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SOME SEQUENCE SPACES AND MATRIX TRANSFORMATIONS IN MULTIPLICATIVE SENSE

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Abstract. In this paper, based on multiplicative calculus matrix transformations in sequence spaces are studied and characterized. Also, we give a brief introduction to $*$- summability based on multiplicative type addition (or just multiplication) and define the multiplicative dual $*$-summability methods using $*$-Stieltjes integral and multiplicative differentiation under the $*$-integral sign.

Keywords: multiplicative calculus; multiplicative differentiation; multiplicative integral.

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1. Introduction

A survey related to the dual summability methods were presented , in detail, by Başar [6] in the sense of additive calculus by the usual way. In this study, the corresponding analysis and results to Başar [6] in the sense of multiplicative calculus were given. Finally, we give a short conclusion on multiplicative calculus and the classes of matrix transformations.

2. Preliminaries, background and notations

Multiplicative calculus is an alternative for classical Newton-Leibnitz calculus and has numerous theory and applications in the literature. In general, calculuses like multiplicative calculus are considered a kind of non-Newtonian calculus. We basically concentrate on matrix transformations between two sequence spaces using multiplicative calculus operations. In this paper we shall use *- symbol for notations of multiplicative calculus and then arithmetic operations of multiplicative calculus turn out

$$\begin{array}{llclcl}
\text{*-addition} & x \oplus y & = & e^{(\ln x + \ln y)} & = & x \cdot y \\
\text{*-subtraction} & x \ominus y & = & e^{(\ln x - \ln y)} & = & x \div y,\ y \neq 0 \\
\text{*-multiplication} & x \odot \times y & = & e^{(\ln x \times \ln y)} & = & x^{\ln y} = y^{\ln x} \\
\text{*-division} & x \oslash y & = & e^{(\ln x \div \ln w)} & = & x^{\frac{1}{\ln y}},\ y \neq 1
\end{array}$$

Multiplicative order or $*$-order is same as the classical ordering operation because the generator function of e^x is a monotonically increasing function.

Throughout the text, for the sake of abbreviation we use CC for the classical calculus, MC for the multiplicative calculus and NNC for the non-Newtonian calculus. It can be easily observed that the basic principles of multiplicative calculus in [7-9]. Plenty of advance studies in MC could be found in several articles such as [1-4, 11, 13, 14]. The main subject of this study is infinite matrix transformations in some new sequence spaces.

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An arithmetic is any system that satisfies the whole of the ordered field axioms has a domain that is a subset of $\mathbb{R}$. There are infinitely-many arithmetics, all of them are isomorphic, that is, structurally equivalent. Nevertheless the fact that two systems are isomorphic does not exclude their separate uses. In [7], it is shown that each ordered pair of arithmetics gives rise to a calculus by a sensible use of the first arithmetic or arguments of function and the second arithmetic for values of function.

Let α and β be arbitrarily selected generators and $(\alpha - arithmetic, \beta - arithmetic)$ is the ordered pair of arithmetics. The following table may be useful for the notations used in α-arithmetic and β-arithmetic:

Definitions for α-arithmetic are also valid for β-arithmetic. For example, β-convergence is defined by means of β-intervals and their β-interiors.

TABLE 1. Notation in α-arithmetic and β-arithmetic.

	α-arithmetic	β-arithmetic
Realm	A	B
Addition	$\dot{+}$	$\ddot{+}$
Substraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{\le}$	$\ddot{\le}$

In the NNC, α-arithmetic is used on arguments and β-arithmetic is used on values; in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of the NNC are applied only to functions with arguments in A and values in B.

The isomorphism from α-arithmetic to β-arithmetic is the unique function ι (iota) which has the following three properties:

(i) ι is one to one.
(ii) ι is from A onto B.
(iii) For any numbers u and v in A,

$$\begin{gathered}\iota(u\dot{+}v) = \iota(u)\ddot{+}\iota(v),\\ \iota(u\dot{-}v) = \iota(u)\ddot{-}\iota(v),\\ \iota(u\dot{\times}v) = \iota(u)\ddot{\times}\iota(v),\\ \iota(u\dot{/}v) = \iota(u)\ddot{/}\iota(v);\ v \neq \dot{0},\\ u\dot{\le}v \Leftrightarrow \iota(u)\ddot{\le}\iota(v).\end{gathered}$$

It turns out that $\iota(x) = \beta[\alpha^{-1}(x)]$ for all x in A, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n.

Since, for example, $u\dot{+}v = \iota^{-1}[\iota(u)\ddot{+}\iota(v)]$, it should be clear that any statement in α-arithmetic can readily be transformed into a statement in β-arithmetic.

It is convenient to indicate the uniform relationships between the corresponding notions of the $*$-calculus and classical calculus.

For each number $a \in A$, let $\bar{a} = \alpha^{-1}(a)$. Let f be a function from A into B, and set $\bar{f}(t) = \beta\{f[\alpha(t)]\}$.

Then $* - \lim_{x\to a} f(x)$ and $\lim_{t\to\bar{a}} \bar{f}(t)$ coexist, and if they exist, we have

$$* - \lim_{x\to a} f(x) = \beta\left[\lim_{t\to\bar{a}} \bar{f}(t)\right].$$

Furthermore, f is $*$-continuous at a if and only if $\bar{f}$ is classically continuous at $\bar{a}$.

If $G_a^{*b} f$ is the $*$-gradient of f over $\dot{[a,b]}$, then $G_a^{*b} f = \beta(G_{\bar{a}}^{\bar{b}})\bar{f}$, where $G_{\bar{a}}^{\bar{b}}\bar{f}$ is the classical gradient of $\bar{f}$ over $\dot{[a,b]}$.

The derivative $(D^* f)(a)$ and $D\bar{f}(\bar{a})$ coexist, and if they exist, we have $(D^* f)(a) = \beta[D\bar{f}(\bar{a})]$. If f is $*$-continuous on $\dot{[a,b]}$, then $M_a^{*b} f = \beta(M_{\bar{a}}^{\bar{b}})\bar{f}$ and

$$\begin{aligned} \int_a^{*b} f(x)dx &= \beta\left[\int_{\bar{a}}^{*\bar{b}} \bar{f}(t)dt\right] = \\ &= \beta\left\{\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}[f(\alpha(x))]dx\right\}. \end{aligned}$$

We only use the case of $\alpha = I$ and $\beta = e^x$ owing to generate MC.

Generally, linear operators are given between two sequence spaces by an infinite matrix. Therefore the knowledge of the theory of matrix transformations is crucial in the study of sequence spaces. We always assume the arguments of all functions and the elements of all matrices are positive real numbers. One can easily obtained the same results for negative real numbers by using the function of $-e^x$ as the generator function instead of e^x.

Suppose $A = (a_{nk})$ is an infinite matrix of reals a_{nk}, where $k, n \in \mathbb{N}$ and $x = (x_k) \in \omega$, the space of all sequences with positive reals. Then, the A-transform of x is given by the matrix product in terms of multiplicative calculus

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = \begin{pmatrix} {a_{11}}^{\ln x_1} \cdot {a_{12}}^{\ln x_2} \cdots \\ {a_{21}}^{\ln x_1} \cdot {a_{22}}^{\ln x_2} \cdots \\ \vdots \\ {a_{n1}}^{\ln x_1} \cdot {a_{n2}}^{\ln x_2} \cdots \\ \vdots \end{pmatrix}.$$

Thus, x is transformed into the sequence $Ax = \{(Ax)_n^*\}$ with

$$(Ax)_n^* = \prod_{k=1}^{\infty} {a_{nk}}^{\ln x_k}, \tag{1}$$

provided that the product on the right hand side of (1) converges for all $n \in \mathbb{N}$.

Let λ and μ be any two sequence spaces. If Ax exists and is in μ for every sequence $x = (x_k) \in \lambda$, then we say that A defines a matrix mapping from λ into μ, and is denoted by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the infinite product on the right hand side of (1) converges for each $n \in \mathbb{N}$ and for every $x \in \lambda$, and we have $Ax = \{(Ax)_n^*\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$.

3. Characterizations of some matrix classes

In this section, our main goal is to characterize some matrix classes in terms of multiplicative calculus. Let $\ell_\infty^*, c^*, c_0^*$ ve ℓ_1^* be the corresponding sequence spaces of ℓ_∞, c, c_0 and ℓ_1 in the usual sense, respectively. That is,

$$\ell_\infty^* := \left\{ x = (x_k) : \sup_{k\in\mathbb{N}} e^{|\ln x_k|} < \infty \right\} = \left\{ x = (x_k) : \sup_{k\in\mathbb{N}} |x_k|^* < \infty \right\},$$

$$c^* := \left\{ x = (x_k) : \exists l \in \mathbb{R}^* \ni \lim_{k\to\infty} e^{\ln x_k} = l \right\} = \left\{ x = (x_k) : \exists l \in \mathbb{R}^* \ni * - \lim_{k\to\infty} x_k = l \right\},$$

$$c_0^* := \left\{ x = (x_k) : \lim_{k\to\infty} e^{|\ln x_k|} = 1 \right\} = \left\{ x = (x_k) : * - \lim_{k\to\infty} x_k = 1 \right\},$$

$$\ell_1^* := \left\{ x = (x_k) : \prod_{k=1}^{\infty} e^{|\ln x_k|} < \infty \right\} = \left\{ x = (x_k) : {}^*\sum_{k=1}^{\infty} |x_k|^* < \infty \right\}.$$

are the spaces of $*$-bounded, $*$-convergent, $*$-null and $*$-absolute sumable sequences, respectively. Çakmak and Başar [15] have investigated the correspondences of those sequence spaces in terms of NNC using real variables and Tekin and Başar [12] have studied on the same using complex variables. We characterize the classes $(\ell_\infty^* : \ell_\infty^*)$, $(c^* : c^*)$ and $(c^* : c^*; p)$ of matrix transformations.

Theorem 3.1. *$A = (a_{nk}) \in (\ell_\infty^* : \ell_\infty^*)$ if and only if*

$$\sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} |a_{nk}|^* = \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} e^{|\ln a_{nk}|} < \infty. \tag{2}$$

Proof. Suppose that the condition (2) holds and $x = (x_k) \in \ell_\infty^*$. Since for every fixed $n \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} |a_{nk} \odot x_k|^* = \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} e^{|\ln a_{nk} \cdot \ln x_k|} < \infty$$

the A-transform of x exists. Therefore,

$$\begin{aligned} \sup_{n\in\mathbb{N}} |(Ax)_n^*|^* &= \sup_{n\in\mathbb{N}} \left| \prod_{k=1}^{\infty} a_{nk} \odot x_k \right|^* = \sup_{n\in\mathbb{N}} \left| \prod_{k=1}^{\infty} e^{\ln a_{nk}\cdot \ln x_k} \right|^* = \\ &= \sup_{n\in\mathbb{N}} e^{|\sum_{k=1}^{\infty} (\ln a_{nk}\cdot \ln x_k)|} \leq \sup_{n\in\mathbb{N}} e^{\sum_{k=1}^{\infty} |\ln a_{nk}\cdot \ln x_k|} \leq \\ &\leq \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} e^{|\ln a_{nk}|} \cdot \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} e^{|\ln x_k|} = \|x\|_\infty^* \odot \left(\sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} |a_{nk}|^* \right) < \infty, \end{aligned}$$

which gives that $Ax \in \ell_\infty^*$, as asserted.

Conversely, assume that $A = (a_{nk}) \in (\ell_\infty^* : \ell_\infty^*)$. Let $x = (x_k) \in \ell_\infty^*$ and observe that $(A)_n^*$ is a sequence of bounded linear operators on ℓ_∞^* such that

$$\sup_{n\in\mathbb{N}} |(Ax)_n^*|^* = \sup_{n\in\mathbb{N}} \left| \prod_{k=1}^{\infty} a_{nk} \odot x_k \right|^* = \sup_{n\in\mathbb{N}} \left| \prod_{k=1}^{\infty} e^{\ln a_{nk}\cdot \ln x_k} \right|^* = \sup_{n\in\mathbb{N}} e^{|\sum_{k=1}^{\infty} (\ln a_{nk}\cdot \ln x_k)|} < \infty.$$

This completes the proof. □

In order to give the necessary and sufficient conditions on an infinite matrix which maps the space of multiplicative convergent sequences into itself, we state and prove a special theorem. We call that *Kojima-Schur theorem in terms of MC* or $*$-*Kojima-Schur theorem*. A matrix satisfying $*$-Kojima-Schur theorem is called a $*$-*conservative matrix.*

Theorem 3.2. *$A = (a_{nk}) \in (c^* : c^*)$ if and only if (2) holds, and there exist $\alpha_k, \alpha \in \mathbb{C}$ such that*

$$* - \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{3}$$

$$* - \lim_{n\to\infty} \prod_{k=1}^{\infty} a_{nk} = \alpha. \tag{4}$$

Proof. Let us suppose that the conditions (2), (3) and (4) hold, and $x = (x_k) \in c^*$ with $x_k \to l$, as $k \to \infty$. Then, from (2), since

$$\begin{aligned} \prod_{k=1}^{\infty} e^{|\ln a_{nk}\cdot \ln x_k|} &\le \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} e^{|\ln a_{nk}\cdot \ln x_k|} = \\ &= \sup_{n\in\mathbb{N}} \prod_{k=1}^{\infty} |a_{nk} \odot x_k|^* < \infty \end{aligned}$$

for each $n \in \mathbb{N}$, the A-transform of x exists. In this case, the equality

$$\prod_{k=1}^{\infty} a_{nk} \odot x_k = \prod_{k=1}^{\infty} a_{nk} \odot (x_k/l) \cdot l \odot \prod_{k=1}^{\infty} a_{nk} \tag{5}$$

holds for each $n \in \mathbb{N}$. Simplifying (5) we have

$$\left(\prod_{k=1}^{\infty} a_{nk}\right)^{\ln x_k} = \prod_{k=1}^{\infty} a_{nk}^{\ln(x_k/l)} \cdot \left(\prod_{k=1}^{\infty} a_{nk}\right)^{\ln l} \tag{6}$$

holds for each $n \in \mathbb{N}$. In the equation (6), since the first term on the right hand side tends to $\prod_{k=1}^{\infty} \alpha_k^{\ln(x_k/l)}$ by (3) and the second term on the right hand side tends to ℓ^{α} by (6) as $n \to \infty$, we have

$$\left(\prod_{k=1}^{\infty} a_{nk}\right)^{\ln x_k} = \prod_{k=1}^{\infty} a_{nk}^{\ln(x_k/l)} \cdot \alpha^{\ln l}.$$

Therefore, $Ax \in c^*$.

Conversely, suppose that $A \in (c^* : c^*)$. Then, Ax exists for every $x \in c^*$. The necessity of the conditions (3) and (4) immediately follows by taking $x = e_*^{(k)} = (1, \ldots, 1, e, 1, \ldots, 1, \ldots)$ and $x = e_* = (e, e, e, \ldots)$, respectively. Since the inclusion relation $c^* \subset \ell_\infty^*$ [15] holds, the necessity of the condition (2) is obtained from Theorem 2.1.

This completes the proof. □

The necessary and sufficient conditions on an infinite matrix that maps the space c^* into itself, is given by $*$-Silverman-Toeplitz theorem. A matrix satisfying $*$-Silverman-Toeplitz theorem is called a *Toeplitz matrix* or *regular matrix*. We denote the class of Toeplitz matrices by $(c^* : c^*; p)$. The next corollary characterizes the class of $(c^* : c^*; p)$.

Corollary 3.1 (**$*$-Silverman-Toeplitz theorem**). *$A = (a_{nk}) \in (c^* : c^*; p)$ if and only if (2) holds, and (3), (4) also hold with $\alpha_k = 1$ for all $k \in \mathbb{N}$, $\alpha = e$, respectively.*

4. Multiplicative dual summability methods

In this section following Başar [6], we give a brief survey on the multiplicative dual summability methods. Lorentz [10] studied the dual summability methods dependent on a Stieltjes integral. We summarize first, the relationship between multiplicative dual summability methods and $*$-Steiltjes integral. Secondly, we indicate the relation between multiplicative dual summability methods. Finally, we also give the theory of multiplicative usual summability methods between sequence spaces.

In mathematics, Riemann-Stieltjes integral is a generalization of the Riemann integral, named after Bernhard Riemann and Thomas Joannes Stieltjes. The definition of this integral was first published in 1894 by Stieltjes. It serves as an instructive and useful precursor of the Lebesgue integral. We shall define this integral in terms of MC. The multiplicative Stieltjes integral of a real-valued function f of a real variable with respect to a real function g is denoted by

$$\int_a^{*b} f(x)dg(x) = \exp\left\{\int_a^{*b} \ln f(x)dg(x)\right\} \tag{7}$$

and defined to be the limit, as the mesh of the partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of the interval $[a, b]$ approaches zero, of the approximating $*$-sum (or the approximating product)

$$S(P, f, g) = \prod_{k=0}^{n-1} [f(c_i)]^{[g(x_{i+1})-g(x_i)]} = \exp\left\{\sum_{k=1}^{n-1} \ln f(c_i) \cdot [g(x_{i+1}) - g(x_i)]\right\},$$

where c_i is in the $i - th$ subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the integrand and the integrator. The "limit" is here understood to be a number L (the value of $*$-Stieltjes integral) such that for every $\varepsilon > 1$, there exists $\delta > 0$ such that for every partition P with $mesh(P) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$\left|\frac{S(P, f, g)}{L}\right|^* = e^{\left|\ln \frac{S(P,f,g)}{L}\right|} < \varepsilon.$$

Suppose a and b are constant, and that $f(x)$ involves a parameter α which is constant in the $*$-integration but may vary to form different $*$-integrals. Since $f(x, \alpha)$ be a $*$-continuous function of x and α in the compact set $\{(x, \alpha) : \alpha_0 \leq \alpha \leq \alpha_1$ and $a \leq x \leq b\}$ and that the partial $*$-derivative $f_\alpha(x, \alpha)$ exists and is $*$-continuous then if one defines:

$$\varphi(\alpha) = \int_a^{*b} f(x, \alpha)dx = \exp\left[\int_a^b \ln f(x, \alpha)dx\right]$$

φ may be $*$-differentiated with respect to α by $*$-differentiating under the integral sign;

$$\begin{aligned}
\frac{d^*\varphi}{d\alpha} &= \exp\left[\frac{\varphi'(\alpha)}{\varphi(\alpha)}\right] = \exp\left\{\frac{\exp\left[\int_a^b \ln f(x, \alpha)dx\right] \cdot \int_a^b \frac{f_\alpha(x,\alpha)}{f(x,\alpha)}dx}{\exp\left[\int_a^b \ln f(x, \alpha)dx\right]}\right\} = \\
&= \exp\left[\int_a^b \frac{f_\alpha(x, \alpha)}{f(x, \alpha)}dx\right] = \int_a^{*b} \frac{\partial^*}{\partial\alpha} f(x, \alpha)dx.
\end{aligned}$$

Consider the function $\sigma(z)$ defined by the $*$-Stieltjes integral of a positive real valued function $\rho(z, t)$ of positive real variables z and t with respect to a positive real function $s(t)$ of positive real variable t as

$$\sigma(z) = \int_0^{*+\infty} \rho(z,t)ds(t) = \exp\left[\int_0^{+\infty} \ln\rho(z,t)ds(t)\right], \tag{8}$$

that have to be able to defined for all $z \geq 0$. If the $*$-limit of

$$* - \lim_{z\to+\infty} \sigma(z) = \sigma$$

exists, then $\sigma(z)$ is called the generalized $*$-limit of the function $s(t)$ acquired by the method (8). The $*$-Stieltjes integral

$$\exp\left[\int_0^{c} \ln\rho(z,t)ds(t)\right] \tag{9}$$

must exist in order that the improper integral on the right hand side of (8) exists for all $c > 0$. (8) is the limit of (9) as $c \to \infty$. Then we have two options to guaranteeing the existence of the integral (9):

(i) If $\rho(z,t)$ is a $*$-continuous function with respect to the variable t for every fixed $z \geq 0$ and $s(t)$ is of bounded variation on all intervals $(0,c)$.

(ii) Conversely, s(t) is a $*$-continuous function and $\rho(z,t)$ is of bounded variation on all intervals $(0,c)$.

The necessary and sufficient condition in order to integral (8) exists for the continuous function $s(t)$ such that $* - \lim_{t\to+\infty} s(t)$ to be finite is the function $\rho(z,t)$ is of bounded variation on the interval $0 \leq t < \infty$ for every fixed $z \geq 0$. In this situation, with the help of integration by parts, we have

$$\exp\left[\int_0^{c} \ln\rho(z,t)ds(t)\right] = \exp\left[K - \int_0^{c} s(t)\cdot\frac{\rho_t(z,t)}{\rho(z,t)}dt\right],$$

where K is a positive constant. By applying this integration to (8), (8) is reduced to the transformation

$$\sigma'(z) = \exp\left[\int_0^{+\infty} s(t)\cdot\frac{\rho_{zt}(z,t)}{\rho_z(z,t)}dt\right] = \int_0^{*+\infty} s(t)\cdot d_t\eta(z,t), \tag{10}$$

where $\eta(z,t)$ is a suitable chosen bounded variation function on the interval $0 \leq t < \infty$. The methods is given by (8) and (10) one of which is reduced to the other one by integration by parts are called *multiplicative dual summability methods.*

If one seeks a transformation in the form

$$\sigma_n := \prod_{k=1}^{\infty} \rho_{nk}x_k \quad \text{for all} \quad n \in \mathbb{N}, \tag{11}$$

which contains as a special case of the transformation of the sequence (y_k) with $y_k = \prod_{j=1}^{k} x_j$ for all $k \in \mathbb{N}$. Under the hypothesis (i) the limitation of (8) is discovered.

Consider the step function of

$$s(t) = \begin{cases} y_k & , \quad k < t \leq k+1, \\ 0 & , \quad t = 0. \end{cases} \tag{12}$$

for all $k \in \mathbb{N}$. It is evident that the function $s(t)$ defined by (12) has the jumps x_k at the points $t = k$ and in other place is partially constant. Therefore,

$$\sigma(z) = \int_0^{*+\infty} \rho(z,t)ds(t) = \prod_{k=1}^{\infty} \rho(z,k)^{\ln x_k}$$

holds for the function which corresponds to the transformation (11) replacing n by the new variable z. However, by replacing the function $s(t)$, defined by (12) under the hypothesis (ii), in (8) does not take one to the asserted result. Since, the existence of the integral (8) is assured only for the continuous functions $s(t)$.

The $*$-limit $* - \lim_{t\to\infty} \rho(z,t) = \varrho(z)$ exists for the convergence preserving method (8). For the bouned function $s(t)$, which tends to infinity as $t \to \infty$, we can say for $\varrho(z)$ that if the limit $* - \lim_{z\to\infty} \sigma(z)$ exists, then

$$* - \lim_{t\to\infty} \rho(z,t) := 1, \quad (z > 0). \tag{13}$$

Suppose that $s(0) = 0$ for the function $s(t)$ of bounded variation in (8). Moreover, if (13) also holds, then

$$\sigma'(z) = \left[\int_0^{*+\infty} s(t) \cdot d_t\rho(z,t) \right]^{-1} \tag{14}$$

is obtained by a usual integration by parts. (14) is the similar type method of (10) and is called the multiplicative dual method of the method (8).

Define the function $s(t)$ as

$$s(t) = \begin{cases} 0 & , \quad t \le t_0, \\ 1 & , \quad t > t_0, \end{cases}$$

and replace by the variable z by the integer variable n. Suppose that the function $a(n,t)$ is interpolated for all n and write a_{nk} instead of $a(n,k)$. Hence, we extract the product transformation

$$A: \qquad \sigma_n := \prod_{k=1}^{\infty} {a_{nk}}^{\ln x_k} \text{ for all } n \in \mathbb{N}$$

from (8). Then the multiplicative dual transformation is given by

$$B: \qquad \sigma'_n := \prod_{k=1}^{\infty} {b_{nk}}^{\ln y_k} \text{ for all } n \in \mathbb{N},$$

where $b_{nk} = \frac{a_{nk}}{a_{n,k+1}}$ for all $k, n \in \mathbb{N}$.

Example 4.1. *Let λ be a $*$-sequence space and consider the production matrix $P = (p_{nk})$ is defined by*

$$p_{nk} = \begin{cases} e & , \quad 0 \le k \le n, \\ 1 & , \quad k > n, \end{cases} \tag{15}$$

for all $k \in \mathbb{N}$. Then, the space of series λ_P whose sequences of partial products are in the space λ, is defined by

$$\lambda_P := \left\{ x = (x_k) \in w^* : \left(\prod_{j=1}^{k} x_j \right) \in \lambda \right\}.$$

Define the sequence $y = (y_k)$ *by means of the sequence* $(x_k) \in w^*$ *by*

$$y_k := (Px)_k = \prod_{j=1}^{k} x_j \ \text{for all } k \in \mathbb{N}. \tag{16}$$

Then, since $\lambda_P \cong \lambda$ *it is clear that "*$x \in \lambda_P$ *if and only if* $y \in \lambda$*".*

Let us assume that the infinite matrices $A = (a_{nk})$ *and* $B = (b_{nk})$ *map the sequences* $x = (x_k)$ *and* $y = (y_k)$ *that are connected with the relation (16) to the sequences* $u = (u_n)$ *and* $v = (v_n)$*, respectively, such that*

$$u_n = (Ax)_n^* = \prod_{k=1}^{\infty} {a_{nk}}^{\ln x_k} \ \text{for all } k \in \mathbb{N}, \tag{17}$$

$$v_n = (By)_n^* = \prod_{k=1}^{\infty} {b_{nk}}^{\ln y_k} \ \text{for all } k \in \mathbb{N}. \tag{18}$$

Obviously, the method B *is applied to the* P*-transform of the sequence* $x = (x_k)$ *while the method* A *is applied to the terms of the sequence* $x = (x_k)$*. Therefore, the methods* A *and* B *are essentially different.*

Let us suppose that the matrix product BP *exists. In this case, we say that* A *and* B *in (17) and (18) are multiplicative dual summability methods if* u_n *reduces to* v_n *(or* v_n *reduces to* u_n*) under the assumption of suitable summation by parts. This leads us to the fact that* BP *exists and is equal to* A *and* $Ax = (BP)x = B(Px) = By$ *conventionally holds, provided that one side exists. This is equivalent to the following connection between the elements of the matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$*:*

$$a_{nk} := \prod_{j=k}^{\infty} b_{nj} \ \text{ or } \ b_{nk} := \frac{a_{nk}}{a_{n,k+1}} \ \text{ for all } k \in \mathbb{N}. \tag{19}$$

At this stage, we can give a short analysis on the multiplicative dual summability methods. One can see that v_n reduces to u_n, as follows:

$$v_n = \prod_{k=1}^{\infty} {b_{nk}}^{\ln y_k} = \prod_{j=1}^{\infty} {b_{nk}}^{\ln(\prod_{j=1}^{k} x_j)} = \prod_{j=1}^{\infty} \prod_{k=j}^{\infty} {b_{nk}}^{\ln x_j} = u_n.$$

However, the order of production can not be reversed and thus the methods A and B are not necessarily equivalent.

The partial products of the productions on the right hand side of (17) and (18) are connected by the relationship

$$\prod_{k=1}^{m} {a_{nk}}^{\ln x_k} = \prod_{k=1}^{m-1} \left(\frac{a_{nk}}{a_{n,k+1}} \right)^{\ln y_k} \cdot {a_{nm}}^{\ln y_m} \quad \text{for all } \ m, n \in \mathbb{N}. \tag{20}$$

Therefore, if one of the products on the right hand side of (17) and (18) converges for a given $n \in \mathbb{N}$, then the other side converges if and only if

$$* - \lim_{m \to \infty} {a_{nm}}^{\ln y_m} = z_n \tag{21}$$

for every fixed $n \in \mathbb{N}$. If (21) holds then from (20) we have by letting $m \to \infty$ that

$$u_n = v_n \cdot z_n \quad \text{for all } \ n \in \mathbb{N}. \tag{22}$$

Consequently, if y_n has a $*$-limit by one of the methods A and B then it is multiplicatively limitable by the other one if and only if (21) holds and

$$* - \lim_{n\to\infty} z_n = \alpha. \tag{23}$$

Hence the limits of (u_n) and (v_n) is different from α. Therefore the A- and B-limits of any sequence is multiplicatively limitable by one of them compromise if and only if B multiplicative limitability infers that (23) holds with $\alpha = 1$. A counterpart argument holds with A and B interchanged. It follows by the currency of (23) with $\alpha \neq 1$ that the methods A and B are contradictory, and vice versa.

5. Conclusion

As an alternative of classical Newton-Leibnitz calculus, multiplicative calculus gives more convenient results in some specific problems. We try to step forward to matrix transformations in terms of multiplicative calculus and give some results between certain sequence spaces. This footstep is like milestone in order to understand infinite matrix domains from the point of non-Newtonian calculus.

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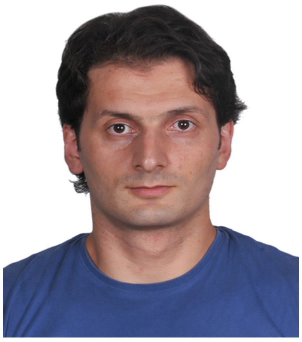

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