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ON THE TWO PARAMETER HOMOTHETIC MOTIONS IN COMPLEX PLANE*

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ABSTRACT. In this article, we investigate two parameter homothetic motions in the complex plane. Also, we obtain some definitions, theorems and corollaries related to the velocities, accelerations and their poles (and hodograph) of a point in complex planar motion.

Keywords: two parameter motion, homothetic motion, planar kinematics, complex plane.

AMS Subject Classification: 53A17.

1. INTRODUCTION

We know that the angular velocity vector has an important role in kinematics of two rigid bodies, especially one rolling on another, [1, 7, 10]. Investigating the geometry of the motion of a line or a point in the motion of plane is important in the study of planar kinematics or planar mechanisms or in physics. Mathematicians and physicists have interpreted rigid body motions in various ways. K. Nomizu has studied the one parameter motions of orientable surface M on tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations of the angular velocity vector of rolling without sliding, [11]. H. H. Hacısalihoğlu showed some properties of one parameter homothetic motions in Euclidean space, [5]. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [2, 4, 12].

Alternative definitions of the imaginary unit i other than $i^2 = -1$ can give rise to interesting and useful complex number systems. Complex numbers were first discovered by Cardan, who called them "fictitious", during his attempts to find solutions to cubic equations, [3]. Müller has introduced one and two parameter planar motions and obtained the relations between absolute, relative, sliding velocity and pole curves of these motions. Moreover, the relations between the complex velocities one parameter motion in the complex plane were provided by [9]. One parameter planar homothetic motion was defined in the complex plane, [8]. In [6] all one parameter motions obtained from two parameter motions on the Euclidean plane are investigated.

In this paper, two parameter homothetic motions in the complex plane are defined. Sliding velocity, pole lines, hodograph and acceleration poles of two parameter complex homothetic motions at the positions of $\forall\,(\lambda, \mu)$ are obtained. Some characteristic properties about the velocity vectors, the acceleration vectors and the pole curves are given. Moreover, in the case of homothetic scale h identically equal to 1, the results given in [13] are obtained as a special case.

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2. Two parameter homothetic motions in complex plane

Let E, E' be moving and fixed complex planes and O, O' be origin points of their coordinate systems, respectively. If $\overrightarrow{O'O} = C'(\lambda,\mu)$ and $h(\lambda,\mu)$ is homothetic constant, then

$$Y(\lambda,\mu) = h(\lambda,\mu)e^{i\theta(\lambda,\mu)}X(\lambda,\mu) + C'(\lambda,\mu), \tag{1}$$

where $\theta(\lambda,\mu)$ is the rotation angle of E with respect to E' and $X(\lambda,\mu) = (X_1(\lambda,\mu),$ $X_2(\lambda,\mu))$ and $Y(\lambda,\mu) = (Y_1(\lambda,\mu), Y_2(\lambda,\mu))$ are the coordinate functions of the moving and fixed plane, respectively. If λ and μ are given by functions of time parameter t, then the complex homothetic motion B_I, which is called the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} is obtained. Here $Y_1(\lambda,\mu), Y_2(\lambda,\mu), X_1(\lambda,\mu), X_2(\lambda,\mu), A(\lambda,\mu), B(\lambda,\mu)$ are complex elements. They can be denoted by

$$Y(\lambda,\mu) = [Y_1 \quad Y_2]^T, \quad X(\lambda,\mu) = [X_1 \quad X_2]^T, \quad C'(\lambda,\mu) = [A \quad B]^T.$$

Without losing generality, we can take $\theta(0,0) = A(0,0) = B(0,0) = 0$ to make two complex planes are congruent at the position $(\lambda,\mu) = (0,0)$.

2.1. **Velocities and Composition of Velocities.** If $\overrightarrow{OO'} = C(\lambda,\mu)$ is taken, then we obtain

$$C'(\lambda,\mu) = -C(\lambda,\mu)e^{i\theta(\lambda,\mu)} \tag{2}$$

and the equality of (2) is substituted into the equality of (1), we get that

$$Y(\lambda,\mu) = [h(\lambda,\mu)X(\lambda,\mu) - C(\lambda,\mu)]e^{i\theta(\lambda,\mu)}. \tag{3}$$

The relative velocity of the point $X(\lambda,\mu)$ is the velocity of the point $X(\lambda,\mu)$ with respect to the moving plane E and the relative velocity vector of the point $X(\lambda,\mu)$ in the moving plane is given by

$$\overrightarrow{X_r} = \dot{X}(\lambda,\mu) = X_\lambda\dot{\lambda} + X_\mu\dot{\mu}. \tag{4}$$

This vector is deduced in the fixed coordinate system as follows

$$\overrightarrow{Y_r} = \overrightarrow{X_r}e^{i\theta(\lambda,\mu)} = \dot{X}(\lambda,\mu)e^{i\theta(\lambda,\mu)} = (X_\lambda\dot{\lambda} + X_\mu\dot{\mu})e^{i\theta(\lambda,\mu)}. \tag{5}$$

The velocity of the point $X(\lambda,\mu)$ with respect to the fixed plane E' is the absolute velocity of the point $X(\lambda,\mu)$. By differentiating the equality of (3) with respect to (λ,μ) and simplifying it, we get

$$\begin{aligned}\overrightarrow{Y_a}=&[\dot{h}(\lambda,\mu) + ih(\lambda,\mu)\dot{\theta}(\lambda,\mu)]X(\lambda,\mu)e^{i\theta(\lambda,\mu)} - [\dot{C}(\lambda,\mu) + \\ &+iC(\lambda,\mu)\dot{\theta}(\lambda,\mu)]e^{i\theta(\lambda,\mu)} + h(\lambda,\mu)Y_r\end{aligned} \tag{6}$$

and the sliding velocity vector of the point $X(\lambda,\mu)$ is given by

$$\begin{aligned}\overrightarrow{Y_f}=&[\dot{h}(\lambda,\mu) + ih(\lambda,\mu)\dot{\theta}(\lambda,\mu)]X(\lambda,\mu)e^{i\theta(\lambda,\mu)} - \\ &-[\dot{C}(\lambda,\mu) + iC(\lambda,\mu)\dot{\theta}(\lambda,\mu)]e^{i\theta(\lambda,\mu)}.\end{aligned} \tag{7}$$

The expressions of the absolute and the sliding velocity vectors with respect to coordinate axis of the moving plane, respectively, are

$$\begin{aligned}\overrightarrow{X_a} = \overrightarrow{Y_a}e^{-i\theta(\lambda,\mu)}=&[\dot{h}(\lambda,\mu) + ih(\lambda,\mu)\dot{\theta}(\lambda,\mu)]X(\lambda,\mu) - \\ &-[\dot{C}(\lambda,\mu) + iC(\lambda,\mu)\dot{\theta}(\lambda,\mu)] + h(\lambda,\mu)\overrightarrow{X_r}\end{aligned} \tag{8}$$

and

$$\begin{aligned}\overrightarrow{X_f} = \overrightarrow{Y_f}e^{-i\theta(\lambda,\mu)}=&[\dot{h}(\lambda,\mu) + ih(\lambda,\mu)\dot{\theta}(\lambda,\mu)]X(\lambda,\mu) - \\ &-[\dot{C}(\lambda,\mu) + iC(\lambda,\mu)\dot{\theta}(\lambda,\mu)].\end{aligned} \tag{9}$$

If the point $X(\lambda,\mu)$ is a fixed point in the moving plane E, $\overrightarrow{X_r}=\overrightarrow{Y_r}=0$. Then the absolute velocity is equal to the sliding velocity. We can give the following theorem from equations (5), (6) and (7).

Theorem 2.1. *The absolute velocity of a point $X(\lambda,\mu)$, where in the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} is equal to addition of the sliding velocity and $h(\lambda,\mu)$ times relative velocity.*

$$\overrightarrow{Y_a}=\overrightarrow{Y_f}+h(\lambda,\mu)\overrightarrow{Y_r}. \tag{10}$$

To avoid from the situations of only rotation and only translation, let us consider

$$\dot{\theta}(\lambda,\mu)=\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\neq 0$$

and

$$h(\lambda,\mu)\neq constant.$$

Now, let us investigate the sliding velocity is equal to zero. Such these points shall be fixed, not only in the moving plane E, but also in the fixed plane E'. In this case, we obtain an equation from the equation (7) as follows;

$$\overrightarrow{Y_f}=(\dot{h}+ih\dot{\theta})Xe^{i\theta}-(\dot{C}+iC\dot{\theta})e^{i\theta}=0$$

and this gives us

$$P(P_1,P_2)=\frac{\dot{C}\dot{h}+Ch\dot{\theta}^2}{\dot{h}^2+h^2\dot{\theta}^2}+i\frac{C\dot{h}\dot{\theta}-\dot{C}h\dot{\theta}}{\dot{h}^2+h^2\dot{\theta}^2} \tag{11}$$

is which is the pole point of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II}. If the sliding velocity of the point $X(\lambda,\mu)$ given by the equation (7) is taken into consideration with the pole point $P(P_1,P_2)$, $\dot{C}$ can be obtained from the following equation

$$P(P_1,P_2)=\frac{\dot{C}+iC\dot{\theta}}{\dot{h}+ih\dot{\theta}}.$$

By substituting the equality of $\dot{C}$ into the equality of (7), we have

$$\overrightarrow{Y_f}=(\dot{h}+ih\dot{\theta})(X-P)e^{i\theta}. \tag{12}$$

Theorem 2.2. *The pole points of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} on the moving plane lie on a line at the position of $\forall\,(\lambda,\mu)$.*

Proof. If we write the equality of $P(P_1,P_2)$ clearly and $C=\begin{bmatrix}-Ae^{-i\theta}\\ -Be^{-i\theta}\end{bmatrix}$ is regarded that is in this equation, we obtain

$$P=\begin{bmatrix}P_1\\ P_2\end{bmatrix}=\frac{1}{\dot{h}+ih\dot{\theta}}\begin{bmatrix}-\dot{A}e^{-i\theta}+i\dot{\theta}Ae^{-i\theta}\\ -\dot{B}e^{-i\theta}+i\dot{\theta}Be^{-i\theta}\end{bmatrix}+\frac{i\dot{\theta}}{\dot{h}+ih\dot{\theta}}\begin{bmatrix}-Ae^{-i\theta}\\ -Be^{-i\theta}\end{bmatrix}.$$

Then

$$P_1=-\frac{e^{-i\theta}}{\dot{h}+ih\dot{\theta}}\dot{A} \tag{13}$$

and

$$P_2=-\frac{e^{-i\theta}}{\dot{h}+ih\dot{\theta}}\dot{B} \tag{14}$$

are obtained. Here $\frac{\dot{\lambda}}{\dot{\mu}}$ is taken from the equality of P_2 and then substituted in the equality of P_1, we get

$$\frac{-e^{-i\theta}B_\mu - P_2h_\mu - iP_2h\theta_\mu}{P_2h_\lambda + iP_2h\theta_\lambda + e^{-i\theta}B_\lambda} = \frac{-e^{-i\theta}A_\mu - P_1h_\mu - iP_1h\theta_\mu}{P_1h_\lambda + iP_1h\theta_\lambda + e^{-i\theta}A_\lambda}$$

thus following line equation is obtained

$$\begin{array}{c}(h_\lambda B_\mu + ih\theta_\lambda B_\mu - B_\lambda h_\mu - ihB_\lambda\theta_\mu)\,P_1 + \\ +\,(A_\lambda h_\mu + ihA_\lambda\theta_\mu - h_\lambda A_\mu - ih\theta_\lambda A_\mu)\,P_2 = (B_\lambda A_\mu - A_\lambda B_\mu)\,e^{-i\theta}.\end{array} \tag{15}$$

□

Corollary 2.1. *If $h(\lambda,\mu)$ is equal to 1, then we obtain a line equation for two parameter motions in the complex plane as follows;[13].*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,P_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,P_2 = i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,e^{-i\theta}, \tag{16}$$

Corollary 2.2. *If $(\lambda,\mu) = (0,0)$ i.e., $A(0,0) = B(0,0) = \theta(0,0) = 0$, then we obtain a line equation for the two parameter motions in the complex plane as follows;[13].*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,P_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,P_2 = i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,, \tag{17}$$

Theorem 2.3. *The pole points of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} on the fixed plane lie on a line at the position of $\forall(\lambda,\mu)$.*

Proof. If the equality of $P(P_1,P_2)$ is substituted into the equality of (1), then $\bar{P}(\bar{P}_1,\bar{P}_2)$ pole point of the fixed plane is obtained. Then the pole point on the fixed plane is

$$\bar{P}_1 = -\frac{h}{\dot{h} + ih\dot{\theta}}\dot{A} + A \tag{18}$$

and

$$\bar{P}_2 = -\frac{h}{\dot{h} + ih\dot{\theta}}\dot{B} + B. \tag{19}$$

Here $\frac{\dot{\lambda}}{\dot{\mu}}$ is taken from the equality of P_2 and then substituted in the equality of P_1, we get

$$\frac{-hB_\mu + Bh_\mu - \bar{P}_2h_\mu + iBh\theta_\mu - i\bar{P}_2h\theta_\mu}{\bar{P}_2h_\lambda + hB_\lambda - Bh_\lambda + i\bar{P}_2h\theta_\lambda - iBh\theta_\lambda} = \frac{-hA_\mu + Ah_\mu - \bar{P}_1h_\mu + iAh\theta_\mu - i\bar{P}_1h\theta_\mu}{\bar{P}_1h_\lambda + hA_\lambda - Ah_\lambda + i\bar{P}_1h\theta_\lambda - iAh\theta_\lambda}. \tag{20}$$

Thus, the following line equation is obtained

$$\begin{array}{l}\left(-hh_\lambda B_\mu + hB_\lambda h_\mu - ih^2\theta_\lambda B_\mu + ih^2B_\lambda\theta_\mu\right)\bar{P}_1 + \\ +\left(-hA_\lambda h_\mu + hh_\lambda A_\mu - ih^2A_\lambda\theta_\mu + ih^2\theta_\lambda A_\mu\right)\bar{P}_2 = \\ = h^2A_\lambda B_\mu - hBA_\lambda h_\mu - ih^2BA_\lambda\theta_\mu - hAh_\lambda B_\mu - ih^2A\theta_\lambda B_\mu - \\ -h^2B_\lambda A_\mu + hAB_\lambda h_\mu + ih^2AB_\lambda\theta_\mu + hh_\lambda A_\mu B + ih^2B\theta_\lambda A_\mu.\end{array} \tag{21}$$

□

Corollary 2.3. *If $h(\lambda,\mu)$ is equal to 1, then we obtain a line equation for the two parameter motions in the complex plane as follows [13]*

$$\begin{array}{c}(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,\bar{P}_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,\bar{P}_2 = \\ = BA_\lambda\theta_\mu + A\theta_\lambda B_\mu - AB_\lambda\theta_\mu - B\theta_\lambda A_\mu + i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,,\end{array} \tag{22}$$

Corollary 2.4. *If $(\lambda,\mu) = (0,0)$ that is, $A(0,0) = B(0,0) = \theta(0,0) = 0$, then we obtain a line equation for the two parameter motions in the complex plane as follows;[13].*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,\bar{P}_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,\bar{P}_2 = i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,, \tag{23}$$

Corollary 2.5. *The pole lines of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} on the the moving and fixed plane, at the position of $\lambda = \mu = 0$, are congruent.*

If the pole line of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} is y axis on the moving plane, then the equation $\dot{A}(\lambda,\mu) = A_\lambda \dot{\lambda} + A_\mu \dot{\mu}$ vanishes. Since $\dot{\lambda}$ and $\dot{\mu}$ are independent motion parameters and they never vanish. Then A_λ and A_μ should be equal to zero at the position of $\lambda = \mu = 0$. Then, we obtain

$$P_1{=}0 \tag{24}$$

and

$$P_2{=}-\frac{1}{\dot{h}+ih\dot{\theta}}\dot{B}. \tag{25}$$

Therefore, there is a relation between the pole lines of the fixed plane and the pole lines of the moving plane as follows;

$$\bar{P}_1{=}0 \tag{26}$$

and

$$\bar{P}_2{=}hP_2. \tag{27}$$

If the y-axis is chosen as pole axis, that is, $A_\lambda = A_\mu = 0$ is taken, then the sliding velocity of any fixed point $Q(X_1, X_2)$ on the moving plane at the position of $\lambda = \mu = 0$ is equal to absolute velocity.

Theorem 2.4. *In the complex homothetic motion B_I obtained from the complex homothetic motion B_{II}, let y-axis be the pole axis at the position of $\lambda = \mu = 0$. Then, there is a relation between the pole ray $\overrightarrow{PQ} = (Q-P)e^{i\theta}$ going from the pole point $P(0,P_2)$ to the point $Q(X_1,X_2)$ and the sliding velocity $\overrightarrow{Y_f}$ of the point $Q(X_1,X_2)$ as follows;*

$$\left\langle \overrightarrow{Y_f}, \overrightarrow{PQ} \right\rangle = \dot{h}X_1^2 + \dot{h}X_2^2 - 2\dot{h}X_2P_2 + \dot{h}P_2^2. \tag{28}$$

Proof. The equation of the sliding velocity vector with the pole point P is

$$\overrightarrow{Y_f} = \dot{h}\left[(X_1-P_1) + i(X_2-P_2)\right]e^{i\theta} + h\dot{\theta}\left[-(X_2-P_2) + i(X_1-P_1)\right]e^{i\theta}$$

and then

$$\overrightarrow{Y_f} = \left[\dot{h}(X_1-P_1) - h\dot{\theta}(X_2-P_2), \dot{h}(X_2-P_2) + h\dot{\theta}(X_1-P_1)\right]e^{i\theta}.$$

$P_1 = 0$ and $\lambda = \mu = 0$ will be regarded, then the last equation will be

$$\overrightarrow{Y_f} = \left[\dot{h}X_1 - h\dot{\theta}(X_2-P_2), \dot{h}(X_2-P_2) + h\dot{\theta}X_1\right].$$

On the other hand $\overrightarrow{PQ} = (X_1, X_2-P_2)$. Then we obtain

$$\left\langle \overrightarrow{Y_f}, \overrightarrow{PQ} \right\rangle = \dot{h}X_1^2 + \dot{h}X_2^2 - 2\dot{h}X_2P_2 + \dot{h}P_2^2.$$

□

Corollary 2.6. *When the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} is at the position of $\lambda = \mu = 0$ and y-axis is the pole axis, if $h(\lambda,\mu)$ is a constant different from zero, then the pole ray from the point $P(P_1,P_2)$ to the point $Q(X_1,X_2)$ and the sliding velocity vector $\overrightarrow{Y_f}$ of the point $Q(X_1,X_2)$ are perpendicular.*

Proof. If $h(\lambda, \mu)$ is a constant different from zero, then $\dot{h}(\lambda, \mu) = 0$. Therefore, the equation (28) will be as follows;

$$\left\langle \overrightarrow{Y_f}, \overrightarrow{PQ} \right\rangle = 0 \tag{29}$$

and it gives us the pole ray and the sliding velocity vector are perpendicular. □

Theorem 2.5. *The length of the sliding velocity vector* $\overrightarrow{Y_f}$ *of the complex homothetic motion* B_I *obtained from the complex homothetic motion* B_{II} *is*

$$\left\| \overrightarrow{Y_f} \right\| = \sqrt{\dot{h}^2 + h^2\dot{\theta}^2} \left\| \overrightarrow{PQ} \right\| \tag{30}$$

at the position of $\forall (\lambda, \mu)$.

Proof. It is known that

$$\begin{aligned}
\left\langle \overrightarrow{a} e^{i\theta}, \overrightarrow{b} e^{i\theta} \right\rangle &= \langle (a_1 + ia_2)(\cos\theta + i\sin\theta), (b_1 + ib_2)(\cos\theta + i\sin\theta) \rangle = \\
&= \left\langle \begin{matrix} (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta), \\ (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta) \end{matrix} \right\rangle = \\
&= a_1b_1 + a_2b_2 = \\
&= \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle.
\end{aligned}$$

Therefore, since $\overrightarrow{Y_f} = \left(\dot{h} + ih\dot{\theta}\right)(X - P)\, e^{i\theta}$, the length of the sliding velocity vector $\overrightarrow{Y_f}$ with the pole point is

$$\left\| \overrightarrow{Y_f} \right\| = \sqrt{\dot{h}^2 + h^2\dot{\theta}^2} \sqrt{(X_1 - P_1)^2 + (X_2 - P_2)^2}$$

and then we obtain

$$\left\| \overrightarrow{Y_f} \right\| = \sqrt{\dot{h}^2 + h^2\dot{\theta}^2} \left\| \overrightarrow{PQ} \right\|.$$

□

Theorem 2.6. *For the complex homothetic motion* B_I *obtained from the complex homothetic motion* B_{II}*, let* Ψ *be an angle between the pole ray* $\overrightarrow{PQ} = (X - P)\, e^{i\theta}$ *going from the pole point* $P = (P_1, P_2)$ *to the point* $Q(X_1, X_2)$ *and the sliding velocity vector* $\overrightarrow{Y_f}$*. Then, there is a relation as follows;*

$$\cos\Psi(\lambda, \mu) = \frac{\dot{h}}{\sqrt{\dot{h}^2 + h^2\dot{\theta}^2}} \tag{31}$$

at the position of $\forall(\lambda, \mu)$.

Proof. Since $\overrightarrow{Y_f} = \left[\dot{h}(X_1 - P_1) - h\dot{\theta}(X_2 - P_2), \dot{h}(X_2 - P_2) + h\dot{\theta}(X_1 - P_1)\right] e^{i\theta}$ and $\overrightarrow{PQ} = \left[(X_1 - P_1)\, e^{i\theta}, (X_2 - P_2)\, e^{i\theta}\right]$, if we get inner product

$$\left\langle \overrightarrow{PQ}, \overrightarrow{Y_f} \right\rangle = \dot{h} \left\| \overrightarrow{PQ} \right\|^2.$$

On the other hand, it is known that

$$\left\langle \overrightarrow{PQ}, \overrightarrow{Y_f} \right\rangle = \left\| \overrightarrow{PQ} \right\| \left\| \overrightarrow{Y_f} \right\| \cos\Psi(\lambda, \mu).$$

By comparing these last two equations the proof of the theorem is completed. □

Corollary 2.7. *If* $h(\lambda, \mu)$ *is a constant different from zero, then we obtain an equation for the two parameter motions in the complex plane as follows [13]*

$$\Psi(\lambda, \mu) = \frac{\pi}{2} + 2\mathrm{k}\pi, \quad (\mathrm{k} = 0, 1, 2, \dots), \tag{32}$$

Definition 2.1. *When the sliding velocity vectors of the fixed points are carried to the initial points, without changing the directions, then the locus of the end points of these vectors is a curve, called hodograph.*

Now, we investigate any points (X_1, X_2) of the locus of the hodographs in all the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} at the position of $\forall(\lambda,\mu)$. For this let $\dot{\lambda}^2+\dot{\mu}^2=1$. By differentiating the equality of (3) with respect to λ and μ, we have

$$\overrightarrow{Y_a}=\overrightarrow{Y_f}=\left(\dot{h}+ih\dot{\theta}\right)Xe^{i\theta}-\left(\dot{C}+iC\dot{\theta}\right)e^{i\theta}$$

and

$$\overrightarrow{Y_f}=\left[\left(\dot{h}+ih\dot{\theta}\right)X_1e^{i\theta}+\dot{A},\left(\dot{h}+ih\dot{\theta}\right)X_2e^{i\theta}+\dot{B}\right].$$

Then we obtain

$$\dot{Y}_1=\left(h_\lambda X_1e^{i\theta}+ih\theta_\lambda X_1e^{i\theta}+A_\lambda\right)\dot{\lambda}+\left(h_\mu X_1e^{i\theta}+ih\theta_\mu X_1e^{i\theta}+A_\mu\right)\dot{\mu},$$

$$\dot{Y}_2=\left(h_\lambda X_2e^{i\theta}+ih\theta_\lambda X_2e^{i\theta}+B_\lambda\right)\dot{\lambda}+\left(h_\mu X_2e^{i\theta}+ih\theta_\mu X_2e^{i\theta}+B_\mu\right)\dot{\mu}.$$

If the equations are written

$$\dot{Y}_1=m_1\dot{\lambda}+m_2\dot{\mu},$$
$$\dot{Y}_2=m_3\dot{\lambda}+m_4\dot{\mu}$$

and the method of Cramer is applied to

$$\Gamma=\begin{vmatrix} m_1 & m_2 \\ m_3 & m_4 \end{vmatrix}=m_1m_4-m_2m_3$$

at the position of $\lambda=\mu=0$ and after it is substituted into the equation of $\dot{\lambda}^2+\dot{\mu}^2=1$, we obtain

$$\begin{aligned}\Gamma=&A_\lambda h_\mu X_2+iA_\lambda h\theta_\mu X_2+B_\mu h_\lambda X_1+iB_\mu h\theta_\lambda X_1+A_\lambda B_\mu-A_\mu h_\lambda X_2-\\ &-iA_\mu h\theta_\lambda X_2-B_\lambda h_\mu X_1-iB_\lambda h\theta_\mu X_1-A_\mu B_\lambda\end{aligned} \tag{33}$$

and

$$\dot{\lambda}=\frac{\begin{vmatrix} \dot{Y}_1 & m_2 \\ \dot{Y}_2 & m_4 \end{vmatrix}}{\Gamma},\qquad \dot{\mu}=\frac{\begin{vmatrix} m_1 & \dot{Y}_1 \\ m_3 & \dot{Y}_2 \end{vmatrix}}{\Gamma},\qquad \dot{\lambda}^2+\dot{\mu}^2=1.$$

Then, we get

$$\begin{aligned}&\frac{\left[(h_\mu X_2+ih\theta_\mu X_2+B_\mu)\dot{Y}_1-(h_\mu X_1+ih\theta_\mu X_1+A_\mu)\dot{Y}_2\right]^2}{\Gamma^2}+\\ &+\frac{\left[(h_\lambda X_1+ih\theta_\lambda X_1+A_\lambda)\dot{Y}_2-(h_\lambda X_2+ih\theta_\lambda X_2+B_\lambda)\dot{Y}_1\right]^2}{\Gamma^2}=1.\end{aligned}$$

From this last equation, we obtain

$$\begin{aligned}&\left[(h_\mu X_2+ih\theta_\mu X_2+B_\mu)^2+(h_\lambda X_2+ih\theta_\lambda X_2+B_\lambda)^2\right]\dot{Y}_1^2+\\ &+\left[(h_\mu X_1+ih\theta_\mu X_1+A_\mu)^2+(h_\lambda X_1+ih\theta_\lambda X_1+A_\lambda)^2\right]\dot{Y}_2^2-\\ &-2\begin{bmatrix}(h_\mu X_2+ih\theta_\mu X_2+B_\mu)(h_\mu X_1+ih\theta_\mu X_1+A_\mu)\\ +(h_\lambda X_1+ih\theta_\lambda X_1+A_\lambda)(h_\lambda X_2+ih\theta_\lambda X_2+B_\lambda)\end{bmatrix}\dot{Y}_1\dot{Y}_2=\Gamma^2\end{aligned} \tag{34}$$

and this is the equation of the hodograph at the position of $\forall(\lambda,\mu)$.

Theorem 2.7. *The hodograph of any points (X_1, X_2) in the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} at the position of $\lambda=\mu=0$ is an ellipse.*

Proof. Taking the conic general form

$$KX^2 + 2LXY + MY^2 + 2DX + 2EY + F = 0$$

we obtain

$$\begin{aligned} K &= \left[(h_\mu X_2 + ih\theta_\mu X_2 + B_\mu)^2 + (h_\lambda X_2 + ih\theta_\lambda X_2 + B_\lambda)^2\right], \\ L &= -\begin{bmatrix} (h_\mu X_2 + ih\theta_\mu X_2 + B_\mu)(h_\mu X_1 + ih\theta_\mu X_1 + A_\mu) \\ + (h_\lambda X_1 + ih\theta_\lambda X_1 + A_\lambda)(h_\lambda X_2 + ih\theta_\lambda X_2 + B_\lambda) \end{bmatrix}, \\ M &= \left[(h_\mu X_1 + ih\theta_\mu X_1 + A_\mu)^2 + (h_\lambda X_1 + ih\theta_\lambda X_1 + A_\lambda)^2\right]. \end{aligned}$$

From here we have

$$\begin{vmatrix} K & L \\ L & M \end{vmatrix} = \begin{bmatrix} (h_\mu X_2 + ih\theta_\mu X_2 + B_\mu)(h_\lambda X_1 + ih\theta_\lambda X_1 + A_\lambda) \\ - (h_\lambda X_2 + ih\theta_\lambda X_2 + B_\lambda)(h_\mu X_1 + ih\theta_\mu X_1 + A_\mu) \end{bmatrix}^2 > 0$$

and this indicates the equation of an ellipse. □

Corollary 2.8. *If $h(\lambda,\mu)$ is a constant different from zero, then the equation (34) is an ellipse equation as follows*

$$\begin{aligned} &\left[(ih\theta_\mu X_2 + B_\mu)^2 + (ih\theta_\lambda X_2 + B_\lambda)^2\right]\dot{Y}_1^2 + \\ +&\left[(ih\theta_\mu X_1 + A_\mu)^2 + (ih\theta_\lambda X_1 + A_\lambda)^2\right]\dot{Y}_2^2 - \\ -2&\begin{bmatrix} (ih\theta_\mu X_2 + B_\mu)(ih\theta_\mu X_1 + A_\mu) \\ + (ih\theta_\lambda X_1 + A_\lambda)(ih\theta_\lambda X_2 + B_\lambda) \end{bmatrix}\dot{Y}_1\dot{Y}_2 = \Gamma^2. \end{aligned} \tag{35}$$

Corollary 2.9. *If $h(\lambda,\mu) = 1$ is written in the equation (34), then we obtain an equation for the two parameter motions in the complex plane as follows [13]*

$$\begin{aligned} &\left[(i\theta_\mu X_2 + B_\mu)^2 + (i\theta_\lambda X_2 + B_\lambda)^2\right]\dot{Y}_1^2 + \\ +&\left[(i\theta_\mu X_1 + A_\mu)^2 + (i\theta_\lambda X_1 + A_\lambda)^2\right]\dot{Y}_2^2 - \\ -2&\begin{bmatrix} (i\theta_\mu X_2 + B_\mu)(i\theta_\mu X_1 + A_\mu) \\ + (i\theta_\lambda X_1 + A_\lambda)(i\theta_\lambda X_2 + B_\lambda) \end{bmatrix}\dot{Y}_1\dot{Y}_2 = \Gamma^2, \end{aligned} \tag{36}$$

2.2. **Accelerations and Composition of Accelerations.** The relative acceleration vector of the point $X(\lambda,\mu)$ is the acceleration vector of the point $X(\lambda,\mu)$ with respect to the moving plane. When the vectorial velocity $\overrightarrow{X_r}$ is derived with respect to λ and μ, then the relative acceleration vector is obtained. Therefore, from the equation (4) it is written that

$$\overrightarrow{b_r} = \dot{X}_r = \ddot{X}(\lambda,\mu) = X_{\lambda\lambda}\ddot{\lambda} + X_\lambda\ddot{\lambda} + X_{\lambda\mu}\dot{\lambda} + X_{\mu\lambda}\dot{\mu} + X_{\mu\mu}\dot{\mu} + X_\mu\ddot{\mu} \tag{37}$$

and this vector is expressed with respect to the fixed coordinate plane as follows,

$$\overrightarrow{b_r}' = \overrightarrow{b_r}e^{i\theta} = \ddot{X}e^{i\theta}. \tag{38}$$

The absolute acceleration vector of the point $X(\lambda,\mu)$ is the acceleration vector of the point $X(\lambda,\mu)$ with respect to the fixed plane. By taking the equations (5) and (12) in the equation (10), we have the absolute velocity as follows;

$$\overrightarrow{Y_a} = \overrightarrow{Y_f} + h\overrightarrow{Y_r} = (\dot{h} + ih\dot{\theta})(X - P)e^{i\theta} + h\dot{X}e^{i\theta}.$$

When this absolute velocity is derived respect to λ and μ, then the absolute acceleration vector of the point $X(\lambda,\mu)$ is obtained. Therefore,

$$\overrightarrow{b_a}' = \left[\ddot{h} - h\dot{\theta}^2 + i(h\ddot{\theta} + 2\dot{h}\dot{\theta})\right](X - P)e^{i\theta} - \left(\dot{h} + ih\dot{\theta}\right)\dot{P}e^{i\theta} + 2\dot{X}\left(\dot{h} + ih\dot{\theta}\right)e^{i\theta} + h\overrightarrow{b_r}'. \tag{39}$$

Here, the sliding acceleration vector of the point $X(\lambda,\mu)$ is

$$\overrightarrow{b_f}' = \left[\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})\right](X-P)\,e^{i\theta}-\left(\dot{h}+ih\dot{\theta}\right)\dot{P}e^{i\theta} \tag{40}$$

and the Coriolis acceleration vector of the point $X(\lambda,\mu)$ is

$$\overrightarrow{b_c}' = 2\dot{X}\left(\dot{h}+ih\dot{\theta}\right)e^{i\theta}. \tag{41}$$

Hence, the sliding acceleration vector is the acceleration of the fixed point in the moving system with respect to the fixed system. Therefore, the composition of these accelerations can be given from the equations (38), (39), (40) and (41) with the following theorem.

Theorem 2.8. *There is the following relation between the acceleration vectors of any points of two parameter complex motions.*

$$\overrightarrow{b_a}' = \overrightarrow{b_f}' + \overrightarrow{b_c}' + h\overrightarrow{b_r}' \tag{42}$$

where

$$\overrightarrow{b_a} = \overrightarrow{b_a}'e^{-i\theta} = \left[\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})\right](X-P)-\left(\dot{h}+ih\dot{\theta}\right)\dot{P}+2\dot{X}\left(\dot{h}+ih\dot{\theta}\right)+h\overrightarrow{b_r} \tag{43}$$

$$\overrightarrow{b_f} = \overrightarrow{b_f}'e^{-i\theta} = \left[\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})\right](X-P)-\left(\dot{h}+ih\dot{\theta}\right)\dot{P} \tag{44}$$

and

$$\overrightarrow{b_c} = \overrightarrow{b_c}'e^{-i\theta} = 2\dot{X}\left(\dot{h}+ih\dot{\theta}\right) \tag{45}$$

are the equations of the absolute, the sliding and the Coriolis acceleration vectors with respect to the moving system, respectively.

Theorem 2.9. *The acceleration pole at the position of $\forall(\lambda\mu)$, which angular velocity in the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} is different from zero, is*

$$X = P + \frac{\left(\dot{h}+ih\dot{\theta}\right)\dot{P}}{\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})}. \tag{46}$$

Proof. Let us search the points where the sliding accelerations are zero at the position of $\forall(\lambda\mu)$. From the equation (40) we can say

$$\left[\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})\right](X-P)\,e^{i\theta}-\left(\dot{h}+ih\dot{\theta}\right)\dot{P}e^{i\theta}=0$$

and from here

$$X = P + \frac{\left(\dot{h}+ih\dot{\theta}\right)\dot{P}}{\ddot{h}-h\dot{\theta}^2+i(h\ddot{\theta}+2\dot{h}\dot{\theta})}$$

is obtained. □

Corollary 2.10. *If $h(\lambda,\mu)=1$, then we obtain the following acceleration pole for the two parameter motions in the complex plane [13]*

$$X = P+\frac{i\dot{\theta}\dot{P}}{i\ddot{\theta}-\dot{\theta}^2}, \tag{47}$$

Theorem 2.10. *If $\dot{\lambda}=\dot{\mu}=0$, then the acceleration poles of the complex homothetic motion B_I obtained from the complex homothetic motion B_{II} at the position of $\lambda=\mu=0$ are on the following line*

$$\begin{aligned}&(h_\lambda B_\mu + ih\theta_\lambda B_\mu - ih\theta_\mu B_\lambda - h_\mu B_\lambda)\,P_{i_1}+\\ &+(h_\mu A_\lambda + ih\theta_\mu A_\lambda - ih\theta_\lambda A_\mu - h_\lambda A_\mu)\,P_{i_2} = A_\lambda B_\mu - A_\mu B_\lambda.\end{aligned} \tag{48}$$

Proof. From the differentiation of the equality of (6) we obtain

$$\overrightarrow{b_f'} = \left(\ddot{h} + i\dot{h}\dot{\theta} + ih\ddot{\theta}\right) X e^{i\theta} + i\dot{\theta}(\dot{h} + ih\dot{\theta}) X e^{i\theta} - \left(\ddot{C} + i\ddot{\theta}C + i\dot{\theta}\dot{C}\right) e^{i\theta} - i\dot{\theta}\left(\dot{C} + i\dot{\theta}C\right) e^{i\theta}$$

and if $(\lambda,\mu) = (0,0)$ and $\dot{\lambda} = \dot{\mu} = 0$ are substituted into the last equation and simplified it, then $\overrightarrow{b_f}' = \left(\ddot{h} + ih\ddot{\theta}\right) X - \ddot{C}$ is obtained. Hence, the acceleration pole is

$$P_{i_1} = X_1 = \frac{-A_\lambda\ddot{\lambda} - A_\mu\ddot{\mu}}{h_\lambda\ddot{\lambda} + h_\mu\ddot{\mu} + ih\left(\theta_\lambda\ddot{\lambda} + \theta_\mu\ddot{\mu}\right)} \tag{49}$$

and

$$P_{i_2} = X_2 = \frac{-B_\lambda\ddot{\lambda} - B_\mu\ddot{\mu}}{h_\lambda\ddot{\lambda} + h_\mu\ddot{\mu} + ih\left(\theta_\lambda\ddot{\lambda} + \theta_\mu\ddot{\mu}\right)}. \tag{50}$$

Here, if $\frac{\ddot{\lambda}}{\ddot{\theta}}$ is taken from the equality of P_{i_2} and substituted into the equality of P_{i_1}, then we obtain

$$\frac{-B_\mu - ihP_{i_2}\theta_\mu - P_{i_2}h_\mu}{P_{i_2}h_\lambda + ihP_{i_2}\theta_\lambda + B_\lambda} = \frac{-A_\mu - ihP_{i_1}\theta_\mu - P_{i_1}h_\mu}{P_{i_1}h_\lambda + ihP_{i_1}\theta_\lambda + A_\lambda}$$

and from here we get the following line equation

$$\begin{aligned} &\left(-h_\lambda B_\mu - ih\theta_\lambda B_\mu + ih\theta_\mu B_\lambda + h_\mu B_\lambda\right) P_{i_1} + \\ &+ \left(-h_\mu A_\lambda - ih\theta_\mu A_\lambda + ih\theta_\lambda A_\mu + h_\lambda A_\mu\right) P_{i_2} = A_\lambda B_\mu - A_\mu B_\lambda. \end{aligned} \tag{51}$$

□

Corollary 2.11. *If $h(\lambda,\mu) = 1$, then we obtain the following acceleration pole for the two parameter motions in the complex plane*

$$\left(\theta_\lambda B_\mu - \theta_\mu B_\lambda\right) P_{i_1} + \left(\theta_\mu A_\lambda - \theta_\lambda A_\mu\right) P_{i_2} = i\left(A_\lambda B_\mu - A_\mu B_\lambda\right) \tag{52}$$

[13], and this acceleration pole and the pole lines of the fixed and the moving plane are congruent.

3. Conclusion

The results we have presented deal with complex homothetic motions in which position of the moving object depend on two parameter. Hodograph of two parameter complex homothetic motions was obtained. Hodograph is the locus of the end points of the velocity of a particle and it is the solution of the first order equation which is Newton's Law. The locus of the hodograph of complex homothetic motion was found as an ellipse in this study.

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