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SOME INCLUSION RELATIONSHIPS FOR CERTAIN SUBCLASSES OF p-VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTION*

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Abstract. In this paper, we investigate several inclusion relationships of certain subclasses of meromorphically p-valent functions which are defined here by means of a linear operator involving the generalized hypergeometric function . We introduce and investigate several new subclasses of p-valent starlike, p-valent convex, p-valent close-to-convex and p-valent quasi-convex meromorphic functions.

Keywords: p-valent meromorphic functions, Hadamard product, hypergeometric function, inclusion relationships.

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1. Introduction

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. We also denote $\Sigma_{p,1-p} = \Sigma_p$. A function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphically p-valent starlike functions of order α in U if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in U; 0 \leq \alpha < p). \tag{2}$$

Also a function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex of order α in U if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in U; 0 \leq \alpha < p). \tag{3}$$

It is easy to observe from (2) and (3) that

$$f(z) \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{4}$$

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For a function $f \in \Sigma_{p,m}$, we say that $f \in \Sigma K_p(\beta,\alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) < -\beta \quad (z \in U; 0 \le \alpha, \beta < p)\,. \tag{5}$$

Functions in the class $\Sigma K_p(\beta,\alpha)$ are called meromorphically p-valent close-to-convex functions of order β and type α. We also say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma K_p^*(\beta,\alpha)$ of meromorphically quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C_p(\alpha)$ such that

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) < -\beta \quad (z \in U; 0 \le \alpha, \beta < p)\,. \tag{6}$$

It follows from (5) and (6) that

$$f(z) \in \Sigma K_p^*(\beta,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma K_p(\beta,\alpha)\,, \tag{7}$$

where $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ are, respectively, the classes of meromorphically p-valent starlike functions of order α and meromorphically p-valent convex functions of order α $(0 \le \alpha < p)$(see Aouf [2]).
For a function $f(z) \in \Sigma_{p,m}$, given by (1) and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k\,, \tag{8}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = (f*g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g*f)(z) \quad (p \in \mathbb{N}). \tag{9}$$

For real or complex numbers

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}; j = 1,2,...,s),$$

we consider the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by (see, for example, [13, p.19])

$${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} \tag{10}$$

$$(q \le s+1; q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}\backslash\{0\} = \mathbb{C}^*), \\ \theta(\theta-1)...(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{11}$$

Corresponding to the function $\phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ given by

$$\phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = z^{-p}\ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)\ , \tag{12}$$

we introduce a function $\phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ defined by

$$\phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * \phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = \frac{1}{z^p(1-z)^{\mu+p}}$$

$$(\mu > -p; z \in U^*). \tag{13}$$

We now define a linear operator $H_{p,q,s}^{m,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s):\Sigma_{p,m}\to\Sigma_{p,m}$ by

$$H_{p,q,s}^{m,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=\phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*f(z) \tag{14}$$

$$\left(\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-;i=1,...,q,j=1,...,s\ ;\mu>-p,f\in\Sigma_{p,m};z\in U^*\right).$$

For convenience, we write

$$H_{p,q,s}^{m,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)=H_{p,q,s}^{m,\mu}(\alpha_1)$$

and

$$H_{p,q,s}^{1-p,\mu}(\alpha_1)=H_{p,q,s}^{\mu}(\alpha_1)\quad(\mu>-p).$$

If $f(z)$ is given by (1), then from (14), we deduce that

$$H_{p,q,s}^{m,\mu}(\alpha_1)f(z)=z^{-p}+\sum_{k=m}^{\infty}\frac{(\mu+p)_{p+k}(\beta_1)_{p+k}...(\beta_s)_{p+k}}{(\alpha_1)_{p+k}...(\alpha_q)_{p+k}}a_kz^k\quad(\mu>-p;z\in U^*). \tag{15}$$

It easily follows from (15) that

$$z\left(H_{p,q,s}^{m,\mu}(\alpha_1)f(z)\right)'=(\mu+p)H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)-(\mu+2p)H_{p,q,s}^{m,\mu}(\alpha_1)f(z) \tag{16}$$

and

$$z\left(H_{p,q,s}^{m,\mu}(\alpha_1+1)f(z)\right)'=\alpha_1H_{p,q,s}^{m,\mu}(\alpha_1)f(z)-(\alpha_1+p)H_{p,q,s}^{m,\mu}(\alpha_1+1)f(z). \tag{17}$$

The linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ was introduced by Patel and Palit [12] .

We note that the linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [7] for analytic functions and is essentially motivated by the operators defined and studied in [6].

Specializing the parameters $\mu,\alpha_i(i=1,2,...q),\beta_j(j=1,2,...s),q$ and s we obtain the following:

(i) $H_{p,2,1}^{m,0}(p,p;p)f(z)=H_{p,2,1}^{m,1}(p+1,p;p)f(z)=f(z)$;

(ii) $H_{p,2,1}^{m,1}(p,p;p)f(z)=\frac{2pf(z)+zf'(z)}{p}$;

(iii) $H_{p,2,1}^{m,2}(p+1,p;p)f(z)=\frac{(2p+1)f(z)+zf'(z)}{p+1}$;

(iv) $H_{p,1,1}^{m,1-p}(c+1,1;c)f(z)=J_{c,p}(f)(z)=\frac{c}{z^{c+p}}\int\limits_0^z t^{c+p-1}f(t)dt$ $(c>0;z\in U^*)$, this integral operator is defined by (34);

(v) $H_{p,2,2}^{m,0}(p+1,p;p)f(z)=\frac{p}{z^{2p}}\int\limits_0^z t^{2p-1}f(t)dt$; $(p\in\mathbb{N};z\in U^*)$;

(vi) $H_{p,2,1}^{1-p,n}(a,1;a)f(z)=\frac{1}{z^p(1-z)^{n+p}}=D^{n+p-1}f(z)$ $(n>-p)$, the operator D^{n+p-1} studied by Ganigi and Uralegaddi [8], Yang [15], Aouf [1], Aouf and Srivastava [3] and Uralegaddi and Patil [14];

(vii) $H_{p,2,1}^{m,\mu}(c,p+\mu;a)f(z)=L_p(a,c)f(z)$ $(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-;\mu>-p)$ (see Liu [10]);

(viii) $H_{1,2,1}^{o,\mu}(\mu+1,n+1;\mu)f(z)=I_{n,\mu}f(z)$ $(\mu>0;n>-1)$ (see Yuan et al. [16]).

We also observe that, for $m=0$, $p=1$ replacing μ by $\mu-1$, we have the operator $H_{1,\mu,q,s}^{0}(\alpha_1)f(z)=H_{\mu,q,s}(\alpha_1)f(z)$ defined by Cho and Kim [5].

We now define the following subclasses of the meromorphic function class $\Sigma_{p,m}$ by means of operator $H_{p,q,s}^{m,\mu}(\alpha_1)$.

Definition 1.1. *In conjunction with (2) and (15),*

$$\Sigma S_{p,q,s}^{*m,\mu}(\alpha_1;\alpha)=\left\{f:f\in\Sigma_{p,m}\ \textit{and}\ H_{p,q,s}^{m,\mu}(\alpha_1)f(z)\in\Sigma S_p^*(\alpha)\,,0\le\alpha<p,p\in\mathbb{N}\right\}. \tag{18}$$

Definition 1.2. *In conjunction with (3) and (15),*

$$\Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)=\left\{f: f\in\Sigma_{p,m} \text{ and } H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\in\Sigma C_p(\alpha)\,, 0\le\alpha<p, p\in\mathbb{N}\right\}. \tag{19}$$

Definition 1.3. *In conjunction with (5) and (15),*

$$\Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha) = \{f: f\in\Sigma_{p,m} \text{ and } H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\in\Sigma K_p(\beta,\alpha)\,, 0\le\alpha,\beta<p, p\in\mathbb{N}\ \}. \tag{20}$$

Definition 1.4. *In conjunction with (6) and (15),*

$$\Sigma K^{*m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha) = \{f: f\in\Sigma_{p,m} \text{ and } H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\in\Sigma K^*_p(\beta,\alpha)\,, 0\le\alpha,\beta<p, p\in\mathbb{N}\ \}. \tag{21}$$

In order to establish our main results, we need the following lemma due to Miller and Mocanu [11].

Lemma 1.1. *[11]. Let $\theta(u,v)$ be a complex-valued function such that*

$$\theta: D\to\mathbb{C},\ D\subset\mathbb{C}\times\mathbb{C}\qquad(\mathbb{C} \text{ is the complex plane})$$

and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions:
(i) $\theta(u,v)$ is continuous in D;
(ii) $(1,0)\in D$ and $\operatorname{Re}\{\theta(1,0)\}>0$;
(iii) for all $(iu_2,v_1)\in D$ such that

$$v_1\le-\frac{1}{2}(1+u_2^2)\ ,\ \operatorname{Re}\{\theta(iu_2,v_1)\}\le 0\,.$$

Let

$$q(z)=1+q_1z+q_2z^2+... \tag{22}$$

be analytic in U such that $(q(z),zq'(z))\in D\,(z\in U)$. If

$$\operatorname{Re}\left\{\theta(q(z),zq'(z))\right\}>0\quad(z\in U)\,,$$

then

$$\operatorname{Re}\{q(z)\}>0\quad(z\in U)\,.$$

In this paper, we investigate several inclusion relationships and integral-preserving properties of certain subclasses of meromorphically p-valent functions which are defined here by means of a linear operator involving the generalized hypergeometric function.

2. The main results

In this section, we give several inclusion relationships for meromorphic function classes, which are associated with the operator $H^{m,\mu}_{p,q,s}(\alpha_1)$.

Theorem 2.1. *Let $\mu>-p$ and $0\le\alpha<p, p\in\mathbb{N}$. Then*

$$\Sigma S^{*m,\mu+1}_{p,q,s}(\alpha_1;\alpha)\subset\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)\subset S^{*m,\mu}_{p,q,s}(\alpha_1+1;\alpha).$$

Proof. We first show that

$$\Sigma S^{*m,\mu+1}_{p,q,s}(\alpha_1;\alpha)\subset\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)\quad(\mu>-p;0\le\alpha<p;p\in\mathbb{N})\,. \tag{23}$$

Let $f(z)\in\Sigma S^{*m,\mu+1}_{p,q,s}(\alpha_1;\alpha)$ and set

$$\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)f(z))'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}=-\alpha-(p-\alpha)q(z)\,, \tag{24}$$

where $q(z)$ is given by (22). By using the identity (16), we obtain

$$(\mu+p)\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} = -\alpha-(p-\alpha)q(z)+(\mu+2p)\,. \tag{25}$$

Differentiating (25) logarithmically with respect to z, we obtain

$$\begin{aligned}\frac{z(H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z))'}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= \frac{z(H_{p,q,s}^{m,\mu}(\alpha_1)f(z))'}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} + \frac{(p-\alpha)zq'(z)}{(p-\alpha)q(z)+\alpha-(\mu+2p)} = \\ &= -\alpha-(p-\alpha)q(z)+\frac{(p-\alpha)zq'(z)}{(p-\alpha)q(z)+\alpha-(\mu+2p)}\,.\end{aligned}$$

Let

$$\theta(u,v)=(p-\alpha)u-\frac{(p-\alpha)v}{(p-\alpha)u+\alpha-(\mu+2p)} \tag{26}$$

with $u=q(z)=u_1+u_2$ and $v=zq'(z)=v_1+iv_2$. Then

(i) $\theta(u,v)$ is continuous in $D=\left(\mathbb{C}\backslash\left\{\frac{\mu+2p-\alpha}{p-\alpha}\right\}\right)\times\mathbb{C}$;

(ii) $(1,0)\in D$ with $\Re\{\theta(1,0)\}=p-\alpha>0$.

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{1}{2}\left(1+u_2^2\right)$, we have

$$\begin{aligned}Re\{\theta(iu_2,v_1)\} &= Re\left\{\frac{-(p-\alpha)v_1}{(p-\alpha)iu_2+\alpha-(\mu+2p)}\right\} = \\ &= \frac{(p-\alpha)\left[\mu+2p-\alpha\right]v_1}{(p-\alpha)^2u_2^2+(\alpha-\mu-2p)^2} \leq \\ &\leq -\frac{(p-\alpha)(1+u_2^2)(\mu+2p-\alpha)}{2\left([(p-\alpha)u_2]^2+(\mu+2p-\alpha)^2\right)}<0\,,\end{aligned}$$

which shows that $\theta(u,v)$ satisfies the hypotheses of Lemma 1. Consequently, we easily obtain the inclusion relationship (23).

By using arguments similar to those detailed above, together with (17), we can also prove the right part of Theorem 2.1, that is, that

$$\Sigma S_{p,q,s}^{*m,\mu}(\alpha_1;\alpha)\subset\Sigma S_{p,q,s}^{*m,\mu}(\alpha_1+1;\alpha)\quad(\mu>-p;0\leq\alpha<p;p\in\mathbb{N})\,. \tag{27}$$

Combining the inclusion relationships (23) and (25), we complete the proof of Theorem 2.1. □

Theorem 2.2. *Let $\mu>-p$ and $0\leq\alpha<p$, $p\in\mathbb{N}$. Then*

$$\Sigma C_{p,q,s}^{m,\mu+1}(\alpha_1;\alpha)\subset\Sigma C_{p,q,s}^{m,\mu}(\alpha_1;\alpha)\subset\Sigma C_{p,q,s}^{m,\mu}(\alpha_1+1;\alpha)\,.$$

Proof. Let $f\in\Sigma C_{p,q,s}^{m,\mu+1}(\alpha_1;\alpha)$. Then, by Definition 1.2, we have

$$H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)\in\Sigma C_p(\alpha)\quad(\mu>-p;0\leq\alpha<p;p\in\mathbb{N})\,.$$

Furthermore, in view of the relationship (4), we find that

$$-\frac{z}{p}(H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z))'\in\Sigma S_p^*(\alpha)\,,$$

that is, that

$$H_{p,q,s}^{m,\mu+1}(\alpha_1)\left(-\frac{zf'(z)}{p}\right)\in\Sigma S_p^*(\alpha)\,.$$

Thus, by Definition 1.1 and Theorem 1.1, we have

$$-\frac{z}{p}f'(z)\in\Sigma S_{p,q,s}^{*m,\mu+1}(\alpha_1;\alpha)\subset\Sigma S_{p,q,s}^{*m,\mu}(\alpha_1;\alpha)\,,$$

which implies that

$$\Sigma C^{m,\mu+1}_{p,q,s}(\alpha_1;\alpha) \subset \Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)\,.$$

The right part of Theorem 2.2 can be proved by using similar arguments. The proof of Theorem 2.2 is thus completed. □

Theorem 2.3. *Let* $\mu > -p$ *and* $0 \le \alpha, \beta < p$, $p \in \mathbb{N}$. *Then*

$$\Sigma K^{m,\mu+1}_{p,q,s}(\alpha_1;\beta,\alpha) \subset \Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha) \subset \Sigma K^{m,\mu}_{p,q,s}(\alpha_1+1;\beta,\alpha)\,.$$

Proof. Let us begin by proving that

$$\Sigma K^{m,\mu+1}_{p,q,s}(\alpha_1;\beta,\alpha) \subset \Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha) \quad (\mu > -p; 0 \le \alpha,\beta < p; p \in \mathbb{N})\,. \tag{28}$$

Let $f(z) \in \Sigma K^{m,\mu+1}_{p,q,s}(\alpha_1;\beta,\alpha)$. Then there exists a function $\Psi(z) \in \Sigma S^*_p(\alpha)$ such that

$$Re\left(\frac{z(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z))'}{\Psi(z)}\right) < -\beta \quad (z \in U)\,.$$

We put

$$H^{m,\mu+1}_{p,q,s}(\alpha_1)g(z) = \Psi(z)\ ,$$

so that we have

$$g(z) \in \Sigma S^{*m,\mu+1}_{p,q,s}(\alpha_1;\alpha) \text{ and } Re\left(\frac{z(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z))'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)g(z)}\right) < -\beta \quad (z \in U)\,.$$

We next put

$$\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)f(z))'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} = -\beta - (p-\beta)q(z)\,, \tag{29}$$

where $q(z)$ is given by (22). Thus, by using the identity (16), we obtain

$$\begin{aligned}
\frac{z(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z))'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)g(z)} &= \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)(zf'(z))}{H^{m,\mu+1}_{p,q,s}(\alpha_1)g(z)} = \\
&= \frac{z\left[H^{m,\mu}_{p,q,s}(\alpha_1)(zf'(z))\right]' + (\mu+2p)H^{m,\mu}_{p,q,s}(\alpha_1)(zf'(z))}{z(H^{m,\mu}_{p,q,s}(\alpha_1)g(z))' + (\mu+2p)H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} = \\
&= \frac{\dfrac{z\left[H^{m,\mu}_{p,q,s}(\alpha_1)(zf'(z))\right]'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} + (\mu+2p)\dfrac{H^{m,\mu}_{p,q,s}(\alpha_1)(zf'(z))}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)}}{\dfrac{z\left[H^{m,\mu}_{p,q,s}(\alpha_1)g(z)\right]'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} + (\mu+2p)}\,.
\end{aligned}$$

Since $g(z) \in \Sigma S^{*m,\mu+1}_{p,q,s}(\alpha_1;\alpha)$, by Theorem 2.1, we can put

$$\frac{z\left[(H^{m,\mu}_{p,q,s}(\alpha_1)(g(z)\right]'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} = -\alpha - (p-\alpha)G(z)\,,$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \text{ and } Re(G(z)) = g_1(x,y) > 0 \quad (z \in U)\,.$$

Then

$$\frac{z\left[(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right]'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)g(z)} = \frac{\dfrac{z\left[H^{m,\mu}_{p,q,s}(\alpha_1)(zf'(z))\right]'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} - (\mu+2p)\left[\beta + (p-\beta)q(z)\right]}{-\alpha - (p-\alpha)G(z) + (\mu+2p)}\,. \tag{30}$$

We thus find from (29) that

$$z(H_{p,q,s}^{m;\mu}(\alpha_1)f(z))^{'} = -H_{p,q,s}^{m;\mu}(\alpha_1)g(z)\left[\beta + (p-\beta)q(z)\right]. \tag{31}$$

Differentiating both sides of (31) with respect to z, we obtain

$$\frac{z\left[H_{p,q,s}^{m,\mu}(\alpha_1)(zf^{'}(z))\right]^{'}}{H_{p,q,s}^{m,\mu}(\alpha_1)g(z)} = -(p-\beta)zq^{'}(z) + \left[\alpha + (p-\alpha)G(z)\right]\left[\beta + (p-\beta)q(z)\right]. \tag{32}$$

By substituting (32) into (30), we have

$$\frac{z\left[(H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)\right]^{'}}{H_{p,q,s}^{m,\mu+1}(\alpha_1)g(z)} + \beta = -\left\{(p-\beta)q(z) - \frac{(p-\beta)zq^{'}(z)}{(p-\alpha)G(z)+\alpha-(\mu+2p)}\right\}.$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq^{'}(z) = v_1 + iv_2$, we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = (p-\beta)u - \frac{(p-\beta)v}{(p-\alpha)G(z)+\alpha-(\mu+2p)}, \tag{33}$$

where $(u,v) \in D = (\mathbb{C}\backslash D^*) \times \mathbb{C}$ and

$$D^* = \left\{z : z \in C \text{ and } Re(G(z)) = g_1(x,y) \geq 1 + \frac{\mu+p}{p-\alpha}\right\}.$$

Then it follows from (33) that
(i) $\Phi(u,v)$ is continuous in D;
(ii) $(1,0) \in D$ and $Re\{\Phi(1,0)\} = p - \beta > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$Re\{\Phi(iu_2,v_1)\} = Re\left\{-\frac{(p-\beta)v_1}{(p-\alpha)G(z)+\alpha-\mu-2p}\right\} =$$

$$\begin{aligned} &= \frac{(p-\beta)v_1\left[\mu+2p-\alpha-(p-\alpha)g_1(x,y)\right]}{\left[(p-\alpha)g_1(x,y)+\alpha-\mu-2p\right]^2 + \left[(p-\alpha)g_2(x,y)\right]^2} \leq \\ &\leq -\frac{(p-\beta)(1+u_2^2)\left[\mu+2p-\alpha-(p-\alpha)g_1(x,y)\right]}{2\left[(p-\alpha)g_1(x,y)+\alpha-\mu-2p\right]^2 + 2\left[(p-\alpha)g_2(x,y)\right]^2} < 0, \end{aligned}$$

which shows that $\Phi(u,v)$ satisfies the hypotheses of Lemma 1.1. Thus, in light of (29), we easily deduce the inclusion relationship (28). □

The remainder of our proof of Theorem 2.3 would make use of the identity (17) in an analogous manner. We, therefore, choose to omit the details involved.

Theorem 2.4. *Let* $\mu > -p$ *and* $0 \leq \alpha, \beta < p$, $p \in \mathbb{N}$. *Then*

$$\Sigma K_{p,q,s}^{*m,\mu+1}(\alpha_1;\beta,\alpha) \subset \Sigma K_{p,q,s}^{*m,\mu}(\alpha_1;\beta,\alpha) \subset \Sigma K_{p,q,s}^{*m,\mu}(\alpha_1+1;\beta,\alpha).$$

Proof. Just as we derived Theorem 2.2 as a consequence of Theorem 2.1 by using the equivalence (4), we can also prove Theorem 2.4 by using Theorem 2.3 in conjunction with the equivalence (7). □

3. A set of integral-preserving properties

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $J_{c,p}(f)$ defined by

$$J_{c,p}(f)(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \quad (c > 0; f \in \Sigma_{p,m}) , \tag{34}$$

which satisfies the following relationship :

$$z\left(H_{p,q,s}^{m,\mu}(\alpha_1)J_{c,p}(f)(z)\right)' = cH_{p,q,s}^{m,\mu}(\alpha_1)f(z) - (c+p)H_{p,q,s}^{m,\mu}(\alpha_1)J_{c,p}(f)(z) . \tag{35}$$

In order to obtain the integral-preserving properties involving the integral operator $J_{c,p}(f)$, we also need the following lemma which is popularly known as Jack's lemma.

Lemma 3.1. *[9]. Let $w(z)$ be a non-constant function analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = \zeta w(z_0) ,$$

where $\zeta \geq 1$ is a real number.

Theorem 3.1. *Let $c > -p$, $\mu > -p$ and $0 \leq \alpha < p$, $p \in \mathbb{N}$. If $f(z) \in \Sigma S_{p,q,s}^{*m,\mu}(\alpha_1; \alpha)$, then*

$$J_{c,p}(f)(z) \in \Sigma S_{p,q,s}^{*m,\mu}(\alpha_1; \alpha) .$$

Proof. Suppose that $f(z) \in \Sigma S_{p,q,s}^{*m,\mu}(\alpha_1; \alpha)$ and let

$$\frac{z\left(H_{p,q,s}^{m,\mu}(\alpha_1)J_{c,p}(f)(z)\right)'}{H_{p,q,s}^{m,\mu}(\alpha_1)J_{c,p}(f)(z)} = -\frac{p + (p-2\alpha)w(z)}{1 - w(z)} , \tag{36}$$

where $w(0) = 0$. Then, by using (35) and (36), we have

$$\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)J_{c,p}(f)(z)} = \frac{c - (c + 2p - 2\alpha)w(z)}{c(1 - w(z))} . \tag{37}$$

Differentiating (37) logarithmically with respect to z, we obtain

$$\begin{aligned} \frac{z\left(H_{p,q,s}^{m,\mu}(\alpha_1)f(z)\right)'}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} &= -\frac{p + (p-2\alpha)w(z)}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} - \\ &\quad -\frac{(c + 2p - 2\alpha)zw'(z)}{c - (c + 2p - 2\alpha)w(z)} , \end{aligned} \tag{38}$$

so that

$$\begin{aligned} \frac{z\left(H_{p,q,s}^{m,\mu}(\alpha_1)f(z)\right)'}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} + \alpha &= \frac{(\alpha - p)(1 + w(z))}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} - \\ &\quad -\frac{(c + 2p - 2\alpha)zw'(z)}{c - (c + 2p - 2\alpha)w(z)} . \end{aligned} \tag{39}$$

Now, assuming that $\max\limits_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \, (z_0 \in U)$ and applying Jack's lemma, we have

$$z_0 w'(z_0) = \zeta w(z_0) \quad (\zeta \geq 1) . \tag{40}$$

If we set $w(z_0) = e^{i\theta}$ $(\theta \in R)$ in (3.6) and observe that

$$Re\left\{\frac{(\alpha - p)(1 + w(z_0))}{1 - w(z_0)}\right\} = 0 ,$$

then we obtain

$$\begin{aligned} Re\left\{\frac{z_0\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z_0)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z_0)}+\alpha\right\} &= Re\left\{\frac{z_0w'(z_0)}{1-w(z_0)}-\frac{(c+2p-2\alpha)z_0w'(z_0)}{c-(c+2p-2\alpha)w(z_0)}\right\}= \\ &= Re\left\{-\frac{2(p-\alpha)\zeta e^{i\theta}}{(1-e^{i\theta})\left[c-(c+2p-2\alpha)e^{i\theta}\right]}\right\}= \\ &= \frac{2\zeta(p-\alpha)(c+p-\alpha)}{c^2-2c(c+2p-2\alpha)\cos\theta+(c+2p-2\alpha)^2}\geq 0\,, \end{aligned}$$

which obviously contradicts the hypothesis $f(z)\in\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)$. Consequently, we can deduce that $|w(z)|<1\,(z\in U)$, which, in view of (36), proves the integral-preserving property asserted by Theorem 5. □

Theorem 3.2. *Let $c>-p$, $\mu>-p$, and $0\leq\alpha<p$, $p\in\mathbb{N}$. If $f(z)\in\Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)$, then*

$$J_{c,p}(f)(z)\in\Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)\,.$$

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned} f(z) &\in \Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)\Leftrightarrow\frac{-zf'(z)}{p}\in\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha) \\ &\Rightarrow J_{c,p}\left(\frac{-zf'(z)}{p}\right)\in\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha) \\ &\Leftrightarrow -\frac{z}{p}\left(J_{c,p}f(z)\right)'\in\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha) \\ &\Rightarrow J_{c,p}(f)(z)\in\Sigma C^{m,\mu}_{p,q,s}(\alpha_1;\alpha)\,, \end{aligned}$$

which proves Theorem 3.2. □

Theorem 3.3. *Let $c>-p$, $\mu>-p$, $0\leq\alpha,\beta<p$ and $p\in\mathbb{N}$. If $f(z)\in\Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha)$, then*

$$J_{c,p}(f)(z)\in\Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha).$$

Proof. Suppose that $f(z)\in\Sigma K^{m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha)$. Then, by Definition 3, there exists a function $g(z)\in\Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)$ such that

$$Re\left(\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)}\right)<-\beta\quad(z\in U)\,.$$

Thus, upon setting

$$\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)}+\beta=-(p-\beta)q(z)\,, \tag{41}$$

where $q(z)$ is given by (22), we find from (35) that

$$\begin{aligned} \frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} &= -\frac{H^{m,\mu}_{p,q,s}(\alpha_1)(-zf'(z))}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)}= \\ &= -\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(-zf'(z)))'+(c+p)H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(-zf'(z))}{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(g(z)))'+(c+p)H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)}= \\ &= -\frac{\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(-zf'(z)))'}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)}+(c+p)\frac{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(-zf'(z))}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)}}{\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z))'}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)}+(c+p)}\,. \end{aligned}$$

Since $g(z) \in \Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)$, we know from Theorem 5 that $J_{c,p}g(z) \in \Sigma S^{*m,\mu}_{p,q,s}(\alpha_1;\alpha)$. So we can set

$$\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z))^{'}}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)} + \alpha = -(p-\alpha)G(z)\,, \tag{42}$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \text{ and } Re(G(z)) = g_1(x,y) > 0 \quad (z \in U)\,.$$

Then we have

$$\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)f(z))^{'}}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} = \frac{\frac{z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}(-zf^{'}(z)))^{'}}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)} + (c+p)\left[\beta + (p-\beta)q(z)\right]}{\alpha + (p-\alpha)G(z) - (c+p)}\,. \tag{43}$$

We also find from (41) that

$$z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}f(z))^{'} = (-H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z))\left[\beta + (p-\beta)q(z)\right]\,. \tag{44}$$

Differentiating both sides of (44) with respect to z, we obtain

$$\begin{aligned} z\left[z\left(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}f(z)\right)^{'}\right]^{'} &= -z(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z))^{'}\left[\beta + (p-\beta)q(z)\right] - \\ &\quad -(p-\beta)zq^{'}(z)H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)\,, \end{aligned} \tag{45}$$

that is,

$$\begin{aligned} \frac{z\left[z\left(H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}f(z)\right)^{'}\right]^{'}}{H^{m,\mu}_{p,q,s}(\alpha_1)J_{c,p}g(z)} &= -(p-\beta)zq^{'}(z) + \\ &\quad + \left[\alpha + (p-\alpha)G(z)\right]\left[\beta + (p-\beta)q(z)\right]\,. \end{aligned} \tag{46}$$

Substituting (46) into (43), we find that

$$\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)^{'}}{H^{m,\mu}_{p,q,s}(\alpha_1)g(z)} + \beta = -(p-\beta)q(z) + \frac{(p-\beta)zq^{'}(z)}{(p-\alpha)G(z) + \alpha - (c+p)}\,. \tag{47}$$

Then, by setting

$$u = q(z) = u_1 + iu_2 \text{ and } v = zq^{'}(z) = v_1 + iv_2\,,$$

we can define the function $\theta(u,v)$ by

$$\theta(u,v) = (p-\beta)u - \frac{(p-\beta)v}{(p-\alpha)G(z) + \alpha - (c+p)}\,.$$

□

The remainder of our proof of Theorem 3.3 is similar to that of Theorem 2.3, so we choose to omit the analogous details involved.

Theorem 3.4. *Let* $c > -p,\ \mu > -p, 0 \le \alpha, \beta < p,\ p \in \mathbb{N}$. *If* $f(z) \in \Sigma K^{*m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha)$, *then*

$$J_{c,p}(f)(z) \in \Sigma K^{*m,\mu}_{p,q,s}(\alpha_1;\beta,\alpha)\,.$$

Proof. Just as we derived Theorem 6 from Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 from Theorem 7.

Remark 3.1. (i) *Putting* $m=0, q=2, s=1, \alpha_1 = \mu+1, \alpha_2 = n+1 (n > -1)$ *and* $\beta_1 = \mu(\mu > 0)$ *in the above results, we obtain the results obtained by Aouf and Xu [4],*

(ii) *Putting* $m=0, p=1, q=2, s=1, \alpha_1 = \mu+1, \alpha_2 = n+1 (n > -1)$ *and* $\beta_1 = \mu(\mu > 0)$ *in the above results,we obtain the results obtained by Yuan et al. [16].*

□

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