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SOFT IDEALS OF SOFT WS-ALGEBRA

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Abstract. The notions of prime soft ideals, irreducible soft ideals and maximal soft ideals, prime soft idealistic over WS-algebras, irreducible soft idealistic over WS-algebras and maximal soft idealistic over WS-algebras are introduced, and several examples are given to illustrate. Relations between prime soft idealistic, irreducible soft idealistic and maximal soft idealistic over WS-algebras are investigated.

Keywords: maximal soft ideal, prime soft ideal, irreducible soft ideal.

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1. Introduction

Dealing with uncertainties is a main problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with classical methods. Because, these classical methods have their inherent difficulties. To overcome these kinds of difficulties, Molodtsov [11] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Molodtsov [11], Jun [3], Jun and Park [4] pointed out several directions for the applications of soft sets. Then, Aktaş and Çağman [1] defined the notion of soft groups. Some researcher Maji at al [9] studied several operations on the theory of soft sets. Finally, Jun, Park, Öztürk [12] applied the notion of soft sets by Molodtsov to the theory of WS-algebras. The notion of soft WS-algebras, subalgebras and soft ideal (deductive systems) have introduced, and their basic properties have derived. At the present, works on the soft set theory are progressing rapidly. Some researchers described the application of soft set theory to a decision making problem. Ideal theory plays an important role in general algebra. Ideal theory in subtraction algebras is defined and studied in [6]. Soft prime ideals, soft ideals and soft maximal ideals in subtraction algebras are studied in [5]. Every subtraction algebra is WS-algebra. The main aim of this paper is to study several ideal structure of ideals of WS-algebras. So we may summarize the contents of the paper. Chapter 2 deals with preliminaries that will be used in the paper. In Chapter 3 soft WS-algebras and union and intersections of soft WS-subalgebras are considered. In Chapter 4 we give the characterization of prime ideals of WS-algebras and every maximal ideal is irreducible and we give an example its converse need not be true. Also we define prime soft ideals, irreducible soft ideals and maximal soft ideals of soft WS-algebras and discussed the algebraic properties of these ideals of soft

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WS-algebras. In chapter 5 we study irreducible idealistic soft WS-algebras. Finally, prime soft idealistic WS-algebras and maximal soft idealistic WS-algebras are defined and derived their basic properties in chapter 6.

2. Preliminaries

2.1. **S-algebras and WS-algebras.** In this section we give some basic definitions and notions to be used in our work for an easy reference by readers.

We start with a well known definition, [14], [5], [8].

Definition 2.1. Let X be a set with a binary operation "$-$". Then $(X;-)$ is called *a subtraction algebra(shortly, S-algebra)* if it satisfies the following conditions:

SA-1 $(\forall x, y, z \in X)$, $x - (y - x) = x$.
SA-2 $(\forall x, y, z \in X)$, $x - (x - y) = y - (y - x)$.
SA-3 $(\forall x \in X)$, $(x - y) - z = (x - z) - y$.

Let X be a S-algebra. By SA-3 condition we omit parentheses in expressions of the form $(x - y) - z$. For any $x \in X$ we define the element 0 of X by $x - x = 0$. Then 0 does not depend x. We summarize some equalities that S-algebras satisfy and we need and use in this work.

Lemma 2.1. [6] Let $(X;-)$ be an S-algebra. Then followings hold for $x, y, z \in X$

S-1. $(x - y) - y = x - y$.
S-2. $x - 0 = x$ and $0 - x = 0$.
S-3. $(x - y) - x = 0$.
S-4. $((x - y) - (x - z)) - (z - y) = 0$.
S-5. $(x - (x - y)) - y = 0$.
S-6. $x - y = 0$ and $y - x = 0$ imply $x = y$.
S-7. $(x - y) - z = (x - z) - (y - z)$.

K. J. Lee, Y.B. Jun and Y.H. Kim [6] introduced to the notion of weak subtraction algebras.

Definition 2.2. Let X be a non empty set with a binary operation $-$ and zero element $0 \in X$. Then $(X;-,0)$ is called *a weak subtraction algebra(shortly, WS-algebra)* if it satisfies the following conditions $\forall x, y, z \in X$:

WSA-1 $(x - y) - z = (x - z) - y$.
WSA-2 $(x - y) - z = (x - z) - (y - z)$.
WSA-3 $x - 0 = x$.
WSA-4 $x - x = 0$.

Lemma 2.2. Let $(X;-,0)$ be a WS-algebra. Then the following conditions hold in X $\forall x, y \in X$:

(1) $0 - x = 0$;
(2) $(x - y) - x = 0$;
(3) $x - y = 0 \Rightarrow (x - z) - (y - z) = 0$.

Proof. For proof see [7] □

Note that every S-algebra is a WS-algebra. There exist WS-algebras that are not S-algebra. Several examples are given in [8]. For the reader's convenience we describe one of them.

Example 2.1. Let $X = \{0, a, b, c\}$ be a set with the following " $-$ " operation table. Then $(X;0)$ is a WS-algebra.

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then $b - (b - c) = b$ and $c - (c - b) = c$ holds in X. But $b \neq c$. **SA-2** condition of subtraction algebra is not satisfied. Therefore $(X; 0)$ is a WS-algebra but not S-algebra.

Let $(X; 0)$ be a WS-algebra. A nonempty subset A of X is called *a subalgebra of* X if for all $x, y \in A$, $x - y \in A$. A nonempty subset A of X is called *an ideal* in X if it satisfies $0 \in A$ and for any $x \in X$ and $y \in A$ from $x - y \in A$ it follows that $x \in A$.

Lemma 2.3. Let $(X; 0)$ be a WS-algebra and I an ideal of X. For any $x, y \in I$ we have $x - y \in I$.

Proof. Let $x, y \in I$. Then $(x - y) - x \overset{WSA-1}{=} (x - x) - y \overset{WSA-4}{=} 0 - y \overset{Lemma 2.2}{=} 0 \in I$. Hence $x - y \in I$ since $x \in I$ and I is an ideal.

□

The operation " $-$ " on a WS-algebra $(X; 0)$ determines a relation on X: For any $x, y \in X$

$$x \leq y \quad \text{if and only if} \quad x - y = 0. \tag{1}$$

Note that the relation (1) may not be an order relation on a WS-algebra (see Example 3.2 in [8]). We will consider following condition (2) to make the relation (1) be a order relation. For a WS-algebra $(X; 0)$ we let for all $x,y \in X$,

$$(x - (x - y) = y - (y - x)). \tag{2}$$

In [8] it is proved that the relation (1) on any WS-algebra $(X; 0)$ having condition (2) defines an order relation on X.

In any WS-algebra X, for any $x, y \in X$ we let the meet relation

$$x \wedge y = x - (x - y). \tag{3}$$

Definition 2.3. Let $(X; 0)$ be WS-algebra. If $(X; 0)$ satisfies $x \wedge y = y \wedge x$ for all $x, y \in X$, it is called *commutative WS-algebra*, equivalently $(X; 0)$ satisfies the condition (2).

Note that subtraction algebras are commutative by condition **SA-2**.

Lemma 2.4. Let $(X; 0)$ be a WS-algebra satisfying $a - (b - c) = b - (a - c)$for any $a, b, c \in X$. Then X consists of only zero element $X = \{0\}$.

Proof. Let $x \in X$. Then $0 \overset{WSA-4}{=} 0 - 0 \overset{WSA-2}{=} (x - x) - (x - x) \overset{hypothesis}{=} x - ((x - x) - x) \overset{WSA-4}{=} x - (0 - x) \overset{Lemma 2.2}{=} x - 0 \overset{WSA-3}{=} x$ □

Lemma 2.5. Let $(X; 0)$ be a WS-algebra with the meet relation " $\wedge$ " defined by (2.3). Then $x \wedge y \leq x$ and $x \wedge y \leq y$.

Proof. To prove $x \wedge y \leq x$ we need to show that $(x \wedge y) - x = 0$. So $(x \wedge y) - x = (x - (x - y)) - x \overset{WSA-2}{=} (x - x) - ((x - y) - x) = 0 - ((x - y) - x) \overset{Lemma 2.2}{=} 0$. Similarly we prove $(x \wedge y) - y = 0$. So $(x \wedge y) - y = (x - (x - y)) - y \overset{WSA-2}{=} (x - y) - ((x - y) - y) \overset{WSA-2}{=} (x - y) - ((x - y) - (y - y)) \overset{WSA-4}{=} (x - y) - ((x - y) - 0) \overset{WSA-3}{=} (x - y) - (x - y) \overset{WSA-4}{=} 0$. □

Recall that *the infimum* of a subset S, denoted by $\inf S$, of an ordered set $(P, \leq)$ is an element a of P such that

(1) $a \leq x$ for all x in S and
(2) for all y in P, if for all x in S, $y \leq x$, then $y \leq a$.

Lemma 2.6. Let $(X;0)$ be a commutative WS-algebra. Then for any $x, y \in X$, $x \wedge y = \inf\{x, y\}$.

Proof. Let X be a WS-algebra and x, $y \in X$. We show that $x \wedge y = \inf\{x, y\}$. By Lemma 2.5 we have proved that the equalities $x \wedge y \leq x$ and $x \wedge y \leq y$ always hold in any WS-algebras. Then $x \wedge y \leq \inf\{x, y\}$. Assume now $t \leq x$ and $t \leq y$. So $t - x = 0$ and $t - y = 0$. By definition $t = t - 0 = t - (t - x) \overset{commut.}{=} x - (x - t)$. Similarly $t = t - 0 = t - (t - y) = y - (y - t)$. Then $t = x - (x - t) = x - (x - (y - (y - t)))$. Since $\inf\{x, y\} \leq x$ and $\inf\{x, y\} \leq y$, to complete the proof it is enough to show $t \leq x \wedge y$ equivalently $t - x \wedge y = 0$. Now $t - (x \wedge y) = x - (x - y)) = t - (x - (x - y)) = (x - (x - t)) - (x - (x - y)) \overset{WSA-1}{=} (x - (x - (x - y)) - (x - t) \overset{commut.}{=} (x - y) - ((x - y) - x) - (x - t) \overset{Lemma 2.2}{=} (x - y) - (x - t) \overset{WSA-1}{=} (x - (x - t)) - y \overset{commut.}{=} (t - (t - x)) - y = t - y = 0$. That is $t \leq x \wedge y$. Thus $x \wedge y = \inf\{x, y\}$. □

3. Soft WS-algebras

In [11] Molodtsov defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and A a subset of E.

Definition 3.1. [11] A pair (F, A) is called *a soft set over* U, where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). For illustration, Molodtsov considered several examples in [3]. At present, works on the soft theory are prossing rapidly. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Y.B. Jun et all [12] presented the soft WS-algebras. As a remainder we will give set operations of soft sets [9].

Definition 3.2. [9] Let (F, A) and (G, B) be two soft sets over a common universe U. *The intersection* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:
(i) $C = A \cap B$,
(ii) $(\forall\ e\ \in\ C)$ $(H(e) = F(e)$ or $G(e)$, (as both are same sets)).
In this case, we write $(F; A) \,\tilde{\cap}\, (G; B) = (H; C)$.

Definition 3.3. [9] Let (F, A) and (G, B) be two soft sets over a common universe U. *The union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:
(i) $C = A \cup B$
(ii) for all e $\in$ C, $H(e) = \begin{cases} \text{F(e)}, & \text{if e} \in \text{A} \setminus \text{B}; \\ \text{G(e)}, & \text{if e} \in \text{B} \setminus \text{A} ; \\ \text{F(e)} \cup \text{G(e)}, & \text{if e} \in A \cap B. \end{cases}$
In this case, we write $(F; A) \,\tilde{\cup}\, (G; B) = (H; C)$.

Definition 3.4. [9] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) AND (G, B)" is denoted by $(F, A)\ \tilde{\wedge}\ (G, B)$ is defined by

$$(F, A)\tilde{\wedge}(G, B) = (H, A \times B),$$

where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.5. [9] If (F, A) and (G, B) be two soft sets over a common universe U, then "(F, A) OR (G, B)" is denoted by $(F, A)\ \widetilde{\vee}\ (G, B)$ is defined by

$$(F, A)\widetilde{\vee}(G, B) = (H, A \times B),$$

where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.6. [12] Let $(X; 0)$ be a WS-algebra and let (F, A) be a soft set over X. Then (F, A) is called *a soft WS-algebra* over X if $F(x)$ is a subalgebra of X for all $x \in A$, that is for each a, $b \in F(x)$ we have $a - b \in F(x)$.

Theorem 3.1. Let (F,A) and (G;B) be two soft WS-algebras over X.
(1) If $A \cap B \neq \emptyset$ then the intersection $(F, A)\widetilde{\cap}(G; B)$ is a soft WS-algebra over X.
(2) If A and B are disjoint, then the union $(F, A)\widetilde{\bigcup}(G, A)$ is a soft WS-algebra.
(3) $(F, A)\widetilde{\bigwedge}(G; B)$ is a soft WS-algebra.

Proof. For a proof see [12]. □

Definition 3.7. Let (F, A) be a soft set over X. Then (F, A) is called *an idealistic soft WS-algebra over X* (it is also called an soft ideal or soft deductive system in [12]) if $F(x)$ is an ideal of X for all $x \in A$.

Theorem 3.2. Let (F,A) and (G;B) be two soft idealistic of WS-algebra X.
(1) If $A \cap B \neq \emptyset$ then the intersection $(F, A)\widetilde{\cap}(G; B)$ is a soft idealistic of WS-algebra X.
(2) If A and B are disjoint, then the union $(F, A)\widetilde{\bigcup}(G, A)$ is a soft idealistic of WS-algebra X.
(3) $(F, A)\widetilde{\bigwedge}(G; B)$ is a soft idealistic of WS-algebra X.

Proof. For a proof see [12]. □

Definition 3.8. [2] *The bi-intersection* of two soft sets (F, A) and (G, B) over common universe U is defined to be the soft set (H, C), where $C = A \cap B$ and $H : C \to P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$

Theorem 3.3. *Let (F,A) and (G,B) be two soft idealistic of WS-algebra* $(X; 0)$. *Then* $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$ *soft ideal of WS-algebra.*

Proof. Since (F,A) and (G,B) are soft ideals of WS-algebra X, $F(x)$ and $G(x)$ are ideals of X for all $x \in A \cap B$. So $F(x) \cap G(x)$ is an ideal of X. □

4. Irreducible soft ideals, prime soft ideals and maximal soft ideals

In this section we study soft ideals, prime soft ideals and maximal soft ideals of WS-algebras. These ideals are defined and investigated for subtraction algebras in [5]. We continue studying general properties of these ideals in WS-algebras. For this reason we will recall the definitions of prime ideals, irreducible ideals and maximal ideals of WS-algebras.

Definition 4.1. Let (X; 0) be a WS-algebra. A proper ideal I of X is called *a prime ideal* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for any x, $y \in X$.

Definition 4.2. Given a WS-algebra $(X; 0)$, a proper ideal I of X is called *a maximal ideal* if I is not a proper subset of any proper ideal of X.

Theorem 4.1 generalizes Theorem 3.8 in [5].

Theorem 4.1. Let $(X; 0)$ be a WS-algebra. Every prime ideal of X is maximal.

Proof. Let $(X;0)$ be a WS-algebra and P an ideal. Assume that P is a prime ideal but not maximal in X. We may find an ideal I containing P properly. Let $x \in X$ and $y \in I \setminus P$. Then $(x-y)\wedge y = (x-y)-((x-y)-y) \overset{WSA-2}{=} (x-y)-((x-y)-(y-y)) \overset{WSA-3}{=} (x-y)-(x-y) \overset{WSA-4}{=} 0 \in P$. Since P is prime and $(x-y)\wedge y \in P$ and $y \notin P$, we have $x-y \in P \subset I$. Since I is an ideal and $x-y \in I$ and $y \in I$ we have $x \in I$. Thus $I = X$ and so P is maximal. □

In [5], the following theorem is proved for subtraction algebras.

Theorem 4.2. Let P be a maximal ideal of a subtraction algebra X. For any x, $y \in P$ we have $x-y \in P$ or $y-x \in P$.

We extend Theorem 4.2 to WS-algebras and prove.

Theorem 4.3. Let $(X;0)$ be a WS-algebra and P a maximal ideal of $(X;0)$. For any x, $y \in X$ we have $x-y \in P$ or $y-x \in P$.

Proof. Let P be a maximal ideal of a WS-algebra $(X;0)$ and x, $y \in X$. Some cases arise. If x, $y \in P$, by Lemma 2.3 $(x-y)-x \in P$ since P is an ideal. Assume that $x \in P$ and $y \in X \setminus P$. By Lemma 2.2 $(x-y)-x = 0 \in P$, hence $x-y \in P$ since P is an ideal and $x \in P$. Similarly, if $y \in P$ and $x \in X \setminus P$, then $y-x \in P$. So assume that x, $y \in X \setminus P$ and that $y-x \notin P$. Let $Q := \{z \in X \mid z-(y-x) \in P\}$. We claim that Q is an ideal of the WS-algebra X and it contains $y-x$ and P. Let $a \in X$ and $b \in Q$. Assume that $a-b \in Q$. Then $(a-b)-(y-x) \in P$. By WSA-2 we have $(a-b)-(y-x) = (a-(y-x))-(b-(y-x) \in P$. Since P is ideal and $(b-(x-y) \in P$, we have $a-(y-x) \in P$. It follows that $a \in Q$. From Lemma 2.2 $0-(y-x) = 0 \in P$ from which we have $0 \in Q$. Hence Q is an ideal. $(y-x)-(y-x) = 0 \in P$ so $y-x \in Q$. Let $z \in P$. By Lemma 2.2 and WSA-1, $(z-(y-x))-z = (z-z)-(y-x) = 0-(y-x) = 0 \in P$. Since P is an ideal and $z \in P$, $z-(y-x) \in P$. Hence $z \in Q$ for all $z \in P$. So P is strictly contained in Q. Since P is maximal ideal, $Q = X$. On the one hand $(x-y)-(y-x) \in P$. And on the other hand $x-y = (x-y)-0 \overset{Lemma 2.2}{=} (x-y)-[(y-x)-y] \overset{WSA-2}{=} [x-(y-x)]-y \overset{WSA-1}{=} (x-y)-(y-x)$ implies $x-y \in P$. This completes the proof. □

Theorem 4.4. Let $(X;0)$ be a WS-algebra and P an ideal of X. Consider following conditions.
(1) P is prime.
(2) P is maximal.
(3) For all x, $y \in P$ we have $x-y \in P$ or $y-x \in P$.
Then (1) $\Leftrightarrow$ (2)$\Rightarrow$ (3).

Proof. Clear from Theorems 4.1 and 4.3. □

Example 4.1 shows that there exist WS-algebras in which Theorem 4.4 (3) $\Rightarrow$ (1) does not hold in general.

Example 4.1. Let $X = \{0, a, b, c\}$ be a set with the following " $-$ " operation table. Then $(X;0)$ is a WS-algebra([8]).

$-$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

Let $I = \{0, a\} \subseteq X$. It is easy to check that I is an ideal of X. Observe that $a-b = 0 \in I$, $a-c = 0 \in I$, $b-c = 0 \in I$. So for any x, $y \in X$, $x-y \in I$ or $y-x \in I$. Since

$c \wedge b = c - (c - b) = c - c = 0 \in I$ and $b \notin I$, $c \notin I$, I is not a prime ideal of $(X;0)$. In fact I is not maximal ideal since $J = \{0, a, b\}$ is an ideal of X containing I properly.

Let I be an ideal in a subtraction algebra $(X;0)$ and $w \in X$. It is proved in [5] that the set $\mathrm{I}_w^\wedge = \{x \in X \mid w \wedge x \in \mathrm{I}\}$ is an ideal in $(X;0)$. Example 4.2 shows that this not the case when $(X;0)$ is a WS-algebra.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with the following " $-$ " operation table.

−	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	0	0	c
d	d	d	d	d	0

It is a routine to check that $(X;0)$ is a WS-algebra and $I = \{0, a\}$ is an ideal. Also it is easily seen that $\mathrm{I}_c^\wedge = \{0, a, d\}$. Now $c \wedge b = c \notin \mathrm{I}$, and so $b \notin \mathrm{I}_c^\wedge$; $c \wedge c = c \notin \mathrm{I}$, and so $c \notin \mathrm{I}_c^\wedge$. Since $d \in \mathrm{I}_c^\wedge$ and $b - d = 0 \in \mathrm{I}_c^\wedge$ and $b \notin \mathrm{I}_c^\wedge$, $\mathrm{I}_c^\wedge$ can not be an ideal in the WS-algebra X.

Theorem 4.5. *Let I be a nonempty subset of a WS-algebra $(X;0)$ such that*
(1) $x \in I$ and $y - x = 0$ imply $y \in I$.
(2) For $x, y \in I$, there exists $z \in I$ such that $x - z = 0$ and $y - z = 0$.
Then I is an ideal of $(X;0)$.

Proof. We may carry over here verbatim proof of Theorem 3.5 in [5]. But for the convenience of the reader we complete the proof for WS-algebras. Since I is nonempty there exists an $a \in I$. By Lemma 2.2 $0 - a = 0$. By (1) $0 \in I$. Let x, $y \in X$. Assume that $y \in I$ and $x - y \in I$. To complete the proof we show $x \in I$. For y and $x - y$ by (2) there exists $z \in X$ such that $y - z = 0$ and $(x - y) - z = 0$. Hence $x - z \overset{WSA-3}{=} (x - z) - 0 = (x - z) - (y - z) \overset{WSA-2}{=} (x - y) - z = 0$. So $x - z = 0$. Since $z \in I$ and $x - z = 0 \in I$, by (1) we have $x \in I$. $\square$

Definition 4.3. A proper ideal I of a WS-algebra $(X;0)$ is called *an irreducible ideal* whenever $I = J \cap K$ for ideals J, K of X we necessarily have $I = J$ or $I = K$.

It is clear that every maximal ideal is irreducible. There are irreducible ideals in WS-algebras that are not maximal. Let $(X;0)$ denote the WS-algebra in Example 4.1 and consider the ideal $I = \{0, a\}$ of $(X;0)$. Since $J = \{0, a, b\}$ is the only ideal of X containing I properly, $I = \{0, a\}$ is an irreducible ideal but not maximal.

Definition 4.4. Let (F, A) be a soft WS-algebra over X. A soft set (G, I) over X is called *a soft ideal* of (F, A), denoted by $(G, I) \mathrel{\widetilde{\lhd}} (F, A)$, if it satisfies:

i) $I \subset A$,

ii) $G(x)$ is an ideal of $F(x)$, for all $x \in I$.

Definition 4.5. Let (F, A) be a soft WS-algebra over X and (G, I) is a non-whole soft ideal of (F, A).

i) (G, I) is called *an irreducible soft ideal* of (F, A) if $G(x)$ is an irreducible ideal of $F(x)$ for all $x \in I$.

ii) (G, I) is called *a prime soft ideal* of (F, A) if $G(x)$ is a prime ideal of $F(x)$ for all $x \in I$.

Let (G, I) be a maximal soft ideal of a soft WS-algebra (F, A) over X, that is, $G(x)$ is a maximal ideal of $F(x)$ for each $x \in I$. Since every maximal ideal is irreducible, it is clear that (G, I) is an irreducible soft ideal of (F, A) over X.

Theorem 4.6. Let (F, A) be a soft WS-algebra over X. Let (G_1, I_1) *and* (G_2, I_2) be two irreducible soft ideals of (F, A).

i) If $I_1 \cap I_2 \neq \emptyset$, then $(G_1, I_1) \widetilde{\cap} (G_2, I_2)$ is an irreducible soft ideal of (F, A).

ii) $I_1 \cap I_2 = \emptyset$ then $(G_1, I_1) \widetilde{\cup}(G_2, I_2)$ is an irreducible soft ideal of (F, A).

Proof. Using Definition 3.2, we can write $(G_1, I_1) \widetilde{\cap} (G_2, I_2) = (G, I)$ where $I = I_1 \cap I_2$ and $G(x) = G_1(x)$ or $G(x) = G_2(x)$ for all $x \in X$. By [4] (G, I) is an soft ideal of (F, A). Since (G_1, I_1) and (G_2, I_2) are irreducible ideals of (F, A), we have that $G(x) = G_1(x)$ is an irreducible ideal of $F(x)$ or $G(x) = G_2(x)$ is an irreducible ideal of $F(x)$ for all $x \in X$. Hence $(G_1, I_1) \widetilde{\cap} (G_2, I_2) = (G, I)$ is an irreducible soft ideal of (F, A). This complete the proof of (i). Let $D = I_1 \cup I_2$. Define T on D by

$$T(x) = \begin{cases} G_1(x), & \text{if } x \in I_1 \backslash I_2 \\ G_2(x), & \text{if } x \in I_2 \backslash I_1 \end{cases}$$

Then it is easily checked that $(G_1, I_1) \widetilde{\cup} (G_2, I_2) = (T, D)$ and $(T, D) \widetilde{\lhd} (F, A)$. Since $I_1 \cap I_2 = \emptyset$, either $x \in I_1 \backslash I_2$ or $x \in I_2 \backslash I_1$ for all $x \in D$. Let $x \in D$. If $x \in I_1 \backslash I_2$, then $T(x) = G_1(x)$ is an irreducible ideal of $F(x)$ since (G_1, I_1) is an irreducible soft ideal of (F, A). If $x \in I_2 \backslash I_1$, $T(x) = G_2(x)$ is an irreducible ideal of $F(x)$ since (G_2, I_2) is an irreducible soft ideal of (F, A). Hence $T(x)$ is an irreducible ideal of $F(x)$ for all $x \in D$, and so $(G_1, I_1) \widetilde{\cup} (G_2, I_2)$ is an irreducible ideal of (F, A). $\square$

Theorem 4.7. Let (F, A) be a soft WS-algebra over X. Let (G_1, I_1) *and* (G_2, I_2) be two prime soft ideals of (F, A).

i) If $I_1 \cap I_2 \neq \emptyset$, then $(G_1, I_1) \widetilde{\cap} (G_2, I_2)$ is a prime soft ideal of (F, A) .

ii) $I_1 \cap I_2 = \emptyset$ then $(G_1, I_1) \widetilde{\cup}(G_2, I_2)$ is a prime soft ideal of (F, A).

Proof. Using Definition 3.2, we can write $(G_1, I_1) \widetilde{\cap} (G_2, I_2) = (G, I)$ where $I = I_1 \cap I_2$ and $G(x) = G_1(x)$ or $G(x) = G_2(x)$ for all $x \in X$. By [4] (G, I) is an soft ideal of (F, A). Since (G_1, I_1) and (G_2, I_2) are prime ideals of (F, A), we have that $G(x) = G_1(x)$ is a prime ideal of $F(x)$ or $G(x) = G_2(x)$ is a prime ideal of $F(x)$ for all $x \in X$. Hence $(G_1, I_1) \widetilde{\cap} (G_2, I_2) = (G, I)$ is a prime soft ideal of (F, A). This complete the proof of (i). Proof of (ii) is similarly proof of (ii) in Theorem4.6 $\square$

5. Irreducible idealistic soft WS-algebras

Definition 5.1. Let (F, A) be a non-whole an idealistic soft WS-algebra over X. Then (F, A) is called *an irreducible idealistic soft WS-algebra over* X if $F(x)$ is an irreducible ideal of X for all $x \in A$.

Theorem 5.1. Let (F, A) and (G, B) be two irreducible idealistic soft WS-algebras over X. Then

1. If $A \cap B \neq \emptyset$ then the intersection $(F, A) \widetilde{\cap} (G; B)$ is an irreducible idealistic soft WS-algebra over X.
2. If A and B are disjoint, then the union $(F, A) \widetilde{\bigcup} (G, A)$ is an irreducible idealistic soft WS-algebra.

Proof. 1. Let $x \in C = A \cap B$. By Definition 3.2, for all $x \in C$, we have $H(x) = F(x)$ or $H(x) = G(x)$ or $H(x) = F(x) = G(x)$ if $F(x) = G(x)$. For all $x \in C$, since $F(x)$ and $G(x)$ are irreducible $H(x)$ is irreducible too. This completes the proof.

2. Let $C = A \cup B$, $A \cap B = \phi$ be null set and $(H, C) = (F, A) \widetilde{\cup} (G, B)$. By hypothesis

$F(a)$ is an irreducible ideal of X for all $a \in A$ and $G(b)$ is an irreducible ideal of X for all $b \in B$. Let $c \in C$. By Definition 3.3, if $c \in A \setminus B$, then $H(c) = F(c)$ is irreducible ideal. If $c \in B \setminus A$, then $H(c) = G(c)$ is an irreducible ideal of X. Assume that $c \in A \cap B$. By definition $H(c) = F(c) = G(c)$ is an irreducible ideal of X also. This completes the proof. □

Note that "AND" of two irreducible soft ideals may not be an irreducible soft ideal.To prove this idea, we examine the following example.

Example 5.1. Let $X = \{0, 1, 2, 3, 4\}$ be a WS-algebra with the following Cayley table:

-	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

i) Let (H, B) be a soft set over X, where $B = \{1, 2, 4\}$ and $H : B \longrightarrow P(X)$ be a set-valued function defined by

$$H(x) = \{y \in X \mid y - (y - x) = 0\}$$

for all $x \in B$. Then $H(1) = \{0, 2, 4\}$, $H(2) = \{0, 1, 4\}$ and $H(4) = \{0, 1, 2, 3\}$ which are irreducible ideal of X. Therefore (H, B) is an irreducible idealistic soft WS-algebra over X.

ii) Let (K, C) be a soft set over X, where $C = \{2\}$ and $K : C \longrightarrow P(X)$ be a set-valued function defined by

$$K(x) = \{y \in X \mid y - x \in \{0, 1\}\}$$

for all $x \in C$. Then $K(2) = \{0, 1, 2, 3\}$ which is irreducible ideal of X. Hence (K, C) is an irreducible idealistic soft WS-algebra over X.

Use i) and ii), $(H, B)\widetilde{\wedge}(K, C) = (F, D)$, $D = B \times C = \{(1, 2), (2, 2), (4, 2)\}$ and $F(1, 2) = H(1) \cap K(2) = \{0, 2\} \lhd X$, $F(2, 2) = H(2) \cap K(2) = \{0, 1\} \lhd X$, $F(2, 4) = H(4) \cap K(2) = \{0, 1, 2, 3\} \lhd X$. Then

$F(1, 2) = \{0, 2\} = \{0, 1, 2, 3\} \cap \{0, 2, 4\}$ and $F(1, 2) \neq \{0, 1, 2, 3\}$ and $F(1, 2) \neq \{0, 2, 4\}$. Hence $(H, B)\widetilde{\wedge}(K, C) = (F, D)$ is not irreducible idealistic soft WS-algebra over X.

6. Prime idealistic soft WS-algebras

In this section, we introduce the definition of prime soft idealistic over a WS-algebra.

Definition 6.1. Let (F, A) be a soft idealistic WS-algebra over X. (F, A) is called *a prime idealistic soft WS-algebra over X* if $F(x)$ is a prime ideal of X for all $x \in A$.

Definition 6.2. Let (F, A) be a soft idealistic WS-algebra over X. (F, A) is called *a maximal idealistic soft WS-algebra over X* if $F(x)$ is a maximal ideal of X for all $x \in A$.

We know that prime ideals and maximal ideals coincide in any WS-algebra which is clear from Theorem 4.4. Thus prime soft idealistic and maximal soft idealistic are the same over any WS-algebra. To illustrate these idealistics we consider an example.

Example 6.1. Let X={0,a,b,c,d} be a WS-algebra with following Cayley table:

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (F, A) be a soft over X, where $A = \{b, c, d\}$ and $F : A \to P(X)$ is a set-valued function defined by $F(x) = \{y \in X \mid x \wedge y \in I\}$ for all $x \in A$ where $I = \{0, a\} \subset X$. Then $F(b) = \{0, a, c, d\}$, $F(c) = \{0, a, b, d\}$ and $F(d) = \{0, a, b, c\}$ are prime ideals of X. Therefore (F, A) is a prime soft idealistic WS-algebra over X.

Theorem 6.1. Let (F, A) and (G, B) be two prime idealistic soft WS-algebras over X.

1. If $A \cap B \neq \emptyset$, then the intersection $(H, C) = (F, A) \widetilde{\cap} (G, B)$ is a prime idealistic soft WS-algebras over X.
2. If A and B are disjoint, then the union $(H, C) = (F, A) \widetilde{\cup} (G, B)$ is a prime idealistic soft WS-algebras over X.

Proof. Similarly Proof of Theorem 5.1 □

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