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ON THE EXISTENCE OF SOLUTIONS IN COUPLED SYSTEM OF NON-LINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS ON THE HALF LINE*

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Abstract. In the present work we discuss the existence of solutions for a system of nonlinear fractional integro-differential equations with initial conditions. This system involving the Caputo fractional derivative and Riemann−Liouville fractional integral. Our results are based on a fixed point theorem of Schauder combined with the diagonalization method.

Keywords: fractional derivative/integral, Banach space, diagonalization method, uniform norm, Green‘s function, completely continuous operator, equicontinuous set.

AMS Subject Classification: 34LXX, 45JXX.

1. Introduction

During the last two decades fractional calculus has started to appear with many important applications in biology [1], physics [3, 4, 5, 6, 12] and chemistry [18]. As surveys for the theory of fractional integration and differentiation we refer the reader to the books [13, 17, 20] and [22] For more recent details about the theory of fractional dynamical systems and interpretations of fractional integration and differentiation see [1, 2, 7, 8, 10, 13, 15, 16, 19] and [21]. For the basic tools in fixed point theory necessary to obtain our result see [9, 20].
A. Arara et al. [1] have considered a class of boundary value problems involving Caputo fractional derivative on the half line with using the diagonalization process.
This paper is concerned with the existence of solutions for coupled system of nonlinear fractional integro–differential equation:

$$\begin{cases} {}^cD^{\alpha_i}x_i(t) = tI^{\gamma_i}f_i\left(t,\ x_j(t)\right) + f_i\left(t,\ x_j(t)\right),\ i,j=1,2, i\neq j,\ t>0, \\ x_i(0)=x_{i0},\ \textit{and}\ x_i(t)\ \textit{are bounded on}\ [0,\ \infty), \end{cases} \tag{1}$$

where $1<\alpha_i\leq 2$, ${}^cD^{\alpha_i}$ are the Caputo fractional derivative, γ_i are real positive numbers, I_i^{γ} are Remann−Liouville fractional integral and $f_i : [0,\ \infty)\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous functions.

2. Basic tools

We dedicate this section to recall and introduce some notations, definitions and preliminary facts that will be used in the remainder of this paper [11, 17, 20, 22].

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Let $\mathcal{I}_n = [0,\ n]$, $L^1(\mathcal{I}_n,\ \mathbb{R})$ denote the Banach space of functions $x : \mathcal{I}_n \longrightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|x\|_{L^1} = \int_0^n |x(t)|dt.$$

Recall that $C(\mathcal{I}_n,\ \mathbb{R})$ is the Banach space of continuous functions from the interval $\mathcal{I}_n$ in to $\mathbb{R}$ endowed with uniform norm,

$$\|x\|_n = \max\{\,|x(t)|\ :\ t \in \mathcal{I}_n\},$$

and $C^2 = C \times C$ is the Banach space of continuous functions from the interval $[0,\ n]$ in to $\mathbb{R}$ endowed with uniform norm

$$\|(x_1,\ x_2)\|_n = \max\left\{\,\|x_1\|_n,\ \|x_2\|_n\ :\ (x_1,\ x_2) \in C^2,\ t \in \mathcal{I}_n\right\}.$$

The Arzela-Ascoli theorem and Schauder‘s fixed point theorem is used in this manuscript which they have an important roll in this article and the reader can refer to references[9, 11].

Definitions of Caputo and Riemann−Liouville fractional derivative/integral and their relation are given bellow.

Definition 2.1. For a function u defined on an interval $[a,\ b]$, the Remann−Liouville fractional integral of f of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a^+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\ ds, \quad t > a,$$

and Remann−Liouville fractional derivative of u of order $\alpha > 0$ defined by

$$D^{\alpha}_{a^+}x(t) = \frac{d^n}{dt^n}\left\{I^{n-\alpha}_{a^+}x(t)\right\},$$

where $n-1 < \alpha \leq n$ while Caputo fractional derivative of x of order $\alpha > 0$ defined by is defined by

$${}^cD^{\alpha}_{a^+}x(t) = I^{n-\alpha}_{a^+}\left\{x^{(n)}(t)\right\}.$$

An important of relation among of Caputo fractional derivative and Riemann−Lioville fractional derivative is the following expression

$$D^{\alpha}_{a^+}x(t) = {}^cD^{\alpha}_{a^+}x(t) + \sum_{j=1}^{n-1}\frac{x^{(j)}(a)}{\Gamma(j-\alpha+1)}(t-a)^{j-\alpha}. \tag{2}$$

We denote ${}^cD^{\alpha}_{a^+}x(t)$ as ${}^cD^{\alpha}_{a}x(t)$ and $I^{\alpha}_{a^+}x(t)$ as $I^{\alpha}_{a}x(t)$. Further ${}^cD^{\alpha}_{0^+}x(t)$ and $I^{\alpha}_{0^+}x(t)$ are referred as ${}^cD^{\alpha}x(t)$ and $I^{\alpha}x(t)$, respectively.

Theorem 2.1. *Let $y \in C^m([0,\ b],\ \mathbb{R})$ and $\alpha,\ \beta \in (m-1,\ m)$, $m \in \mathbb{N}$ and $x \in C^1([0,\ b],\ \mathbb{R})$. Then* (i) ${}^cD^{\alpha}I^{\alpha}x(t) = x(t)$. (ii) $I^{\alpha}I^{\beta}x(t) = I^{\alpha+\beta}x(t)$. (iii) $\lim_{t\to 0^+}\{{}^cD^{\alpha}y(t)\} = \lim_{t\to 0^+}\{I^{\alpha}y(t)\}$. (iv) $\lim_{t\to 0^+}\{{}^cD^{\alpha}y(t)\} = \lim_{t\to 0^+}\{I^{\alpha}y(t)\}$. (v) $I^{\alpha}\{{}^cD^{\alpha}y(t)\} = y(t) - \sum_{k=0}^{m-1}\frac{y^{(k)}(0)}{k!}t^k$.

Proof. Part (i) and (ii) can be shown by using the semigroup properties of the Caputo derivative and Theorem 3.1 in [22]. For the proof of the last part, the reader is also referred to Theorem 2.22 in [13]. □

Proposition 2.1. *Let $y \in \mathcal{C}([0,\ \infty),\ \mathbb{R})$, $n \in \mathbb{N}$ and $\alpha > 0$, $\beta > 0$. Then* (i) $I^{\alpha}(ty(t)) = tI^{\alpha}y(t) - \alpha I^{\alpha+1}y(t)$. (ii) $I^{\alpha}\{t\ I^{\beta}y(t)\} = tI^{\alpha+\beta}y(t) - \alpha I^{\alpha+\beta+1}y(t)$.

Proof. (i) can be found in [[17], P. 53] and (ii) is an immediate consequence of (i) and Theorem 2.1 (ii). $\square$

Lemma 2.1. *(Lemma 2.22 [22]). Let $\alpha > 0$, then $I^{\alpha}\left({}^{c}D^{\alpha}x(t)\right) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{r-1}t^{r-1}$ for some $c_i \in \mathbb{R},\ i = 0,\ 1,\ \cdots,\ r-1,\ r = [\alpha] + 1$.*

3. Existence results

Consider the system of boundary value problem

$$\begin{cases} {}^{c}D^{\alpha_i}x_i(t) = tI^{\gamma_i}f_i\left(t,\ x_j(t)\right) + f_i\left(t,\ x_j(t)\right),\ i, j = 1, 2, i \neq j,\ t \in \mathcal{I}_n, \\ x_i(0) = x_0^i,\ \textit{and}\ x_i'(n) = 0, \end{cases} \tag{3}$$

where $n \in \mathbb{N},\ \gamma_i > 0$ and $1 < \alpha_i \leq 2$ and $f_i : [0,\ \infty) \times \mathbb{R} \longrightarrow \mathbb{R},\ i = 1,\ 2$ are given continuous functions.

In this section first, we discuss the system of nonlinear fractional differential equation (3) which has at least one solution.

Proposition 3.1. *Assume that $x,\ y \in C\left([0,\ n],\ \mathbb{R}\right)$ then the system of boundary value problem (3a)-(3c), is equivalent the following system of Volterra fractional integral equations*

$$\begin{cases} x_i(t) = -c_0 - c_1 t + tI^{\alpha_i+\gamma_i}f_i(t,\ x_j(t)) - \alpha_i I^{\alpha_i+\gamma_i+1}f(t,\ x_j(t)) + I^{\alpha}f_i(t, x_j(t)), \\ i,\ j = 1,\ 2,\ i \neq j. \end{cases}$$

Proof. By integrating both sides of Eqn. (3) of order α_i respectively and using the Proposition 2.6 with together Lemma 1, the lemma is proved. $\square$

The next lemma shows that the solvability of the system of boundary value problem (4)-(5) is equivalent to the solvability of a system of the fractional integral equation.

Lemma 3.1. *Assume that $f_i \in C\left(\mathcal{I}_n \times \mathbb{R},\ \mathbb{R}\right)$ and consider the linear system of fractional order differential equation*

$$\begin{cases} {}^{c}D^{\alpha_i}x_i(t) = tI^{\gamma_i}f_i(t,\ x_j(t)) + f_i(t,\ x_j(t)),\ i, j = 1,\ 2,\ i \neq j, \\ x_i(0) = x_0^i,\ x_i'(n) = 0, \end{cases} \tag{4}$$

where $t \in \mathcal{I}_n,\ 1 < \alpha_i \leq 2$. Then $x_i \in C\left(\mathcal{I}_n,\ \mathbb{R}\right),\ i = 1,\ 2$ is a solution (4) if and only if $x_i,\ i = 1,\ 2$ is a solution of the system of fractional integral equation:

$$\begin{cases} x_i(t) = x_i(0) + \int\limits_0^n G_n in(t,\ s)\, f_i(s,\ x_j(s))\, ds, \\ i, j = 1,\ 2,\ i \neq j, \end{cases} \tag{5}$$

where $G_{in}(t,\ s),\ i = 1, 2$ are the Green's functions and defined by

$$G_{in}(t,\ s) = \begin{cases} \frac{t(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} - \frac{\alpha_i(t-s)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)} + \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} + \mathcal{G}_i(t,\ s), & 0 \leq s \leq t \leq n, \\ \\ \mathcal{G}_i(t,\ s), & 0 \leq t \leq s \leq n. \end{cases} \tag{6}$$

where

$$\begin{aligned} \mathcal{G}_i(t,\ s) &= \frac{-t(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} - \frac{n(n-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \\ &+ \frac{\alpha_i t(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} - \frac{t(n-s)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}, \quad i = 1, 2. \end{aligned} \tag{7}$$

Proof. Let $x_1,\, x_2 \in C\,(\mathcal{I}_n,\ \mathbb{R})$ be a solution of Eqn. (4). In view of Proposition 2, we have

$$x_i(t) = tI^{\alpha_i+\gamma_i} f(t,\ x_j(t)) - \alpha I^{\alpha_i+\gamma_i+1} f(t,\ x_j(t)) + I_i^{\alpha} f_i(t,\, x_j(t)) - c_{0i} - c_{1i}t,\ i \neq j, \tag{8}$$

for arbitrary constants $c_{0i},\ c_{i1},\ i = 1,\, 2$. By derivation (8) we get

$$x_i'(t) = \int\limits_0^t \left\{ \frac{(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{t(t-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \right. \tag{9}$$

$$\left. - \frac{\alpha_i(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{(t-s)^{\alpha_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} \right\} f_i(s, x_j(s))ds - c_{1i,i\neq j.} \tag{10}$$

$$\tag{11}$$

Hence using the boundary conditions (4), (8) and (9) we obtain $c_{0i} = -x_i(0),\ i = 1,\, 2$ and

$$c_{1i} = \int\limits_0^n \left\{ \frac{(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{n(n-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \right.$$

$$\left. - \frac{\alpha_i(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{(n-s)^{\alpha_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} \right\} f_i(s, x_j(s))ds, i \neq j$$

Substituting values $c_{0i} = -x_i(0),\ i = 1, 2$ and the above values of $c_{1i},\ i = 1, 2$ into (8) we get

$$\begin{aligned} x_i(t) &= x_0 - \int\limits_0^n \left\{ \frac{nt(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i-1)} - \frac{\alpha_i t(n-s)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i)} - \frac{t(n-s)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \right\} f_i(s,\ x_j(s))\, ds + \\ &+ \int\limits_0^t \left\{ \frac{(t-s)^{\alpha_j-1}}{\Gamma(\alpha)} + \frac{t(t-s)^{\alpha_j+\gamma_j-1}}{\Gamma(\alpha_j+\gamma_j)} - \frac{(t-s)^{\alpha_j+\gamma_j}}{\Gamma(\alpha_i+\gamma_i+1)} \right\} f_i(s,\ x_j(s))\, ds, \end{aligned} \tag{12}$$

and then

$$\begin{aligned} x_i(t) &= x_i(0) + \int\limits_t^n \mathcal{G}_i(t,\ s) f_i(s,\ x_j(s))\, ds + \\ &+ \int\limits_0^t \left\{ \mathcal{G}_i(t,\ s) + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha_i)} + \frac{t(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} - \frac{\alpha_i(t-s)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)} \right\} f_i(s,\ x_j(s))\, ds = \\ &= x_i(0) + \int\limits_0^n G_{in}(t,\ s)\, f_i(s,\ x_j(s))\, ds,\ i, j = 1,\, 2 \textit{ and } i \neq j, \end{aligned}$$

where $\mathcal{G}_i(t,\ s)$ being as before.
Conversely, suppose that $x_1,\, x_2 \in C\,(\mathcal{I}_n,\ \mathbb{R})$ satisfying in (5), hence $x_1,\ x_2$ satisfying in Eqn.

(10). By derivation of Eqn. (10) we have,

$$x_i'(t) = \int_t^n \frac{\partial \mathcal{G}_i(t,s)}{\partial t} f_i(s, x_j(s))ds - \mathcal{G}_i(t, t-0) + \mathcal{G}_i(t, t-0) +$$

$$+ \int_0^t \left\{ \frac{\partial \mathcal{G}_i(t,s)}{dt} + \frac{(t-s)^{\alpha_i - 2}}{\Gamma(\alpha_i - 1)} - \frac{(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \right.$$

$$\left. + \frac{t(t-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \frac{\alpha(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} \right\} f_i(s, x_j(s))ds =$$

$$= \int_0^t \left\{ \frac{(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{t(t-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \frac{\alpha(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} \right\} f_i(s, x_j(s))ds -$$

$$- \int_0^n \left\{ \frac{(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{(n-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \frac{(n-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} \right\} f_i(s, x_j(s))ds +$$

$$+ \int_0^t \frac{(t-s)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} f_i(s, x_j(s))ds - \int_0^n \frac{(n-s)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} f_i(s, x_j(s))ds, \ i \neq j.$$

Thus $x_1'(n) = 0, \ x_2'(n) = 0$ and

$${}^cD^{\alpha_i} x_i(t) = {}^cD^{\alpha_i - 1} x_i'(t) = {}^cD^{\alpha_i-1} \int_0^t \frac{(t-s)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} f_i(s, \ x_j(s))\, ds +$$

$$\begin{aligned}
&+ \ {}^cD^{\alpha_i-1} \int_0^t \left\{ \frac{(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} + \frac{t(t-s)^{\alpha_i+\gamma_i-2}}{\Gamma(\alpha_i+\gamma_i-1)} - \frac{\alpha(t-s)^{\alpha_i+\gamma_i-1}}{\Gamma(\alpha_i+\gamma_i)} \right\} f_i(s, \ x_j(s))\, ds = \\
&= \ {}^cD^{\alpha_i-1} \left\{ I^{(\alpha_i-1)} f_i(t, \ x_j(t)) \right\} + {}^cD^{\alpha_i-1} \left\{ \frac{d}{dt} \left[{}^cI^{\alpha_i} \left(t\, I^{\alpha_i} f_i(t, \ x_j(t)) \right) \right] \right\} = \\
&= \ f_i(t, \ x_j(t)) + t I^{\alpha_i} f_i(t, \ x_j(t)), \ i, j = 1, 2, \ i \neq j.
\end{aligned}$$

Hence ${}^cD^{\alpha_i} x_i(t) - t I^{\gamma_i} f_i(t, \ x_j(t)) = f_i(t, \ x_j(t)), \ i \neq j$. The proof is therefore complete. □

Remark 3.1. For each $t \in \mathcal{I}_n$, denote functions

$$g_{in}(t) = \int_0^n |G_{in}(t, \ s)|\, ds, \ i = 1, \ 2.$$

Then g_{in} are continuous on $\mathcal{I}_n$ and hence are bounded. Let

$$\overset{*}{G}_{in} = \max \{ g_{in}(t) \ : \ t \in \mathcal{I}_n \}, \ i = 1, \ 2.$$

Theorem 3.1. *Assume that $f_i(t, \ .)$, $i = 1, 2$ are continuous on $[0, \ \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ and there exist four continuous functions $\omega_i, \ \sigma_i : [0, \ \infty) \longrightarrow \mathbb{R}^+, \ i = 1, 2$ and nondecreasing such that*

(**H1**): *$|f_i(t, \ u)| \leq \omega_i(t) \sigma_i(|u|)$ for each $t \in [0, \ \infty)$ and $u \in \mathbb{R}$,*

(**H2**): *There exist two positive constants $r_i, \ i = 1, 2$ such that*

$$r_i \geq |x_i(0)| + \overset{*}{\omega}_{in} \ \sigma_i(r_i) \ \overset{*}{G}_{in}, \ i = 1, 2, \tag{13}$$

where $\overset{*}{\omega}_{in} = \max\{\omega_i(t) \ : \ t \in \mathcal{I}_n\}$.

Then system (1) has at least one solution $(x_1(t),\ x_2(t))$ *on* $[0,\ \infty)$ *such that* $|x_i(t)| \le r_i,\ i = 1,\ 2.$

Before starting the proof of Theorem 3.1 we need to prove the following lemma.

Lemma 3.2. *Assume that* $f_1(t,\ .)$ *and* $f_2(t,\ .)$ *are continuous on* $[0,\ \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ *and there exist four continuous functions* $\omega_i,\ \sigma_i : [0,\ \infty) \longrightarrow \mathbb{R}^+,\ i = 1,\ 2$ *and nondecreasing such that* $(H1) - (H2)$ *are hold.*
Let, $\mathcal{C} = C(\mathcal{I}_n,\ \mathbb{R}) \times C(\mathcal{I}_n,\ \mathbb{R})$ *and* $\Omega = \{(x,\ y) \in \mathcal{C} : \|(x,\ y)\|_n < \mathcal{R}\}$ *where* $\|(x_1,\ x_2)\|_n = \max\{\|x_1(t)\|_n,\ \|x_2(t)\|_n,\ t \in \mathcal{I}_n\}$ *and* $\mathcal{R} = \max\{r_1,\ r_2\}$ *so that* $r_1,\ r_2$ *are the constants from* $(H2)$. *Consider the operator* $F : \mathcal{C} \longrightarrow \mathcal{C}$ *defied by*

$$(F(x_1,\ x_2))(t) = ((T_1x_1)(t),\ (T_2x_2)(t)),$$

where

$$(T_ix_i)(t) = x_i(0) + \int_0^n G_{in}(t,\ s)\, f_i(s,\ x_j(s))\, ds,\ i,j = 1,\ 2,\ i \ne j. \tag{14}$$

Then the following statements are hold. (i) Ω *is a closed, convex sub set of* $\mathcal{C}$. *(ii)* F *is continuous. (iii)* F *maps* Ω *into a bonded set of* $\mathcal{C}$. *(iv)* F *maps* Ω *into an equicontinuous set of* $\mathcal{C}$. *(v)* F *is completely continuous. (vi)* $F(\Omega) \subset \Omega$.

Proof. (i) is clear so we try to prove (ii). Let $\{(x_{1l},\ y_{1l})\} \in \mathcal{C}$ be a sequence such that $\{(x_{1l},\ y_{1l})\} \to (x_1,\ x_2) \in \mathcal{C}$ and for $i = 1,\ 2$, let $L = \max\{\|x_{il}\| < L_i,\ \|x_1\| < L_3$ and $\|x_2\| < L_4\}$, then for each $t \in \mathcal{I}_n$, it is sufficient to show that $\|T_ix_{il} - Tx_i\|_n \to 0$ as $l \to \infty$. For each $t \in \mathcal{I}_n$ by $(H1)$ we have

$$\begin{aligned} |(T_1x_{1l})(t) - (Tx_1)(t)| &\le \int_0^n |G_{1n}(t,\ s)||f(s,\ x_{1l}(s)) - f(s,\ x_1(s))|\, ds \le \\ &\le \int_0^n \omega_1(s)|G_{1n}(t,\ s)|[\sigma(|x_{1l}(s)|) + \sigma(|x_1(s)|)]\, ds \le \\ &\le 2\, \overset{*}{\omega}_{1n}\, \sigma(R) \int_0^n |G_{1n}(t,\ s)|\, ds \le 2\, \overset{*}{G}_{1n}\, \sigma(R)\, \overset{*}{\omega}_{1n}, \end{aligned}$$

where $\|\ \overset{*}{\omega}_{1n}\ \| = \max\{\, |\omega_1(t)| : t \in \mathcal{I}_n\}$. Thus the Lebesgue dominated convergence theorem implies that $\|T_1x_{1l} - T_1x_1\|_n \to 0$ as $l \to \infty$. Proving continuity of T_2 as similar as proving continuity of T_1 which was done in above.
(iii). Let $(x_1,\ x_2) \in \Omega$ then $\|F(x_1,\ x_2)\|_n = \max\{\, \|T_ix_i\|_n,\ i = 1,\ 2\}$ and for each $t \in \mathcal{I}_n$ and for $i = 1, 2$ using $(H1)$ we have

$$\begin{aligned} |(T_ix_i)(t)| &\le |x_i(0)| + \int_0^n |G_{in}(t,\ s)||f_i(s,\ x_j(s))|\, ds \le \\ &\le |x_i(0)| + \int_0^n |G_{in}(t,\ s)|\omega_i(s)\sigma_i(|x_j(s)|)\, ds \le \\ &\le |x_(0)| + \overset{*}{\omega}_{in}\, \sigma_i(\|x_j\|_n) \int_0^n |G_{in}(t,\ s)|\, ds = |x_i|(0) + \overset{*}{\omega}_{in}\, \sigma_i(\|x_j\|_n)\, \overset{*}{G}_{in} := M_i, \end{aligned}$$

Let $M = \max\{ M_1,\ M_2 \}$ then $\|F(x_1,\ x_2)\|_n \leq M$. That is say, $F(\Omega)$ is uniformly bounded.
(iv) Since $G_{in}(t,\ s), i = 1,\ 2$ are continuous on $\mathcal{I}_n \times \mathcal{I}_n$, they are uniformly continuous on $\mathcal{I}_n \times \mathcal{I}_n$. Thus, for fixed $s \in \mathcal{I}_n$ and for any $\epsilon > 0$, there exists a constant $\delta > 0$, such that any $t_1,\ t_2 \in \mathcal{I}_n$ and $|t_1 - t_2| < \delta$,

$$|G_{in}(t_1,\ s) - G_{in}(t_2,\ s)| < \frac{\epsilon}{2\sqrt{2}n\sigma(R)\ \overset{*}{\omega}_{in}},\ i = 1,\ 2.$$

Then for $i = 1,\ 2$

$$|(T_i x_1)(t_2) - (T_i x_i)(t_1)| \leq \int_0^n |G_{in}(t_2,\ s) - G_{in}(t_1,\ s)||f_i(s,\ x_j(s))|\, ds < \frac{\epsilon}{2\sqrt{2}}. \tag{15}$$

Using $(H1)$, Eqn. (14) and for the Euclidean distance d on $\mathbb{R}^2$, we have that if $t_1,\ t_2 \in \mathcal{I}_n$ are such that $|t_1 - t_2| < \Omega$, then

$$\begin{aligned} d\left(F(x_1,\ x_2)(t_2) - F(x_1,\ x_2)(t_1)\right) &= \left\{ \sum_{i=1}^{2} [(T_i x_i)(t_2) - (T_i x_i)(t_1)]^2 \right\}^{\frac{1}{2}} \leq \\ &\leq \sum_{i=1}^{2} |(T_i x_i)(t_2) - (T_i x_i)(t_1)| < \epsilon. \end{aligned}$$

That is say, $F(\Omega)$ is equicontinuous.
(v) It is a consequence of $(i) - (iii)$ together with Theorem 2.1 Arzela-Ascoli Theorem.
(iv). Let $(x_1,\ x_2) \in \Omega$, that is $\|(x_1,\ x_2)\|_n < \mathcal{R}$ with $\mathcal{R} = \min\{r_1,\ r_2\}$. We prove that $F(x_1,\ x_2) \in \Omega$. For each $t \in \mathcal{I}_n$ and using $(H1) - (H2)$ we have

$$\begin{aligned} \|F(x_1,\ x_2)\|_n &= \max\{ \|T_i x_i\|_n,\ i = 1,\ 2\} \leq \\ \leq\ \max &\left\{ |x_i(0)| + \int_0^n |G_{in}(t,\ s)||f_i(s,\ x_j(s))|ds,\ i = 1,\ 2 \right\} \leq \\ \leq\ \max &\left\{ \overset{*}{\omega}_{in}\ \sigma(\|x_i\|_n)\ \overset{*}{G}_{in},\ i = 1,\ 2 \right\} \leq \max\{r_1,\ r_2\} = \mathcal{R}. \end{aligned}$$

We complete the proof of Lemma 3. □

Proof of Theorem 3.1. Necessary conditions of Schauder's fixed point theorem for the operator $F : \mathcal{C} \longrightarrow \mathcal{C}$ was obtained in Lemma 3, therefore F has fixed points $(x_{1n},\ x_{2n})$ in Ω, hence by Lemma 2, the fixed points of F are solutions the system of boundary value problem:

$$\begin{cases} {}^cD^{\alpha_i} x_i(t) = I^{\gamma_i} f_i\left(t,\ x_j(t)\right) + f_i\left(t,\ x_j(t)\right), & t \in \mathcal{I}_n,\ 1 < \alpha_i \leq 2,\ i, j = 1,\ 2,\ i \neq j, \\ x_i(0) = x_{i0},\ x_i'(n) = 0. \end{cases} \tag{16}$$

Using diagonalization process, we prove the system (1) has a bounded solution on $[0,\ \infty)$.
For $k \in \mathbb{N}$, assume that $(x_{1k},\ x_{2k})$ be a solution of the boundary value problem (15) on $[0,\ n_k]$ and $\{n_k\}_k \in \overset{*}{\mathbb{N}}$ is a sequence satisfying $0 < n_1 < n_2 < \cdots < n_k < \cdots \uparrow \infty$. Let

$$(X_{1k}(t),\ X_{2k}(t)) = \begin{cases} \left(x_{1k}(t),\ x_{2k}(t)\right), & t \in [0,\ n_k], \\ \left(x_1(n_k),\ x_2(n_k)\right), & t \in [n_k,\ \infty). \end{cases} \tag{17}$$

If we consider $\mathcal{S} = \{(X_{1k},\, X_{2k}),\; k = 1,\, 2, \cdots\}$ then for each $t \in [0,\; n_1]$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}\|(X_{1k},\, X_{2k})\| &= \max\{\|X_{1k}\|,\; \|X_{2k}\|\}\\ &= \max\{\max\{\,|x_{ik}(t)|\;:\;t \in [0,\; n_1]\},\; i = 1,\, 2\}\\ &= \max\{\|x_{1k}\|,\; \|x_{2k}\|\} \le \max\{r_1,\; r_2\} = \mathcal{R},\end{aligned}$$

and

$$X_{in_k}(t) = x_i(0) + \int_0^{n_1} G_{in_1}(t,\; s)\, f_i(s,\, X_{jn_k}(s))\, ds,\; i, j = 1, 2, i \ne j. \tag{18}$$

Thus, for each $t,\, \tau \in [0,\; n_1]$ and $k \in \mathbb{N}$, from system (17) and by $(H1) - (H2)$ we get

$$|X_{in_k}(t) - X_{in_k}(\tau)| \le \lambda_1 \int_0^{n_1} [G_{in_1}(t,\; s) - G_{in_1}(\tau,\; s)]\, ds,\; i = 1, 2,$$

where $\lambda_1 = \max\{\overset{*}{\omega}_{i1}\; \sigma_i(r_i),\; i = 1, 2\}$. Hence the Arzela-Ascoli Theorem guarantees that there is a subsequence $\mathcal{N}_1$ of $\mathbb{N}$ and tow functions $u_1,\, v_1 \in C([0,\; n_1],\; \mathbb{R})$ such that $(X_{n_k},\; Y_{n_k}) \to (u_1,\, v_1) \in C([0,\; n_1],\; \mathbb{R})$ as $k \to \infty$ through $\mathcal{N}_1$.

Let $\overset{*}{\mathcal{N}_1} = \mathcal{N}_1 - \{1\}$. Notice that $\|(X_{1n_k},\, X_{2n_k})\| \le \mathcal{R}$ for each $t \in [0,\; n_2]$ and $k \in \mathbb{N}$. With repetition of the above process on interval $[0,\; n_2]$, that is for each $t \in [0,\; n_2]$ and $k \in \mathbb{N}$ from system (17) and by $(H1) - (H2)$ we have

$$|X_{in_k}(t) - X_{in_k}(\tau)| \le \lambda_2 \int_0^{n_2} [G_{in_1}(t,\; s) - G_{in_1}(\tau,\; s)]\, ds,\; i = 1, 2,$$

where $\lambda_2 = \max\{\overset{*}{\omega}_{i2}\; \sigma_i(r_i),\; i = 1,\, 2\}$. Hence the Arzela-Ascoli Theorem guarantees that there is a subsequence $\mathcal{N}_2$ of $\overset{*}{\mathcal{N}_1}$ and tow functions $u_2,\, v_2 \in C([0,\; n_2],\; \mathbb{R})$ such that $(X_{1n_k},\; X_{2n_k}) \to (u_2,\; v_2) \in C([0,\; n_2],\; \mathbb{R})$ as $k \to \infty$ through $\mathcal{N}_2$. It is clear that $(u_1(t),\; v_1(t)) = (u_2(t),\; v_2(t))$ for each $t \in [0,\; n_1]$, as $\mathcal{N}_2 \subseteq \overset{*}{\mathcal{N}_1}$.

Let $\overset{*}{\mathcal{N}_2} = \mathcal{N}_2 - \{2\}$. Proceed inductively to obtain for $m \in \{3,\, 4,\, \cdots\}$ a subsequence $\mathcal{N}_m$ of $\overset{*}{\mathcal{N}}_{m-1}$ and tow functions $u_m,\, v_m \in C([0,\; n_m],\; \mathbb{R})$ such that $(X_{1n_k},\; X_{2n_k}) \to (u_m,\; v_m) \in C([0,\; n_m],\; \mathbb{R})$ as $k \to \infty$ through $\mathcal{N}_m$.

Let $\overset{*}{\mathcal{N}_m} = \mathcal{N}_m - \{m\}$. We define two functions $x_1,\; x_2$ on $(0,\; \infty)$ as follows.

Fix $t \in (0,\; \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Then define $x_1(t) = X_{1m}(t)$ and $x_2(t) = X_{2m}(t)$. Then $x_1,\; x_2 \in C([0,\; \infty),\; \mathbb{R})$, $x_1(0) = x_{10}, x_2(0) = x_{20}$ and $|x_1(t)| \le \mathcal{R}$, $|x_2(t)| \le \mathcal{R}$ for $t \in [0,\; \infty)$. Again fix $t \in [0,\; \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Then for $n \in \overset{*}{\mathcal{N}_m}$ we have

$$X_{in_k}(t) = x_i(0) + \int_0^{n_m} G_{in_m}(t,\; s)\, f_i(s,\, X_{jn_k}(s))\, ds,\; i, j = 1,\, 2,\; i \ne j.$$

Let $n_k \to \infty$, through $\overset{*}{\mathcal{N}_m}$ to obtain

$$X_{im}(t) = x_i(0) + \int_0^{n_m} G_{in_m}(\tau,\; s)\, f_i(s,\, X_{jm}(s))\, ds,\; i, j = 1,\, 2,\, i \ne j,$$

that is

$$x_i(t) = x_i(0) + \int_0^{n_m} G_{im}(\tau,\ s)\, f_i(s,\ x_j(s))\, ds,\ i = 1,\ 2,\ i \neq j.$$

We can use this method for each $\tau \in [0,\ n_m]$, and for each $m \in \mathbb{N}$. Thus

$$^cD^{\alpha_i} x_i(t) = I^{\gamma_i} f_i\left(t,\ x_j(t)\right) + f_i\left(t,\ x_j(t)\right),\ t \in [0,\ n_m],\ i = 1,\ 2,\ i \neq j.$$

for each $m \in \mathbb{N}$ and $\alpha_i \in (1,\ 2]$ and the constructed functions x_1, x_2 are a solution of system (1). This completes the proof of the theorem.

Example 3.1. Consider a coupled system of non-linear fractional integro-differential equations with initial conditions as follows:

$$\begin{cases} {}^cD^{\frac{3}{2}} x_i(t) - tI^{\frac{1}{2}}\left(\frac{\sqrt[3]{x_j(t)}}{1+t^2}\right) = \frac{\sqrt[3]{x_j(t)}}{1+t^2},\ t > 0,\ i = 1,\ 2,\ i \neq j. \\ x_i(0) = 1 \textit{ and } x_i(t) \textit{ are bounded on } [0,\ \infty). \end{cases}$$

Here,

$$f_1(t,\ u) = \frac{\sqrt[3]{u}}{1+t^2},\quad \omega_1(t) = \frac{1}{1+t^2},\quad \sigma_1(u) = \sqrt[3]{u}$$
$$f_2(t,\ u) = \frac{\sqrt{u}}{1+e^t},\quad \omega_2(u) = \frac{1}{1+e^t},\quad \sigma_2(u) = \sqrt{u},$$

f and g are continuous for each $(t,\ u) \in [0,\ \infty)\times\mathbb{R}$. Four functions ω, σ, η and μ are continuous on $[0,\ \infty)$ and satisfying in $(H1)$, that is $|f_i(t,\ u)| \leq \omega_i(t)\sigma_i(|u|)$, $i = 1, 2$ for each $t \in [0,\ \infty)$ and $u \in \mathbb{R}$. We have $\overset{*}{\omega}_{1n}= \sup\{\omega_1(t)\ :\ t \in \mathcal{I}_n\} = 1$ and $\overset{*}{\omega}_{2n}= \sup\{\omega_2(t)\ :\ t \in \mathcal{I}_n\} = \frac{1}{2}$. The Green's functions for this example, by Eqn. (6) we get

$$G_{1n}(t,\ s)ds = \begin{cases} t(t-s) - \frac{\alpha(t-s)^2}{2} + \frac{\sqrt{t-s}}{\Gamma(3/2)} + \mathcal{G}_1(t,\ s), & 0 \leq s \leq t \leq n \\ \\ \mathcal{G}_1(t,\ s), & 0 \leq t \leq s \leq n, \end{cases}$$

where

$$\mathcal{G}_1(t,\ s) = (\alpha_1 - 1)t(n-s) - n - \frac{t}{\Gamma(1/2)\sqrt{n-s}}$$

Hence $\overset{*}{G}_{1n}= \sup\left\{\int_0^n |G_{1n}(t,\ s)|\, ds,\quad t \in \mathcal{I}_n\right\}$ is exist. Since

$$\lim_{M\to\infty} \frac{M}{1+ \overset{*}{\omega}_{1n}\ \sigma_1(M)\ \overset{*}{G}_{1n}} = \lim_{M\to\infty} \frac{M}{\sigma_1(M)} = \lim_{M\to\infty} \frac{M}{\sqrt[3]{M}} = \infty,$$

then there exist $r_1 > 0$ such that

$$\frac{r_1}{1+ \overset{*}{\omega}_{1n}\ \sigma_1(r_1)\ \overset{*}{G}_{1n}} \geq 1.$$

On other hand, Eqn. (6) yields

$$G_{2n}(t,\ s) = \begin{cases} \frac{t(t-s)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})} - \frac{\alpha_2(t-s)^{\frac{5}{3}}}{\Gamma(\frac{8}{3})} + \frac{(t-s)^{\frac{1}{3}}}{\Gamma(\frac{4}{3})} + \mathcal{G}_2(t,\ s), & 0 \leq s \leq t \leq n, \\ \\ \mathcal{G}_2(t,\ s), & 0 \leq t \leq s \leq n, \end{cases}$$

where

$$\mathcal{G}_2(t,s) = \frac{-t(n-s)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)} - \frac{n(n-s)^{\frac{-1}{3}}}{\Gamma\left(\frac{2}{3}\right)} + \frac{2t(n-s)^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)} - \frac{t(n-s)^{-\frac{2}{3}}}{\Gamma\left(\frac{-1}{4}\right)}.$$

Hence $\overset{*}{G}_{2n} = \sup\left\{\int_0^n |G_{2n}(t,\ s)|\, ds\right\}$ is exist. Since

$$\lim_{N\to\infty} \frac{N}{1+\overset{*}{\omega}_{2n}\ \sigma_2(N)\ \overset{*}{G}_{2n}} = \lim_{M\to\infty} \frac{N}{\sigma_2(N)} = \lim_{M\to\infty} \frac{N}{\sqrt{N}} = \infty.$$

Then there exist $r_2 > 0$ such that

$$\frac{r_2}{1+\overset{*}{\omega}_{2n}\ \sigma_1(r_2)\ \overset{*}{G}_{2n}} \geq 1.$$

Hence this example is satisfying in $(H2)$. Therefore by Theorem 3.1 the system of this example has a bounded solution $(x_1,\ x_2) \in \Omega \subseteq \mathcal{C}$.

Remark 3.2. Proposition 2.1 $rm(i)$ can be generalized, that is if p is nonnegative integer, then [5, P. 53]

$$I^{\alpha}(t^p y(t)) = \sum_{k=0}^{p} \binom{-\alpha}{k} \left[D^{(k)} t^n\right] \left[I^{\alpha+k} y(t)\right] = \sum_{k=0}^{p} \binom{-\alpha}{k} \frac{\Gamma(p+1)t^{p-k}}{\Gamma(p-k+1)} I^{\alpha+k} y(t).$$

Hence, using Theorem 2.5 $rm(ii)$ the above equation yields

$$I^{\alpha}\left\{t^p I^{\beta} y(t)\right\} = \sum_{k=0}^{p} \binom{-\alpha}{k} \frac{\Gamma(p+1)t^{p-k}}{\Gamma(p-k+1)} I^{\alpha+\beta+k} y(t),$$

where,

$$\binom{-\alpha}{k} = (-1)^k \times \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} = (-1)^k \times \frac{\Gamma(\alpha+k)}{k!\,\Gamma(\alpha)}.$$

Therefore we can prove that the system of nonlinear fractional differential equation:

$$\begin{cases} {}^cD^{\alpha_i} x(t) = t^{p_i} I^{\gamma_i} f_i\left(t,\ x_j(t)\right) + f_i\left(t,\ x_j(t)\right),\ t > 0,\ i = 1,\ 2,\ i \neq j, \\ x_i(0) = x_{i0} \textit{ and } x_i(t) \textit{ are bounded on } [0,\ \infty), \end{cases}$$

which under conditions $(H1)$ and $(H2)$ has at least one bounded solution on $[0,\ \infty)$, where p_1, p_2 are nonnegative integers.

4. Conclusions

The existence of solutions for the nonlinear fractional integro-differential equations with initial conditions comprising standard Caputo fractional derivative have been discussed in $C([0,+\infty),\mathbb{R})$. In order to obtain the results in this article the diagonalization method had important role. Although the present study provides some insights in the equations encountered in the global existence solutions, this existence theorem may be explored for other classes of fractional differential equations which encounter in the global existence solutions.

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