G(y/2, x/2)$.

Now we prove that G is continuous and has the mixed monotone property. Clearly G is continuous. Let (x, y)$\in X\times X$ be fixed. Suppose that x_1, $x_2\in X$ are such that $x_1<x_2$.

Case 1: If $x_1<y$, then we have $G(x_1, y)=0\leq G(x_2, y)$.

Case 2: If $x_1\geq y$, then we have $G(x_1, y)=x_1^2-y^2\leq x_2^2-y^2=G(x_2, y)$.

Similarly, we can show that if y_1, $y_2\in X$ are such that $y_1<y_2$, then $G(x, y_1)\geq G(x, y_2)$.

Now, we prove that the pair {F, G} satisfies the generalized compatibility hypothesis.

Let (x_n) and (y_n) be two sequences in X such that

$$\lim_{n\to\infty}G(x_n, y_n)=\lim_{n\to\infty}F(x_n, y_n)=t_1,$$
$$\lim_{n\to\infty}G(y_n, x_n)=\lim_{n\to\infty}F(y_n, x_n)=t_2,$$

then we must have $t_1=t_2=0$ and one can easily prove that

$$\lim_{n\to\infty}d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))=0,$$

$$\lim_{n\to\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n)))=0.$$

Now we prove that there exist two elements $x_0, y_0 \in X$ with

$$G(x_0, y_0) \leq F(x_0, y_0) \text{ and } G(y_0, x_0) \geq F(y_0, x_0).$$

Since we have $G(0, 1/2)=0=F(0, 1/2)$ and $G(1/2, 0)=(1/4)\geq(1/16)=F(1/2, 0)$. Next, we shall show that the mappings F and G satisfy the condition (1). Let $x, y, u, v \in X$ such that $G(x, y)\leq G(u, v)$ and $G(y, x)\geq G(v, u)$. Then

$d(F(x, y), F(u, v))$
$=|(x^2-y^2)/4-(u^2-v^2)/4|$
$\leq \ln(|(x^2-y^2)-(u^2-v^2)|+1)$
$\leq \ln(|G(x, y)-G(u, v)|+1)$
$\leq \ln(d(G(x, y), G(u, v))+1)$
$\leq \ln[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}+1]$,

which implies that

$\phi(d(F(x, y), F(u, v)))$
$=\ln[d(F(x, y), F(u, v))+1]$
$\leq \ln[\ln[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}+1]+1]$
$\leq \ln[\ln[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}+1]+1]/(\ln[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}+1]\times\ln[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}+1]$
$\leq \psi(\phi[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}])$
$\times\phi[\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}]$.

Thus the contractive condition (1) is satisfied for all $x, y, u, v \in X$ and $z=(0, 0)$ is a coupled coincidence point of F and G.

Now we prove the uniqueness of the coupled coincidence point. Note that if $(X, \preccurlyeq)$ is a partially ordered set, then we endow the product $X\times X$ with the following partial order relation, for all $(x, y), (u, v)\in X\times X$:

$$(x, y)\preccurlyeq(u, v) \Leftrightarrow G(x, y)\preccurlyeq G(u, v) \text{ and } G(y, x)\succcurlyeq G(v, u),$$

where $G:X\times X\to X$ is one-one.

Theorem 2. In addition to the hypotheses of Theorem 1, suppose that for every (x, y), (x^*, y^*) in $X\times X$, there exists another (u, v) in $X\times X$ which is comparable to (x, y) and (x^*, y^*), then F and G have a unique coupled coincidence point.

Proof. From Theorem 1, the set of coupled coincidence points of F and G is non-empty. Assume that $(x, y), (x^*, y^*)\in X\times X$ are two coupled coincidence points of F and G, that is,

$$F(x, y) = G(x, y) \text{ and } F(y, x)=G(y, x),$$
$$F(x^*, y^*) = G(x^*, y^*) \text{ and } F(y^*, x^*)=G(y^*, x^*).$$

We shall prove that $G(x, y)= G(x^*, y^*)$ and $G(y, x)= G(y^*, x^*)$. By assumption, there exists $(u, v)\in X\times X$, that is, comparable to (x, y) and (x^*, y^*). We define the sequences $\{G(u_n, v_n)\}$ and $\{G(v_n, u_n)\}$ as follows, with $u_0=u$, $v_0=v$:

$$G(u_{n+1}, v_{n+1})=F(u_n, v_n) \text{ and } G(v_{n+1}, u_{n+1})=F(v_n, u_n),\ n\geq 0.$$

Since (u, v) is comparable to (x, y), we may assume that $(x, y)\preccurlyeq(u, v)=(u_0, v_0)$, which implies that $G(x, y)\preccurlyeq G(u_0, v_0)$ and $G(y, x)\succcurlyeq G(v_0, u_0)$. We suppose that $(x, y)\preccurlyeq(u_n, v_n)$ for some n. We shall prove that $(x, y)\preccurlyeq(u_{n+1}, v_{n+1})$. Since F is G-increasing, we have $G(x, y)\preccurlyeq G(u_n, v_n)$ implies $F(x, y)\preccurlyeq F(u_n, v_n)$ and $G(y, x)\succcurlyeq G(v_n, u_n)$ implies $F(y, x)\succcurlyeq F(v_n, u_n)$. Therefore

$$G(x, y)=F(x, y)\preccurlyeq F(u_n, v_n)=G(u_{n+1}, v_{n+1}),$$

and

$$G(y, x)=F(y, x)\succcurlyeq F(v_n, u_n)=G(v_{n+1}, u_{n+1}).$$

Thus, we have

$$(x, y)\preccurlyeq(u_{n+1}, v_{n+1}), \text{ for all } n.$$

Now, by (1) and (i_ϕ), we have

$$\begin{aligned}&\phi(d(G(x, y), G(u_{n+1}, v_{n+1})))\\&=\phi(d(F(x, y), F(u_n, v_n)))\\&\leq\psi(\phi[\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}])\\&\times\phi[\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}],\end{aligned}$$

which, by the fact that $\psi<1$, implies

$$\begin{aligned}&\phi(d(G(x, y), G(u_{n+1}, v_{n+1})))\\&\leq\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}).\end{aligned}$$

Similarly

$$\begin{aligned}&\phi(d(G(y, x), G(v_{n+1}, u_{n+1})))\\&\leq\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}).\end{aligned}$$

Combining them, we get

$$\begin{aligned}&\max\{\phi(d(G(x, y), G(u_{n+1}, v_{n+1}))), \phi(d(G(y, x), G(v_{n+1}, u_{n+1})))\}\\&\leq\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}).\end{aligned}$$

Since ϕ is non-decreasing, it follows that

$$\begin{aligned}&\Phi(\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\})\\&\qquad\leq\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}). \qquad (19)\end{aligned}$$

Now (19) shows that $\{\phi(\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\})\}$ is a non-increasing sequence. Therefore, there exists some $\Delta\geq0$ such that

$$\lim_{n\to\infty}\phi\left(\max\left\{\begin{matrix}d\big(G(x,y), G(u_{n+1}, v_{n+1})\big),\\ d(G(y,x), G(v_{n+1}, u_{n+1}))\end{matrix}\right\}\right)=\Delta. \qquad (20)$$

Since $\psi\in\Psi$, we have $\lim_{r\to\Delta+}\psi(r)<1$ and $\psi(\Delta)<1$. Then there exists $\beta\in[0, 1)$ and $\varepsilon>0$ such that $\psi(r)\leq\beta$ for all $r\in[\Delta, \Delta+\varepsilon)$. From (20), we can take $n_0\geq0$ such that $\Delta\leq\phi[\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\}]\leq\Delta+\varepsilon$ for all $n\geq n_0$. Then, from (1) and (i_ϕ), for all $n\geq n_0$, we have

$$\begin{aligned}&\phi(d(G(x, y), G(u_{n+1}, v_{n+1})))\\&=\phi(d(F(x, y), F(u_n, v_n)))\\&\leq\psi(\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}))\\&\times\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\})\\&\leq\beta\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}).\end{aligned}$$

Thus, for all $n\geq n_0$, we have

$$\begin{aligned}&\phi(d(G(x, y), G(u_{n+1}, v_{n+1})))\\&\leq\beta\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\}).\end{aligned}$$

Similarly, for all $n\geq n_0$, we have

$$\phi(d(G(y, x), G(v_{n+1}, u_{n+1})))$$

$\leq\beta\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\})$.
Combining them, for all $n\geq n_0$, we get
$\max\{\phi(d(G(x, y), G(u_{n+1}, v_{n+1}))), \phi(d(G(y, x), G(v_{n+1}, u_{n+1})))\}$
$\leq\beta\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\})$.
Since ϕ is non-decreasing, it follows that

$$\phi(\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\}) \tag{21}$$

$\leq\beta\phi(\max\{d(G(x, y), G(u_n, v_n)), d(G(y, x), G(v_n, u_n))\})]$,
for all $n\geq n_0$. Letting $n\to\infty$ in the above inequality and using (20), we obtain that $\Delta\leq\beta\Delta$. Since $\beta\in[0, 1)$, therefore $\Delta=0$. Thus by (20), we get

$$\lim_{n\to\infty}\phi\left(\max\begin{Bmatrix} d\big(G(x,y), G(u_{n+1}, v_{n+1})\big), \\ d(G(y,x), G(v_{n+1}, u_{n+1})) \end{Bmatrix}\right)=0. \tag{22}$$

Since $\{\phi(\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\})\}$ is a non-increasing sequence and ϕ is non-decreasing, then $\{\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\}\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\xi\geq0$ such that

$$\lim_{n\to\infty}\max\begin{Bmatrix} d\big(G(x,y), G(u_{n+1}, v_{n+1})\big), \\ d(G(y,x), G(v_{n+1}, u_{n+1})) \end{Bmatrix}= \xi. \tag{23}$$

Since ϕ is non-decreasing, we have

$$\phi(\max\{d(G(x, y), G(u_{n+1}, v_{n+1})), d(G(y, x), G(v_{n+1}, u_{n+1}))\})\geq\phi(\xi).$$

Letting $n\to\infty$ in this inequality, by using (22), we get $0\geq\phi[\xi]$, which, by (ii_ϕ), implies that $\xi=0$. Thus, by (23), we get

$$\lim_{n\to\infty}\max\begin{Bmatrix} d\big(G(x,y), G(u_{n+1}, v_{n+1})\big), \\ d(G(y,x), G(v_{n+1}, u_{n+1})) \end{Bmatrix}= 0,$$

which implies that

$$G(x, y)=\lim_{n\to\infty}G(u_{n+1}, v_{n+1}) \text{ and } G(y, x)=\lim_{n\to\infty}G(v_{n+1}, u_{n+1}).$$

Similarly, we can show that

$$G(x^*, y^*)=\lim_{n\to\infty}G(u_{n+1}, v_{n+1}) \text{ and } G(y^*, x^*)=\lim_{n\to\infty}G(v_{n+1}, u_{n+1}).$$

Thus $G(x, y)= G(x^*, y^*)$ and $G(y, x)= G(y^*, x^*)$.

3. Application to integral equations

As an application of the results established in section 2 of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p)=\int_a^b\big(K_1(p,q) + K_2(p,q)\big)\big[f\big(q,x(q)\big) + g\big(q,x(q)\big)\big]dq + h(p), \tag{24}$$

for all $p\in I=[a, b]$.
Let Θ denote the set of all functions $\theta:[0, +\infty)\to[0, +\infty)$ satisfying
(i_θ) θ is non-decreasing,
(ii_θ) $\theta(p)=\ln(p+1)$.
Assumption 1. We assume that the functions K_1, K_2, f, g fulfill the following conditions:

(i) $K_1(p, q)\geq 0$ and $K_2(p, q)\geq 0$ for all p, q∈I.
(ii) There exists positive numbers λ, μ and θ∈Θ such that for all x, y∈ I with x≥y, the following conditions hold:

$$0\leq f(q, x)-f(q, y)\leq \lambda\theta(x-y), \tag{25}$$

and

$$0\leq g(q, x)-g(q, y)\leq \mu\theta(x-y). \tag{26}$$

(iii) $$\max\{\lambda, \mu\}\sup_{p\in I}\int_a^b (K_1(p,q) + K_2(p,q))dq\leq 1/2. \tag{27}$$

Definition 15. [13]. A pair (α, β)∈X×X with X=C(I, ℝ), where C(I, ℝ) denote the set of all continuous functions from I to ℝ, is called a coupled lower-upper solution of (24) if, for all p∈I,

$\alpha(p)\leq\int_a^b K_1(p,q)[f(q,\alpha(q)) + g(q,\beta(q))]dq+\int_a^b K_2(p,q)\left[f\left(q,\beta(q)\right) + g\left(q,\alpha(q)\right)\right]dq + h(p),$

and

$\beta(q)\geq\int_a^b K_1(p,q)[f(q,\beta(q)) + g(q,\alpha(q))]dq+\int_a^b K_2(p,q)[f(q,\alpha(q)) + g(q,\beta(q))]dq + h(p).$

Theorem 3. Consider the integral equation (24) with K_1, K_2∈C(I×I, ℝ), f, g∈C(I× ℝ, ℝ) and h∈C(I, ℝ). Suppose that there exists a coupled lower-upper solution (α, β) of (24) and that Assumption 1 is satisfied. Then the integral equation (24) has a solution in C(I, ℝ).

Proof. Consider X=C(I, ℝ), the natural partial order relation, that is, for x, y∈C(I, ℝ),

$$x\leq y\Leftrightarrow x(p)\leq y(p),\ \forall p\in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x, y)=\sup_{p\in I}|x(p)-y(p)|.$$

Now define on X×X the following partial order: for (x, y), (u, v)∈X×X,

$$(x, y)\ )\leq(u, v)\Leftrightarrow x(p)\leq u(p) \text{ and } y(p)\geq v(p), \text{ for all } p\in I.$$

Define ϕ:[0, +∞)→[0, +∞) by

$$\phi(t)=\begin{cases}\ln(t+1), \text{for } t\neq 1,\\ \frac{3}{4}, \text{for } t=1,\end{cases}$$

and ψ:[0, +∞)→ [0, 1) defined by

$$\psi(t)=\phi(t)/t, \text{ for all } t\geq 0.$$

Define now the mapping F:X×X→X by

$$F(x, y)(p))=\int_a^b K_1(p,q)[f(q,x(q)) + g(q,y(q))]dq$$
$$+\int_a^b K_2(p,q)[f(q,y(q)) + g(q,x(q))]dq + h(p),$$

for all p∈I. It is not difficult to prove, like in [14], that F is increasing. Now for x, y, u, v∈X with x≤u and y≥v, we have

$F(x, y)(p)-F(u, v)(p)$

$=\int_a^b K_1(p,q)[\left(f\left(q,x(q)\right) - f\left(q,u(q)\right)\right) + (g(q,y(q)) - g(q,v(q)))]dq$

$+\int_a^b K_2(p,q)[\left(f\left(q,y(q)\right) - f\left(q,v(q)\right)\right) + (g(q,x(q)) - g(q,u(q)))]dq$

Thus, by using (25) and (26), we get

$$F(x, y)(p)-F(u, v)(p) \tag{27}$$
$$\leq\int_a^b K_1(p,q)[\lambda\theta(x(q)-u(q))+\mu\theta(y(q)-v(q))]dq$$
$$+\int_a^b K_2(p,q)[\lambda\theta(y(q)-v(q))+\mu\theta(x(q)-u(q))]dq.$$

Since the function θ is non-decreasing and $x\leq u$ and $y\geq v$, we have

$$\theta(x(q)-u(q)) \leq \theta(\sup_{q\in I}|x(q)-u(q)|)=\theta(d(x, u)),$$
$$\theta(y(q)-v(q)) \leq \theta(\sup_{q\in I}|y(q)-v(q)|)=\theta(d(y, v)).$$

Hence by (27), in view of the fact that $K_2(p, q)\leq 0$, we obtain

$$|F(x, y)(p)-F(u, v)(p)|$$
$$\leq\int_a^b K_1(p,q)[\lambda\theta(d(x,u))+\mu\theta(d(y,v))]dq$$
$$+\int_a^b K_1(p,q)[\lambda\theta(d(y,v))+\mu\theta(d(x,u))]dq$$
$$\leq\int_a^b K_1(p,q)[\max\{\lambda,\mu\}\theta(d(x,u))+\max\{\lambda,\mu\}\theta(d(y,v))]dq$$
$$+\int_a^b K_2(p,q)[\max\{\lambda,\mu\}\theta(d(y,v))+\max\{\lambda,\mu\}\theta(d(x,u))]dq$$

as all the quantities on the right hand side of (27) are non-negative. Now, taking the supremum with respect to p we get, by using (26),

$$d(F(x, y), F(u, v))$$
$$\leq \max\{\lambda,\mu\}sup_{p\in I}\int_a^b\big(K_1(p,q)+K_2(p,q)\big)dq[\theta(d(x,u))+\theta(d(y,v))]$$
$$\leq (\theta(d(x, u))+\theta(d(y, v)))/2.$$

Thus

$$d(F(x, y), F(u, v))\leq(\theta(d(x, u))+\theta(d(y, v)))/2. \tag{28}$$

Now, since θ is non-decreasing, we have

$$\theta(d(x, u)) \leq \theta(\max\{d(x, u), d(y, v)\}),$$
$$\theta(d(y, v)) \leq \theta(\max\{d(x, u), d(y, v)\}),$$

which implies, by (ii_θ), that

$$(\theta(d(x, u))+\theta(d(y, v)))/2\leq\theta(\max\{d(x, u), d(y, v)\})$$
$$\leq\ln[\max\{d(x, u), d(y, v)\}+1]. \tag{29}$$

Thus by (28) and (29), we have

$$d(F(x, y), F(u, v))\leq\ln[\max\{d(x, u), d(y, v)\}+1],$$

which implies that

$$\phi(d(F(x, y), F(u, v)))$$
$$=\ln[d(F(x, y), F(u, v))+1]$$
$$\leq\ln[\ln[\max\{d(x, u), d(y, v)\}+1]+1]$$
$$\leq((\ln[\ln[\max\{d(x, u), d(y, v)\}+1]+1])/(\ln[\max\{d(x, u), d(y, v)\}+1]))$$
$$\times\ln[\max\{d(x, u), d(y, v)\}+1]$$
$$\leq\psi(\phi(\max\{d(x, u), d(y, v)\}))\times\phi(\max\{d(x, u), d(y, v)\}),$$

which is the contractive condition in Corollary 4. Now, let $(\alpha, \beta)\in X\times X$ be a coupled upper-lower solution of (24), then we have $\alpha(p)\leq F(\alpha, \beta)(p)$ and $\beta(p)\geq F(\beta, \alpha)(p)$, for all $p\in I$, which shows that all hypothesis of Corollary 4 are satisfied. This proves that F has a coupled fixed point $(x, y)\in X\times X$ which is the solution in $X=C(I, \mathbb{R})$ of the integral equation (24).

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Ümumiləşmiş mizoquçi-takahaşi sixilmasi mənada inikaslarin ümumiləşmiş uyuşan cütü üçün ikiqat təsadüfi nöqtə haqqinda ümumi teorem

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XÜLASƏ

İşdə hissə-hissə nizamlanmış metric fəzada ümumiləşmiş Mizoquçi-Takahaşi sıxılması mənada F, G:X×X→X inikaslarının ümumiləşmiş uyuşan cütü üçün ikiqat təsadüfi nöqtə haqqında teorem quralacaq. Biz həmçinin F inikasının qarışıq monotonluq xassəsi olmadan qeyd olunmuş nöqtənin müəyyən əlaqəli nəticərini göstərəcəyik. Həmçinin nəticələrimizin adekvatlığını göstərən misal da verilmişdir. Tətbiqi olaraq alınmış nəticələrin praktiki əhəmiyyətini illustrasiya etmək üçün inteqral tənliklərin həllinin alırıq. Eyni zamanda biz bir neçə məlum nəticələri təkmilləşdirəcəyik, genişləndirəcəyik və ümumiləşdirəcəyik.

Açar sözlər: ikiqat təsadüfi nöqtə, ümumiləşmiş Mizoquçi-Takahaşi sıxılması, ümumiləşmiş uyuşma, artan inikas, qarışıq monoton inikası.

Общая теорема о двойной неподвижной точке для обобщенной совместимой пары отображений при обобщенным сокращением мизогути-такахаши

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РЕЗЮМЕ

Установлена теорема о сопряженной точке совпадения для обобщенной согласованной пары отображений F, G: X × X → X при обобщенном сокращенииМизогучи-Такахаши на частично упорядоченном метрическом пространстве. Мы также выводим некоторые связанные результаты с неподвижной точкой без смешанного монотонного свойства F: X × X → X. Также приведен пример, подтверждающий наш результат.Мы получаем решение интегральных уравнений для иллюстрации пригодности полученных результатов. Мы улучшаем, расширяем и обобщаем несколько известных результатов.

Ключевые слова:Связанная точка совпадения, обобщенное сокращение Мизогучи-Такахаши, обобщенная совместимость, возрастающее отображение, смешанное монотонное отображение, коммутирующее отображение.