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SPLINE COLLOCATION METHOD FOR SOLUTION OF HIGHER ORDER LINEAR BOUNDARY VALUE PROBLEMS

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Abstract. Spline collocation method for solution of the linear boundary value problems are developed by using Eighth-degree B-spline. Formulation of Eighth degree Bspline are derived and collocation method based on such B-spline formulated. Two numerical examples have been considered to illustrate the efficiency and implementation of the method.

Keywords: linear fifth-order and seventh-order boundary value problems, collocation method, B-spline.

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1. Introduction

Generally, boundary value problems are known to arise in the Mathematics, Physics and Engineering Sciences. Over the years, there are several authors who study on these types of boundary value problems. There are several methods, such as finite difference, Galerkin, Orthogonal splines, Collocation, Finite Element Method and etc, to discrete the boundary value problems [3], [12]. Some references [1], [31] contain theorems which detail the conditions for existence and uniqueness of solutions of such BVPs. Eighth-order differential equations govern the physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability sets in as over stability, it is modeled by an eighth-order ordinary differential equation. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as overstability, it is modeled by twelfth-order boundary value problem [7]. Agarwal [1] presented the theorems which listed the conditions for the existence and uniqueness of solutions of eighth-order boundary-value problems. Boutayeb and Twizell [2] developed the finite difference methods for the solution of eighth-order linear boundary-value problems with different boundary conditions. Twizell et al. [26]-[28] developed numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [28] presented the solution of eighth order boundary value problem using octic spline. Houstis and Christara and Rice [13] presented the solution of second-order boundary value problems using Quartic spline collocation. Siddiqi and Twizell [29] presented the solution of sixth order boundary value problem by using polynomial splines of degree six. Lamini et al. [23] used spline collocation method to solve the sixth-order boundary value problems. Siraj-ul-Islam et al. [33], [34] developed nonpolynomial splines approach to the solution of sixth order and fourth-order boundary value problems. Viswanadham et al. [17], [18] used sixth order and septic B-splines to solve sixth order boundary value problems.

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Inc and Evans [15] presented the solutions of eighth order boundary value problems using adomian decomposition method. Usmani used fourth degree splines to solve a third and fourth order boundary value problem in [32]. The eighth-order boundary value problem using Nonic spline have been solved by Ghazala Akram, Shahid S. Siddiqi [5]. El-Gamel et al. used Sinc-Galerkin method to solve sixth order boundary value problems [21].

In [24] Laminii et al, developed two methods for the solution of special linear and nonlinear fifth order boundary value problem. These methods have extended for the solution of some linear boundary value problems [26]. In the present paper, a spline collocation method based on B-spline which satisfy the same boundary conditions, is developed and analyzed for approximating the solutions of fifth and seventh order boundary value problems.

This paper considers the fifth and seventh order boundary value problems and focuses on the application of extended eighth degree B-spline interpolation in approximating the solutions.

We consider the following linear differential equation

$$y^{(5)}(x) = g(x)y(x) + q(x), \qquad a \le x \le b \tag{1}$$

with boundary conditions

$$y(a) = k_1, \quad y^{'}(a) = k_2, \quad y^{''}(a) = k_3, \quad y(b) = k_4, \quad y^{'}(b) = k_5, \tag{2}$$

and the seventh order boundary value problem

$$y^{(7)}(x) = g(x)y(x) + q(x), \qquad a \le x \le b \tag{3}$$

with boundary conditions

$$\begin{array}{l} y(a) = k_6, \quad y^{'}(a) = k_7, \quad y^{''}(a) = k_8, \quad y^{'''}(a) = k_9, \\ y(b) = k_{10}, \quad y^{'}(b) = k_{11}, \quad y^{''}(b) = k_{12}, \end{array} \tag{4}$$

where g(x), q(x) are given and continuous functions in [a,b] and k_i are constants. We develop the collocation methods based on eighth degree B-spline function.This paper organized as follows. In section 1 the introduction and survey of papers dealing with solution of boundary value problems are given. In section 2 the brief discussion regarding the eighth degree B-spline and the values of its derivatives are given. The collocation method based on such spline for numerical solution of fifth and seventh order boundary value problems are developed in section 3 and 4 respectively. To test the efficiency of the method, in section 5 several numerical examples of fifth and seventh order linear boundary value problems are solved by the proposed method. Numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. In section 6 conclusions are given.

2. The eighth degree B-splines

The theory of spline functions is a very attractive field of approximation theory. Usually, a spline is a piecewise polynomial function defined in region D, such that there exists a decomposition of D into subregions in each of which the function is a polynomial of some degree k. A B-spline of degree K is a spline from $S_k(\triangle_n)$ with minimal support and the partition of unity holding.

The B-spline of degree k is denoted by $B_i^k(x)$, where $i \in Z$, and then we have the following properties:

1. $supp(B_i^k) = [x_i, x_{i+k+1}]$.
2. $B_i^k(x) \ge 0, \quad \forall x \in R$ (non-negativity).

3. $\sum\limits_{i=-\infty}^{\infty} B_i^k(x) = 1, \quad \forall x \in R$ (partition of unity).

Alternative approach to drive the B-spline relations.

We consider equally-spaced knots of a partition $\pi : a = x_0 < x_1 < \ldots < x_n$ on $[a;b]$. The alternative approach for deriving the B-splines which are more applicable with respect to the recurrence relation for the formulations of B-splines of higher degrees. At first, we recall that the kth forward difference $f(x_0)$ of a given function $f(x)$ at x_0, which is defined recursively by the following [16], [22]:

$$\Delta f(x_0) = f(x_1) - f(x_0), \quad \Delta^{k+1} f(x_0) = \Delta^k f(x_1) - \Delta^k f(x_0).$$

Definition 2.1. *The function $(x-t)_+^m$ is defined as follows:*

$$(x-t)_+^m(t) = \begin{cases} (x-t)^m & x \geq t \\ 0 & x < t. \end{cases}$$

It is clear that $(x-t)_+^m$ is $(m-1)$ times continuously differentiable both with respect to t and x.

The B-spline of order m is defined as follows:

$$B_i^m(t) = \frac{1}{h^m} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{m+1-j} (x_{i-2+j} - t)_+^m =$$
$$= \frac{1}{h^m} \Delta^{m+1} (x_{i-2} - t)_+^m.$$

Hence, we can obtain the B-spline of various orders by taking various values of m. let $m = 1$; thus,

$$m = 1 \to B_i^1(t) = \frac{1}{h} \Delta^2 (x_{i-2} - t) + = \frac{1}{h} [(x_{i-2} - t)_+ - 2(x_{i-1} - t) + (x_i - t)_+],$$

$$B_i^1(t) = \frac{1}{h} \begin{cases} (x_i - t) - 2(x_{i-1} - t), & x_{i-2} < t \leq x_{i-1} \\ (x_i - t), & x_{i-1} < t \leq x_i \\ 0, & \text{otherwise,} \end{cases}$$

$$B_i^1(t) = \frac{1}{h} \begin{cases} h - (x_{i-1} - t) = t - x_{i-2}, & x_{i-2} < t \leq x_{i-1} \\ (x_i - t), & x_{i-1} < t \leq x_i \\ 0, & \text{otherwise,} \end{cases}$$

In order to obtain the eighth degree B-spline, let $m = 8$

$$B_i^8(t) = \frac{1}{h^8} \Delta^9 (x_{i-2} - t)_+^8 = \frac{1}{h^8} [(x_{i+7} - t)_+^8 - 9(x_{i+6} - t)_+^8 + 36(x_{i+5} - t)_+^8 -$$

$$-84(x_{i+4} - t)_+^8 + 126(x_{i+3} - t)_+^8) - 126(x_{i+2} - t)_+^8 + 84(x_{i+1} - t)_+^8 -$$

$$-36(x_i - t)_+^8 + 9(x_{i-1} - t)_+^8 - (x_{i-2} - t)_+^8],$$

then the B-splines of eight degree are can be obtained as

$$B_i^8(t)=\frac{1}{h^8}\begin{cases}(x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8+126(x_{i+3}-t)^8\\ -126(x_{i+2}-t)^8+84(x_{i+1}-t)^8-36(x_i-t)^8+9(x_{i-1}-t)^8, & x_{i-2}<t\le x_{i-1},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8+126(x_{i+3}-t)^8\\ -126(x_{i+2}-t)^8+84(x_{i+1}-t)^8-36(x_i-t)^8, & x_{i-1}<t\le x_i,\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8+126(x_{i+3}-t)^8\\ -126(x_{i+2}-t)^8+84(x_{i+1}-t)^8, & x_i<t\le x_{i+1},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8+126(x_{i+3}-t)^8\\ -126(x_{i+2}-t)^8, & x_{i+1}<t\le x_{i+2},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8+126(x_{i+3}-t)^8,\\ & x_{i+2}<t\le x_{i+3},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8-84(x_{i+4}-t)^8 & x_{i+3}<t\le x_{i+4},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8+36(x_{i+5}-t)^8, & x_{i+4}<t\le x_{i+5},\\ (x_{i+7}-t)^8-9(x_{i+6}-t)^8, & x_{i+5}<t\le x_{i+6},\\ (x_{i+7}-t)^8, & x_{i+6}<t\le x_{i+7},\\ 0 & otherwise.\end{cases} \tag{5}$$

To solve linear fifth and seventh-order boundary value problems, $B_i, B_i', B_i'', \ldots, B_i^{(7)}$ evaluated at the nodal points their coefficients are summarized in Table (1.a-d)

Table 1.a The value of $B_i(x)$ and $B_i^{(')}(x)$ and $B_i^{''}(x)$ in the three lines respectively

x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}
1	247	4293	15619	15619	4293	247	1
-8/h	-952/h	-8568/h	-9800/h	9800/h	8568/h	952/h	8/h
56/h^2	3080/ h^2	10584/ h^2	-13720/ h^2	-13720/ h^2	10584/ h^2	3080/ h^2	56/h^2

Table 1.b The value of $B_i^{'''}(x)$ and $B_i^{(4)}(x)$ in the two lines respectively

x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}
-336/h^3	-7728/h^3	3024/h^3	31920/h^3	-31920/h^3	-3024/h^3	7728/h^3	336/h^3
1680/h^4	11760/h^4	-45360/h^4	31920/h^4	31920/h^4	-45360/h^4	11760/h^4	1680/h^4

Table 1.c The value of $B_i^{(5)}(x)$ and $B_i^{(6)}(x)$ in the two lines respectively

x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}
-6720/h^5	6720/h^5	60480/h^5	-168000/h^5	168000/h^5	-60480/h^5	-6720/h^5	6720/h^5
20160/h^6	-100800/h^6	181440/h^6	-100800/h^6	-100800/h^6	181440/h^6	-100800/h^6	20160/h^6

Table 1.d The value of $B_i^{(7)}(x)$

x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}
-40320/h^7	282240/h^7	-846720/h^7	1411200/h^7	-1411200/h^7	846720/h^7	-282240/h^7	40320/h^7

$B_i(x) \equiv 0$ for $x \le x_{i-2}$ and $x \ge x_{i+7}$

3. Collocation method based on B-spline for linear fifth order boundary value problem

We can collocate the solution of equation (1) as:

$$y(x) = \sum_{j=-5}^{n} \alpha_j B_j(x), \tag{6}$$

where α_j, are unknown real coefficients to be determine and $B_j(x)$ are Eighth degree B-spline function and let $x_0, x_1, \ldots, x_n$ be $n+1$ grid points in the interval $[a, b]$, so that

$$x_i = a + ih \quad i = 0, 1, \ldots, n; \quad x_0 = a, \quad x_n = b, \quad h = (b-a)/n \tag{7}$$

with substituting (6) in (1), we get

$$[\sum_{j=-5}^{n} \alpha_j (B_j^{(5)}(x_i) - g(x_i)B_j(x_i)) - q(x_i)] = 0, \quad i = 0, 1, \ldots, n. \tag{8}$$

To evaluate the α_j we using of boundary conditions and (8). there for we need to the following system:

$$[\sum_{j=-5}^{n} \alpha_j (B_j^{(5)}(x_i) - g(x_i)B_j(x_i)) - q(x_i)] = 0, \quad i = 0, 1, \ldots, n, \tag{9}$$

$$\sum_{j=-5}^{n} \alpha_j B_j(x_i) = y(a), \qquad i = n+1, \tag{10}$$

$$\sum_{j=-5}^{n} \alpha_j B_j^{'}(x_i) = y^{'}(a), \qquad i = n+2, \tag{11}$$

$$\sum_{j=-5}^{n} \alpha_j B_j^{''}(x_i) = y^{''}(a), \qquad i = n+3, \tag{12}$$

$$\sum_{j=-5}^{n} \alpha_j B_j^{'}(x_i) = y^{'}(b), \qquad i = n+4, \tag{13}$$

$$\sum_{j=-5}^{n} \alpha_j B_j(x_i) = y(b), \qquad i = n+5. \tag{14}$$

The value of the spline functions at the knots $\{x_i\}_{i=0}^{n}$ are determined using Table (1.a-d) and $(9-14)$ a system of $n+6$ linear equations in the (n+6) unknowns $\alpha_{-5}, \alpha_{-4}, \ldots, \alpha_n$ is thus obtained. This system can be written in matrix-vector form as follows

$$A\alpha = M, \tag{15}$$

where A is an$(n+6)$ dimensional matrix,
$\alpha = [\alpha_{-5}, \alpha_{-4}, \ldots, \alpha_n]^T$,
and
$M = [q(x_0), \ldots, q(x_n), y(a), y^{'}(a), y^{''}(a), y^{'}(b), y(b)]^T$.

By solution the above linear system, $\alpha = [\alpha_{-5}, \alpha_{-4}, \ldots, \alpha_n]^T$ can be obtain and then by using (15) we can obtain the approximate solution of (6).

4. Collocation method based on B-spline for linear seventh order boundary value problem

The approximate solution of equation (3) by collocation method can be approximated as:

$$y(x)=\sum_{j=-7}^{n}\alpha_j B_j(x), \tag{16}$$

where α_j, are unknown real coefficients to be determine and $B_j(x)$ are Eighth degree B-spline function and let $x_0, x_1, \dots, x_n$ be $n+1$ grid points in the interval $[a,b]$, so that

$$x_i = a+ih \quad i=0,1,\dots,n; \quad x_0=a, \quad x_n=b, \quad h=(b-a)/n \tag{17}$$

with substituting (16) in (3), we get

$$[\sum_{j=-7}^{n}\alpha_j(B_j^{(7)}(x_i)-g(x_i)B_j(x_i))-q(x_i)]=0, \quad i=0,1,\dots,n \tag{18}$$

To evaluate the α_j we using of boundary conditions and (18). there for we need to the following system:

$$[\sum_{j=-7}^{n}\alpha_j(B_j^{(7)}(x_i)-g(x_i)B_j(x_i))-q(x_i)]=0, \quad i=0,1,\dots n, \tag{19}$$

$$\sum_{j=-7}^{n}\alpha_j B_j(x_i)=y(a), \qquad i=n+1, \tag{20}$$

$$\sum_{j=-7}^{n}\alpha_j B_j^{'}(x_i)=y^{'}(a), \qquad i=n+2, \tag{21}$$

$$\sum_{j=-7}^{n}\alpha_j B_j^{''}(x_i)=y^{''}(a), \qquad i=n+3, \tag{22}$$

$$\sum_{j=-7}^{n}\alpha_j B_j^{'''}(x_i)=y^{'''}(a), \qquad i=n+4, \tag{23}$$

$$\sum_{j=-7}^{n}\alpha_j B_j^{''}(x_i)=y^{''}(b), \qquad i=n+5, \tag{24}$$

$$\sum_{j=-7}^{n}\alpha_j B_j^{'}(x_i)=y^{'}(b), \qquad i=n+6, \tag{25}$$

$$\sum_{j=-7}^{n}\alpha_j B_j(x_i)=y(b), \qquad i=n+7. \tag{26}$$

The value of the spline functions at the knots$\{x_i\}_{i=0}^{n}$ are determined using Table (1.a-d) and $(19-26)$. a system of $n+8$ linear equations in the (n+8)unknowns $\alpha_{-7}, \alpha_{-6}, \dots, \alpha_n$ is thus obtained. This system can be written in matrix-vector form as follows

$$A\alpha = M, \tag{27}$$

where A is an $(n+8)$ dimensional matrix,

$\alpha=[\alpha_{-7}, \alpha_{-6}, \dots, \alpha_n]^T$

and

$M=[q(x_0),\dots,q(x_n),y(a),y^{'}(a),y^{''}(a),y^{'''}(a),y^{''}(b),y^{'}(b),y(b)]^T.$

By solution the above linear system, $\alpha = [\alpha_{-7}, \alpha_{-6}, \ldots, \alpha_n]^T$ can be obtain and then by using (27) we can obtain the approximate solution of (16).

5. Numerical results

In this section, two linear problems will be tested by using the method discussed in third and forth sections.

Example 1. We first consider the linear boundary value problem [6]

$$y^{(5)}(x) = y(x) - 15e^x - 10xe^x, \quad 0 \le x \le 1 \tag{28}$$

subject to boundary conditions

$$y(0) = 0, \quad y(1) = 0, \quad y^{'}(1) = -e, \quad y^{'}(0) = 1, \quad y^{''}(0) = 0. \tag{29}$$

The proposed method is tested on this problem where the domain [0,1] is divided into 20, 40, 60, 80, 160 equal subintervals. maximum absolute error and approximate solution and exact solutionin for this problem are shown in table 2. for which the theoretical solution is $x(1-x)e^x$.

Table 2.

n	Y_i	y_i	Max absolute error
20	0.39693750	0.39693712	3.7818200×10^{-7}
40	0.43428627	0.43428587	4.0232684×10^{-7}
60	0.43730851	0.43730809	4.2094653×10^{-7}
80	0.43790829	0.43790786	4.2541296×10^{-7}
160	0.43790829	0.43790786	4.2971123×10^{-7}

Example 2. We consider the linear boundary value problem [6]

$$y^{(7)}(x) = y(x) - 7e^x, \quad 0 \le x \le 1 \tag{30}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y(1) = 0, \quad y^{'}(1) = -e, \quad y^{''}(1) = -2e, \\ &y^{'}(0) = 0, \quad y^{''}(0) = -1, \quad y^{'''}(0) = -2. \end{aligned} \tag{31}$$

Table 3.

n	Y_i	y_i	Max absolute error
20	0.35094702	0.35093626	1.0759022×10^{-5}
40	0.80296795	0.80296662	1.3295655×10^{-6}
60	1.0000000	0.99999899	1.0002474×10^{-6}
80	1.0000000	0.99999900	9.9983989×10^{-7}
160	1.0000000	0.99999899	1.0000000×10^{-6}

The proposed method is tested on this problem where the domain [0,1] is divided into 20, 40, 60, 80, 160 equal subintervals. maximum absolute error and approximate solution and exact solution for this problem are shown in table 3. for which the theoretical solution is $(1-x)e^x$.

Table 4. The maximum absolute errors of $y(x_i)$ for Example 1

	$n = 10$	$n = 20$	$n = 40$
Eighth degree B-spline method (this paper)	3.02×10^{-7}	3.78×10^{-7}	4.02×10^{-7}
Cubic B-spline method [25]	1.84×10^{-4}	4.54×10^{-5}	1.14×10^{-5}
Sixth-degree B-spline method [11]	0.1570	0.0747	0.0208
Finite difference method [20]	4.02×10^{-3}	3.91×10^{-3}	1.15×10^{-2}
Sextic spline method [14]	2.76×10^{-3}	2.45×10^{-4}	2.01×10^{-5}
Quartic spline method [30]	3.60×10^{-3}	5.55×10^{-4}	7.66×10^{-5}

where $y = x(1-x)e^x$ is the analytic solution. Example 1 has been solved by sixth degree B-spline method [11] , Cubic B-spline method [25], sextic spline method [14], quartic spline method [30] and finite difference method [20]. The respective maximum absolute errors of $y(x_i)$ are given in Table 4.

6. Conclusion

B-spline collocation methods has been considered for the numerical solution of fifth-order and seventh order linear boundary value problems. The Eighth degree B-spline was tested on two linear problem. Numerical results show the ability and efficient of the present method. Numerical results showed that the method achieved fifth-order accuracy. This method enable us to approximate the solution at every point of the range of domain. Numerical results obtained by the present method are in good agreement with the exact solutions or numerical solutions available in the literature.Table (4) shows that our methods produced better numerical results in the sense that $max \mid e_i \mid$ is minimum in comparison with the methods in [11], [14], [20], [25], [30]. The proposed method can be extended to solve higher (i.e., more than 7th) order boundary value problems by using the B-splines of order more than 8. By using eighth degree B-spline functions and the collocation method to solve a fifth order linear BVP of (1-2) or a seven order linear BVP of (3-4).

References

[1] Agarwal, R.P., (1986), Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore.

[2] Boutayeb, A., Twizell, E.H. (1993), Finite-difference methods for the solution of eighth-order boundary-value problems, International Journal of Computer Mathematics, 48, pp.63-75.

[3] Boor, C. de., Swartz, B., (1973), Collocation at gaussian points, SIAM J. Numer. Anal, 10, pp.582-606.

[4] Akram, G., Shahid, S.S., Iftikhar, I., (2012), Solution of Seventh Order Boundary Value Problems by Variational Iteration Technique, Applied Mathematical Sciences, 6(94), pp.4663-4672.

[5] Akram, G., Shahid, S.S., (2006), Nonic spline solutions of eighth order boundary value problems, Applied Mathematics and Computation, 182(1), pp.829-845.

[6] Ali, J., Islam, S., Shah, I., Khan, H., (2011), The Optimal Homotopy Asymptotic Method for the Solution of Fifth and Sixth Order Boundary Value Problems, World Applied Sciences Journal, 2012, pp.1120-1126

[7] Chandrasekhar, S., (1961), Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, Reprinted: Dover Books, New York.

[8] Maksudov, F.G., Aliev, F.A., (1985), Optimization of pulsed systems with unseparated boundary conditions, Soviet Mathematics-Doklady 31, 159.

[9] Aliev, F.A, Mutallimov, M.M., (2005), Algorithms for Solving the Problem of Optimal Control with Three-point Unseparated Boundary Conditions, Journal of Automation and Information Sciences, 37(7), pp.30-39.

[10] Aliev,F.A., Ismailov, N.A., Temirbekova, L.N., (2012), Methods of Solving the choice of extremal modes for the gas-lift process, Appl. Comput. Math., 11(3), pp.348-357.

[11] Caglar, H.N., Caglar, S.H., Twizell, E.H., (1999), The numerical solution of fifth-order boundary value problems with sixth degree B-spline functions, Appl. Math. Lett., 12, pp.25-30

[12] Djidjeli, K., Twizell, E.H., Boutayeb, A., (1993), Numerical methods for special nonlinear boundary value problems of order 2m, Journal of Computational and Applied Mathematics, 47, pp.35-45.

[13] Houstis, E.N., Christara, C., Rice, J.R., (1988), Quartic-spline collocation methods for two point boundary value problems, International Journal for Numerical Methods in Engineering. 26(4), pp.935-952.

[14] Islam, S., Khan, M.A., (2006), A numerical method based on polynomial sextic spline functions for the solution of special fifth-order boundary-value problems, Appl. Math. Comput., 181, pp.356-361.

[15] Inc, M., Evans, D.J., (2004), An efficient approach to approximate solutions of eighth-order boundary value problems, International Journal of Computer Mathematics, 81(6), pp.685-692.

[16] Rashidinia, J., and Sharifi, Sh., (2011), Survey of B-spline functions to approximate the solution of mathemaical problems, School of Mathematics, Iran University of Sience and Technology.

[17] Kasi Viswanadham, K.N.S., Murali Krishna, P, (2010), Septic B-spline collocation method for sixth order boundary value problems, ARPN Journal of Engineering and Applied Sciences, 5(7) , pp.36-40 .

[18] Kasi Viswanadham, K.N.S., P. Murali Krishna, (2010), Sextic B-spline Galerkin method for sixth order boundary value problems, International J. of Math. Sci. and Engg. Appls, 4(4), pp.377-387.

[19] Kasi Viswanadham, K.N.S., P. Murali Krishna, Prabhakara Raob, C., (2010), Numerical Solution of Fifth Order Boundary Value Problems by Collocation Method with Sixth Order B-Splines, International Journal of Applied Science and Engineering, pp.119-125.

[20] Khan, M.S., (1994), Finite difference solutions of fifth order boundary value problems. Ph.D. thesis, Brunal University, England.

[21] Mohamed El-Gamel, John R. Cannon, Ahmed I zayed, (2004), Sinc-Galerkin method for solving linear sixth order boundary value problems, Mathematics of computation, 73(247), pp.1325-1343.

[22] Printer, P.M., (1975), Splines and varitional Methods, Colorado State University, Wiley Classics Edition published.

[23] Lamini, A, Mraoui, H, Sbibih, D, Tijini, A, Zidna, A, (2008), Spline collocation method for solving linear sixth order boundary-value problems, International Journal of Computer Mathematics, 85(11), pp.1673-1684.

[24] Lamnii, A., Mraoui, H., Sbibih, D., and Tijini, A., (2008), Spline Solution of Fifth-Ordrer Boundary Value Problems, Mathematics and Computaters in Simulation, 77, pp.237-246.

[25] Lang, F.G, Xu, X.P., (2011), A new cubic B-spline method for linear fifth order boundary value problems, J. Appl. Math. Comput, 36, pp.101-116.

[26] Siddiqi, S.S., Twizell, E.H., (1997), Spline solutions of linear twelfth-order boundary value problems, J. Comp. Appl. Math., 78(2), pp.371-390.

[27] Siddiqi, S.S., Twizell, E.H., (1998), Spline solutions of linear tenth-order boundary value problems, Intern. J. Computer Math, 68(3), pp.345-362.

[28] Siddiqi, S.S., Twizell, E.H., (1996), Spline solutions of linear eighth-order boundary value problems, Comp. Math. Appl. Mech. Eng, 131(3-4), pp.309-325.

[29] Siddiqi, S.S., Twizell, E.H., (1996), Spline solutions of linear sixth-order boundary value problems, Intern. J. Computer Math, 60(3), pp.295-304.

[30] Siddiqi, S.S., Akram, G., Elahi, A., (2008), Quartic spline solution of linear fifth order boundary value problems, Appl. Math. Comput, 196, pp.214-220.

[31] Tirmizi, S.I.A, (1991), On numerical solution of third order boundary value problems, Commun. Appl. Numer. Math., 7, 309.

[32] Usmani, Ft.A., (1992), The use of quartic splines in the numerical solution of a fourth-order boundary value problem, J. Uomp. Appl. Maths, 44(2), pp.187-199.

[33] Ul Islam, S., Tirmizi, I.A., Haq, F., Taseer, S.K, (2008), Family of numerical methods based on nonpolynomial splines for solution of contact problems, Commun Nonlinear Sci Numer Simulat, 13, pp.1448-1460.

[34] Ul Islam, S., Tirmizi, I.A., Haq, F., Khan, M.A, (2008), Non-polynomial splines approach to the solution of sixth-order boundary value problems, Appl. Math. Comput., 195(1), pp.270-284.

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