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ON A HOMOTOPY BASED METHOD FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

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Abstract. A new iterative method is proposed to solve systems of linear algebraic equations, $Ax = b$. The method is based on the concept of homotopy. Similar works have been done in this direction and some special cases of convergence have been addressed. After reviewing the literature, here we study the method in a more general framework and present more cases of convergence. A comparative study of the method from computational cost viewpoint and speed of convergence shows that the new presented method competes well with classic iterative techniques. Also using a convergence control parameter (CCP) of prof. Shi Jun Liao an stationary modification of the method is presented. This modification of the method clarifies two issues, one that Liao's CCP may fail to be efficient in a linear equation. Also there are cases where this control parameter can extend the convergence cases of the presented homotopy method.

Keywords: system of linear equations, homotopy analysis method, iterative methods, convergence control parameter, homotopy perturbation method.

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1. Introduction

The system of linear equations $Ax = b$, is probably the most important problem which has led matrix analysis and its variants to its current status. Most phenomena in various fields culminate in the solution of a system of linear equations. Numerical solution of a partial differential equation, control problems of electrical engineering, analysis of stochastic processes and optimization problems are notable examples which could be counted here. Especially it is the spring board for most of modern concepts in advanced matrix analysis.

The solution strategies for the system $Ax = b$, could be classified mainly as direct methods and iterative ones. Direct methods are mostly classic techniques which are only applicable when relatively small size problems are under consideration. But for large systems which are most appealing ones in applied sciences, an iterative method would be of more interest. Reference [7] would be a good overview of these type of methods. In this work we discuss a similar scheme to iterative methods based on the homotopy idea, which has largely been used to handle nonlinear functional equations [4, 5, 2]. Recently the idea has been used to solve systems of linear equations [3, 6, 9]. The very first work originates back to Dr. Keramati [3] who only studied a very special case, he used the identity matrix as a preconditioner. In [9] Yusufoglu also studies very special cases and doesn't address a complete overview of the method. Liu [6] studied the method in a more general framework. He gave convergence results for special matrices with regular splitting.

Although Liu's work is a good contribution to the subject but following gaps are non-avoidable which are not addressed in any of the mentioned papers:

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- In Liu's work, role of the convergence control parameter (CCP), which he denotes by q and the auxiliary matrix is not practically established.
- A comparison of the method with other methods is needed to ensure efficiency of the new presented one. For example the operation count and speed of convergence are issues which should be compared.
- In the so far published papers the cases where a homotopy method converges is restricted to special families and no comments have been presented about possible convergent cases.

With the aim of discussing above mentioned gaps, this paper is organized as follows:
Section 2 gives the basic idea of the method. The convergence criteria are studied in section 3 and section 4 focuses on special convergent cases. In Section 5, the method is compared with classic iterative techniques and in section 6 we present an stationary modification adding a simple convergence-control parameter. Also further discussion and open problems have been stated at the end of paper.

2. The basic idea

Consider system of linear equations, $Ax = b$, where A is an invertible matrix of size n and $x, b \in \mathbb{R}^n$. Invertibility of A implies that the system has a unique solution. Our aim is to obtain this solution by using the homotopy idea. Suppose $v^{(\circ)}$ is an initial guess of the solution, using an *auxiliary matrix*[1] M (probably related to A), we construct a convex combination as follows:

$$\mathcal{A}(x; q) = (1-q)M(x - v^{(\circ)}) + q(Ax - b). \tag{1}$$

This is a homotopy between $v^{(\circ)}$ and x (the exact solution). Enforcing the homotopy to be zero we obtain the following homotopy equation

$$(1-q)M(x - v^{(\circ)}) + q(Ax - b) = 0. \tag{2}$$

Thus for every $q \in [0, 1]$, we have a system of linear equations whose solution is dependent upon q. When $q = 0$ the system is equivalent to $M(x - v^{(\circ)}) = 0$, if M is an invertible matrix, this system leads to the obvious solution $v^{(\circ)}$. In the case where $q = 1$ system (2) will be equivalent to $Ax = b$, i.e. the original system under study.

Now, if we accept that the solution to the homotopy equation could be represented as an infinite series in the form

$$x = x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + x^{(3)}q^3 + \cdots, \tag{3}$$

then substituting series (3) in (2) we would have

$$M(x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + \cdots - v^{(\circ)}) + q\{(A - M)(x^{(\circ)} + x^{(1)}q + \cdots) + Mv^{(0)} - b\} = 0. \tag{4}$$

The above equation holds for every $q \in [0, 1]$, so the left hand side expressions must be independent of q, therefore we have

$$\begin{aligned} &M(x^{(\circ)} - v^{(\circ)}) = 0, \\ &Mx^{(1)} = b - Ax^{(\circ)}, \\ &Mx^{(n)} = (M - A)x^{(n-1)}, \ n \geq 2. \end{aligned} \tag{5}$$

[1] same as "preconditioner matrix ", but here we preferred to use the term "auxiliary" as it is a commonly used one in a homotopy method framework.

If we choose "M" to be an invertible matrix, each of the above systems have a unique solution, these solutions, from a theoretical point of view, are:

$$\begin{aligned} x^{(\circ)} &= v^{(\circ)}, \\ x^{(1)} &= M^{-1}(b - Ax^{(\circ)}), \\ x^{(n)} &= (I - M^{-1}A)x^{(n-1)}, \ n \geq 2. \end{aligned} \tag{6}$$

By this recursive relation, every $x^{(n)}$ could be expressed in terms of the initial guess,

$$\begin{aligned} x^{(\circ)} &= v^{(\circ)}, \\ x^{(n)} &= (I - M^{-1}A)^{(n-1)}M^{-1}(b - Av^{(\circ)}), \ n \geq 1. \end{aligned} \tag{7}$$

However computing the matrix M^{-1} is not recommended here, in general this is an ill-conditioned (and costly) problem. For obtaining solutions (7) we try to solve equations (5), which are equivalent to our system. If M is a matrix which has an easy-computable inverse or it is triangular, then solving equations (5) may reduce to just doing a backward or forward substitution or even much simpler ones.
From solutions (7), we construct the series solution

$$\begin{aligned} x &= x^{(\circ)} + x^{(1)} + x^{(2)} + x^{(3)} + \cdots = \\ &= v^{(\circ)} + \sum_{n=1}^{\infty} (I - M^{-1}A)^{(n-1)}M^{-1}(b - Av^{(\circ)}). \end{aligned} \tag{8}$$

We see that the matrix $I - M^{-1}A$ plays a key role in the homotopy method. Whenever we are concerned with the convergence problem, we will repeatedly refer to this matrix.

3. Discussion on convergence

The approximations (8), obtained by the homotopy method, wouldn't be worthwhile unless we make sure that the series is convergent. In this section we study the conditions under which the solutions (8) is convergent for every choice of initial guess.
From Neumann's series Theorem we have the following corollary:

Corollary 3.1. *If we find a matrix norm such that $\|I - M^{-1}A\| < 1$, then the series $\sum_{n=0}^{\infty}(I - M^{-1}A)^n$ will converge to $A^{-1}M$.*

So we come to the following theorem:

Theorem 3.1. *If the auxiliary matrix, M, is chosen such that the spectral radius of $I - M^{-1}A$ is less than one, i.e. $\rho(I - M^{-1}A) < 1$, then the solution series obtained by homotopy method, for solving system $Ax = b$, will converge to the exact solution.*

Proof. If $\rho(I - M^{-1}A) < 1$, then there exists a matrix norm for which $\|I - M^{-1}A\| < 1$, so according to corollary 3.1:

$$\sum_{n=0}^{\infty}(I - M^{-1}A)^n = A^{-1}M.$$

Thus for solution x, obtained from (8), we have

$$\begin{aligned} Ax &= A(v^{(\circ)} + \sum_{n=1}^{\infty} (I - M^{-1}A)^{(n-1)}M^{-1}(b - Av^{(\circ)}) = \\ &= Av^{(\circ)} + A(A^{-1}M)M^{-1}(b - Av^{(\circ)}) = \\ &= b. \end{aligned}$$

□

Note. *The convergence rate of the solution series is dependent upon $\rho(I - M^{-1}A)$, the smaller the spectral radius the faster the convergence rate.*

3.1. **Requirements for the auxiliary matrix.** The auxiliary matrix M, plays a crucial role in the presented homotopy method. In order to make the method more efficient and easy to use, we require M to satisfy following conditions:

(1) Systems (5) must be easy-to-solve,
(2) $\rho(I - M^{-1}A) < 1$, i.e. insures convergence of the method,
(3) $\rho(I - M^{-1}A) \ll 1$, i.e. insures rapid convergence of the solution series.

If one chooses M to be a diagonal, triangular or even tridiagonal matrix, then the first condition is fulfilled, but in order to make the other requirements satisfied we need to calculate spectral radius of $I - M^{-1}A$, which is not an economical advice. In the next section we will study some special cases of matrices for which the second condition is automatically fulfilled, but first let's present a special case.

3.2. **Identity matrix as the auxiliary matrix.** This is the case which is studied by Dr. Keramati in [3]. With choosing M to be an identity matrix, the solution series, according to (8), would be of the form:

$$\begin{aligned} x &= x^{(\circ)} + x^{(1)} + x^{(2)} + x^{(3)} + \cdots = \\ &= v^{(\circ)} + \sum_{n=1}^{\infty} (I - A)^{(n-1)}(b - Av^{(\circ)}). \end{aligned} \tag{9}$$

The convergence criteria of this special case is: $\rho(I - A) < 1$. Choosing $v^{(\circ)}$ to be the zero vector, the case which is studied by Dr. Keramati, if the coefficient matrix is strictly diagonally dominant then by rewriting the system in a suitable form we could reach convergence conditions.

Theorem 3.2. [3] *If $A = [a_{ij}]$ is a strictly diagonally dominant and we define $B = [b_{ij}]$ to be*

$$b_{ij} = \begin{cases} 1 & , \ i = j \\ \frac{a_{ij}}{a_{ii}} & , \ i \neq j. \end{cases}$$

Then $\|I - B\|_\infty < 1$.

Proof. see [3]. □

The case proposed by Dr. Keramati is very simple and easy to use, but it lacks the great flexibility we have in choosing the M part. Even for the same examples, other choices of M may result in better approximations. As an example, we can refer to the first example in [3], if we solve that system by putting M as the lower triangular part of A, then the 5th order approximation would be the exact solution, upon rounding up to 4 digits, which is more accurate than the solution presented in [3].

4. Some special cases of convergence

In this section we mention some special cases where the convergence condition for the homotopy method is satisfied. These cases are:

(1) Diagonally dominant matrices,
(2) Hermitian positive definite matrices,
(3) M-matrices,
(4) Nonnegative matrices with regular splitting

From the above list the 4th case has been discussed in [6], we focus on other cases.

4.1. **Diagonally dominant matrices.** There are two classes of diagonally dominant matrices where the homotopy method results in convergent series.
First class: Strictly row diagonally dominant matrices (SRDD)
Second class: Irreducibly row diagonally dominant matrices (IRDD)

For definitions of the mentioned families please refer to [7].

Theorem 4.1. *If A is SRDD and M is the lower (or upper) triangular (with diagonal) part of A, then the homotopy method for solving the system $Ax = b$ is convergent.*

Proof. It suffices to prove $\rho(I - M^{-1}A) < 1$. We do the proof for the lower triangular case, the other one is similar. Let λ be an arbitrary eigenvalue of $I - M^{-1}A$ and x be the corresponding eigenvector, without loss of generality we can assume that $\|x\|_\infty = 1$, therefore we have

$$(I - M^{-1}A)x = \lambda x \Longrightarrow Mx - Ax = \lambda Mx.$$

Since M is the lower triangular part of A, for every i, $1 \le i \le n$, we have:

$$-\sum_{j=i+1}^{n} a_{ij}x_j = \lambda \sum_{j=1}^{i} a_{ij}x_j \Longrightarrow$$

$$\Longrightarrow \lambda a_{ii}x_i = -\lambda \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j,$$

now we can choose an index k such that

$$|x_k| \ge |x_j|, 1 \le j \le n,$$

therefore

$$|\lambda||a_{kk}| \le |\lambda| \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}|,$$

then diagonally dominance of A results in:

$$|\lambda| \le \frac{\sum\limits_{j=k+1}^{n} |a_{kj}|}{|a_{kk}| - \sum\limits_{j=1}^{k-1} |a_{kj}|} < 1,$$

since λ was an arbitrary eigenvalue, this completes the proof. □

Here may arise one question: *Are the triangular and diagonal parts of the coefficient matrix, the only matrices that can be chosen as auxiliary matrix in homotopy method?* The answer would be: *No.* Actually it is possible to choose other auxiliary matrices that not even keep the simplicity of systems (5), but also have faster convergence rates compared to other cases.

We now turn to IRDD matrices after mentioning a lemma.

Lemma 4.1. [7] *IRDD matrices are nonsingular.*

Proof. see [7] □

Theorem 4.2. *If the coefficient matrix of the system $Ax = b$ is IRDD, then the homotopy method is convergent.*

Proof. If we choose the auxiliary matrix M to be the lower triangular (with diagonal) part of A and let $N = M - A$, then just similar to the proof of the SRDD case (Theorem 4.1), we can conclude that $\rho(I - M^{-1}A) \leq 1$. But we must further prove that this inequality is strict. For this aim we use a proof by contradiction, assume that λ is an eigenvector of $I - M^{-1}A = M^{-1}N$ with $|\lambda| = 1$. Then the matrix $M^{-1}N - \lambda I$ would be singular and as a result $A' = N - \lambda M$ would also be singular. But $N - \lambda M$ is an IRDD matrix and this would contradict Lemma 4.2. $\square$

4.2. **Hermitian positive definite matrices.** In the special case of Hermitian positive definite (HPD) matrices, the following theorem states that it is possible to ensure convergence of the homotopy method in some cases.

Theorem 4.3. [8] *Let A be a HPD matrix and suppose that there exist a splitting $A = M - N$ where $M + N^*$ is also positive definite, then $\rho(I - M^{-1}A) < 1$.*

Proof. See [8] $\square$

So if we can split a HPD matrix A, in a way which satisfies the conditions stated in the above theorem, then the homotopy method, for solving $Ax = b$, would converge. Of course we should keep in mind that this matrix splitting must result in easily solvable systems according to (5).

Although it seems that the splitting mentioned in theorem 4.4, is a bit hard to find, it may be possible to find several auxiliary matrices satisfying these properties. For example, when the coefficient matrix is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 26 \end{pmatrix}$, then matrices $M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 7 & 26 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 26 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 26 \end{pmatrix}$ satisfy our requirements.

4.3. M**-matrices.** M-matrices are a family of square matrices which arise naturally in modeling and studying large class of biological processes such as population growth, they also play a key role in analyzing *Markov Chains* and studying *queuing theory.*

Definition 4.1. *A square matrix $A = [a_{ij}]$ of size n is called an M-matrix whenever it satisfies following conditions:*

(1) *Diagonal entries are positive (i.e. $a_{ii} > 0$ for $i = 1...n$),*
(2) *non-diagonal entries are non-positive (i.e. $a_{ij} \leq 0$ for $i, j = 1...n$ and $i \neq j$),*
(3) *if D is diagonal of A, then $\rho(I - D^{-1}A) < 1$.*

In the case where the coefficient matrix is an M-matrix we can assure convergence of the homotopy method. By the definition of M-matrices, it suffices to choose the auxiliary matrix M to be the diagonal part of A, i.e $M = D$.

Now we come to the last comment on convergence problem, actually the cases we studied here are not the only cases where the homotopy method converges, there are examples of matrices which doesn't fit in any of the considered families but still the homotopy method is convergent on them, for example we can mention Example 5.3, which will be studied in the next section. So the question: "*For which classes of matrices, the homotopy method is convergent?*" is still an *open* problem.

Multiplications	
a forward or backward substitution	$m\frac{n(n+1)}{2}$
matrix-vector multiplication	mn^2
total	$m(\frac{3n^2+n}{2})$
Summations	
a forward or backward substitution	$m\frac{n(n-1)}{2}$
matrix-vector multiplication	$m(n^2-n)$
summations for $M-A$	n^2
summation in computing $x^{(1)}$	n^2
summation for constructing app_m	mn
total	$m(\frac{3n^2-n}{2})+2n^2$

Table 1. Computational cost of the homotopy method

5. Comparison with classic iterative methods

In this section, we first compute the computational cost of the homotopy method applied to a linear system of equations, $Ax=b$. We will see that the amount of algebraic operations needed in homotopy method have the same order as classic iteration methods, thus comparable to them. Then it is verified that in cases where classic methods like Jacobi or Gauss-Seidel are convergent, so is our homotopy method. Moreover we will show that there are cases where iterative techniques may diverge while homotopy method successfully solves the system.

5.1. **Computational cost.** According to equations (5), for obtaining each $x^{(i)}$ $(i \geq 2)$, we need to do one vector-matrix multiplication ($(M-A)x^{(i-1)}$) and (at most) a backward or forward substitution. However, other than these operations, one matrix substraction $(M-A)$ must be done in the very beginning which costs only n^2 summations. Moreover for $x^{(1)}$ according to $Mx^{(1)} = b - Ax^{(\circ)}$, we need n^2 multiplications and n^2 summations as well as one backward or forward substitution. The computational cost for obtaining an approximation of order m $(m \geq 2)$, i.e. $app_m = x^{(\circ)} + \cdots + x^{(m)}$, is presented in Table 1.

5.2. **Superiority of the homotopy technique.** Most of the iterative methods are based on a matrix splitting like $A = B - C$. Always we have an iteration matrix $G = B^{-1}C$ which is used for convergence study of the method. The necessary and sufficient condition for convergence of the iterative scheme is $\rho(G) < 1$. For example in Jacobi and Gauss-Seidel techniques the coefficient matrix splitting is of the form $A = D - L - U$, where D is the diagonal part of A, having nonzero entries, $-L$ is the strict lower triangular part and $-U$ is the strict upper diagonal part of A. In Jacobi we get advantage of the iteration matrix $G_j = D^{-1}(L+U)$ and the iteration matrix of Gauss-Seidel is $G_g = (D-L)^{-1}U$. So the convergence criteria would be $\rho(G_j) < 1$ and $\rho(G_g) < 1$. The following theorems show the superiority of the homotopy method.

Theorem 5.1. *If the Jacobi method, for solving the system $Ax = b$, converges then so is the homotopy method.*

Proof. Suppose the Jacobi method is convergent, so by the terminology presented earlier we have $\rho(D^{-1}(L+U)) < 1$. Now, in homotopy method, we choose the diagonal matrix D as the auxiliary matrix M, thus according to matrix splitting $A = D - L - U$ we would have:

$$I - M^{-1}A = I - D^{-1}A = I - D^{-1}(D - L - U) = D^{-1}(L + U),$$

therefore the convergence criteria for the homotopy method is the same as the Jacobi method. $\square$

A similar theorem relates the Gauss-Seidel method to the homotopy method.

Theorem 5.2. *If the Gauss-Seidel method, for the solution of the system $Ax = b$, is convergent then the homotopy method is also convergent.*

Similar theorems and corollaries could be stated for other iterative methods, like Richardson's method (which would be the same as Dr. Keramat's case [3]) and SOR method. In general if an iteration method uses the matrix splitting $A = B - C$ (thus the iteration matrix: $G = B^{-1}C$), then in applying the homotopy method, one can use M to be the same as B. With this choice of the auxiliary matrix the convergence criteria of the homotopy method would be the same as the criteria in the iterative method.

The superiority of the homotopy method arises in the freedom we have in choosing the auxiliary matrix M, with this freedom the homotopy method is able to give convergent series even in cases where an iterative technique diverges.

Example 5.3. We consider the system of linear equations:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix},$$

the exact solution of this system is $x = (3, -2, 1)^t$. The Jacobi and standard Gauss-Seidel methods are divergent but if we choose M to be the upper triangular part of A including its diagonal then the homotopy method converges. The convergence rate is so high that the second order approximation gives the exact solution.

6. Adding a convergence control parameter

We can add a convergence control parameter, $\alpha \in \mathbb{R}$, to the homotopy equation (1), just like what is proposed by Liao in HAM [4], and we would gain new results which ensures convergence in some cases.

New homotopy equation would be:

$$(1 - q)M(x - v^{(\circ)}) + \alpha q(Ax - b) = 0,$$

thus the systems (6) would change to

$$\begin{aligned} &M(x^{(\circ)} - v^{(\circ)}) = 0, \\ &Mx^{(1)} = \alpha(b - Ax^{(\circ)}), \\ &Mx^{(n)} = (M - \alpha A)x^{(n-1)}, \ n \geq 2. \end{aligned} \tag{10}$$

The convergence condition in this *modified homotopy method* would change to $\rho(I - \alpha M^{-1}A) < 1$. So the main question we are concerned with, can be stated as follows:

In solving a system of equations $Ax = b$, using modified homotopy method, can we ensure the convergence by suitably choosing the auxiliary matrix M and the auxiliary parameter α.

If the answer to this question is "*Yes*", then for an arbitrary matrix B we must be able to find a scalar α such that $\rho(I - \alpha B) < 1$. Here we show, through an example, that the general answer to the above mentioned question is "*No*".

Example 6.1. If $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we have

$$I - \alpha B = \begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1+\alpha & 0 \\ 0 & 0 & 1-2\alpha \end{pmatrix}.$$

So $\rho(I - \alpha B) = max\{|1-\alpha|, |1+\alpha|, |1-2\alpha|\} > 1$.

Remark. When homotopy analysis method of prof. Liao is applied to a nonlinear equation, is is verified by numerous examples that the convergence control parameter can control and extend the convergence region of the solution series. The above example shows that the parameter may not be able to affect the solution series.

But there are cases where αs could be found to fulfill our requirements. In the forthcoming paragraphs we will show, when $M = I$ is chosen, there do exist such αs in some special cases (e.g. positive definite matrices). Our reasoning is based on a theorem in [1], which we will also recall here.

Suppose the coefficient matrix A is a nonsingular matrix of size n with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Let's denote its Shur decomposition with $A = Q^*UQ$, so Q is a unitary matrix and U is an upper triangular matrix whose diagonal elements are eigenvalues of A, i.e. $u_{jj} = \lambda_j$.
Now we define $B = I - \alpha U$, where α is an arbitrary scalar, this matrix is *similar* to $I - \alpha A$:

$$I - \alpha A = Q^*Q - \alpha(Q^*UQ) = Q^*(I - \alpha U)Q = Q^*BQ.$$

Therefore $I - \alpha U$ and $I - \alpha A$ have the same spectrum, hence for satisfying $\rho(I - \alpha A) < 1$ (the convergence condition of the modified homotopy method) it suffices to have $\rho(I - \alpha U) < 1$, which may be a simpler condition to fulfill. However $I - \alpha U$ is an upper triangular matrix, so its eigenvalues are exactly its diagonal elements. Thus for satisfying $\rho(I - \alpha U) < 1$ it suffices to have $|1 - \alpha\lambda_j| < 1$ for $j = 1, ..., n$. Such an α depends upon λ_j, so for every λ_j we try to find an α_j which satisfies $|1 - \alpha_j\lambda_j| < 1$. Once all α_js are determined we can choose $\alpha = \min\limits_{1\leq j\leq n} \alpha_j$ (or *max* depending on the occasion).
Let's denote real and imaginary parts of the eigenvalue λ_j by $Re(\lambda_j)$ and $Im(\lambda_j)$, then we have

$$\begin{aligned} & |1 - \alpha_j Re(\lambda_j) - i\alpha_j Im(\lambda_j)| < 1 \\ \Longrightarrow\ & 1 + \alpha_j^2 Re^2(\lambda_j) - 2\alpha_j Re(\lambda_j) + \alpha_j^2 Im^2(\lambda_j) < 1 \\ \Longrightarrow\ & \alpha_j^2 |\lambda_j|^2 - 2\alpha_j Re(\lambda_j) < 0. \end{aligned} \tag{11}$$

According to different values for $Re(\lambda_j)$ the following cases could be considered:

(1) There exist at least one j with $Re(\lambda_j) = 0$, i.e. we have one eigenvalue which is pure imaginary. Then there would be no α to satisfy (11). In this case the modified homotopy method with $M = I$ fails to ensure the convergence.
(2) $Re(\lambda_j) > 0$ for all $j = 1, \cdots, n$, i.e. the real parts of all eigenvalues are positive. Therefore if we choose α_j in the interval $0 < \alpha_j < \dfrac{2Re(\lambda_j)}{|\lambda_j|^2}$, condition (5.1.12) is fulfilled, then by choosing $\alpha = \min\limits_{1\leq j\leq n} \alpha_j$ the condition $\rho(I - \alpha_j U) < 1$ is satisfied. This is exactly the convergence condition of the modified homotopy method.
(3) $Re(\lambda_j) < 0$ for all $j = 1, \cdots, n$, i.e. the real parts of all eigenvalues are negative. Therefore condition (12) is satisfied whenever α_j is chosen from the interval $\dfrac{2Re(\lambda_j)}{|\lambda_j|^2} <$

$\alpha_j < 0$. Then by putting $\alpha = \max\limits_{1\leq j\leq n} \alpha_j$ the condition $\rho(I - \alpha_j U) < 1$ is satisfied. This is the α that ensures convergence condition of the modified homotopy method.

We conclude that whenever the coefficient matrix is a *definite* one then there exists an α which satisfies the convergence condition of the modified homotopy method.

Definition 6.1. *A square matrix B is said to be definite if the real part of all eigenvalues of B are positive (negative).*

Theorem 6.1. [1] *If A is a definite matrix then there exist a scalar α such that $\rho(I - \alpha A) < 1$.*

Although the above theorem ensures the ability of the modified homotopy method for solving the system of linear equations $Ax = b$, in the case where A is a definite matrix, but we still have problem on finding α. The above presented reasoning needs to determine all eigenvalues of the coefficient matrix, which is not recommended. However there are cases where we can choose, in advance, suitable αs according to the coefficient matrix at hand. Here we show that there are simple choices for α when the coefficient matrix is symmetric positive definite or SRDD and IRDD.

Theorem 6.2. *In the system of linear equations $Ax = b$, if A is a symmetric positive definite matrix, then by choosing $M = I$ and $0 < \alpha < \dfrac{2}{\|A\|_\infty}$, the modified homotopy method converges.*

Proof. For a symmetric positive definite matrix all eigenvalues are real and positive so the condition (11) is equivalent to the inequality $0 < \alpha_j < \frac{2}{\|A\|_\infty}$. Every matrix norm is an upper bound for spectrum of matrix, with this fact we have $|\lambda_j| \leq \rho(A) \leq \|A\|_\infty$ which results in $\frac{1}{\|A\|_\infty} \leq \frac{1}{\lambda_j}$. Therefore if α satisfies $0 < \alpha < \frac{2}{\|A\|_\infty}$ then the convergence criteria of the modified homotopy method is fulfilled. $\square$

Remark. Analytically speaking, every α which satisfies $0 < \alpha < \frac{2}{\|A\|_\infty}$, results in a convergent series solution, but numerically we can't let α to take small values. As α approaches zero the matrix $I - \alpha A$ tends to the identity matrix, thus the spectral radius gets closer to 1. So this would greatly reduce the speed of convergence.

Theorem 6.3. *If A is an IRDD or SRDD matrix with positive diagonal elements, then by choosing $M = I$ and $0 < \alpha < \min\limits_j \dfrac{2}{a_{jj}}$, the modified homotopy method converges.*

Note that if the matrix is SRDD or IRDD then the system would be equivalent with the one with positive diagonal elements.

Proof. With positivity of diagonal elements we conclude that $\forall j Re\lambda_j > 0$. So A would be a definite matrix, thus there are αs which satisfies convergent condition. According to previous discussions such an α must satisfy $0 < \alpha < \min\limits_{1\leq j\leq n} \dfrac{2Re(\lambda_j)}{|\lambda_j|^2}$. Now referring to Gerschgorin's theorem, we have for every j $\frac{1}{a_{jj}} < \frac{2Re(\lambda_j)}{|\lambda_j|^2}$, so if we choose $\alpha < \frac{1}{a_{jj}}$, for every j, then our convergence condition is satisfied. $\square$

7. Further discussion

The presented homotopy based method competes well with classic iterative methods like Richardson, Jacobi and Gauss-Seidel. However there is a gap here to compare this new idea with more powerful methods like Krylov subspace methods. But before commenting on this issue one should answer two open questions, one is "for which classes of matrices the homotopy method is convergent?" which is stated in section 4. The second question concerns the modified method which uses the CCP , which states "For which cases the CCP can ensure convergence of the homotopy method?".

These are the questions that could be considered as research topics for future studies.

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