TWMS J. Pure Appl. Math. V.4, N.2, 2013, pp.194-203

ON THE QUATERNIONIC CURVES ACCORDING TO PARALLEL TRANSPORT FRAME*

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Abstract. In this paper, we have studied parallel transport frame for a quaternionic curve in $\mathbb{E}^3$ and $\mathbb{E}^4$. Firstly, we have defined a new kind of slant helix with respect to parallel transport frame and given some necessary and sufficient conditions for the quaternionic slant helix in $\mathbb{E}^3$. We have introduced a new definition of harmonic curvature functions in terms of $\boldsymbol{M}_3$ according to parallel transport frame and defined quaternionic $\boldsymbol{M}_3-$slant helix by using the new harmonic curvature functions in $\mathbb{E}^4$.

Keywords: Parallel transport frame, quaternionic curves, quaternionic $\boldsymbol{M}_3-$slant helix, rotations, quaternionic space.

AMS Subject Classification: 11R52, 53A04, 14H45.

1. Introduction

In 1843, quaternions were invented by William Rowan Hamilton who extended 3-dimensional vector algebra for inclusion of multiplications and divisions, [6]. Mathematically, quaternions provide us with a simple and elegant representation for describing finite rotations in space. They are defined with the aid of one real and three imaginary components; $+1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ where ${\boldsymbol{e}_1}^2 = {\boldsymbol{e}_2}^2 = {\boldsymbol{e}_3}^2 = -1$.

Özdamar and Hacısalihoğlu defined harmonic curvature functions. They generalized the inclined curves in $\mathbb{E}^3$ to $\mathbb{E}^n$, $n > 3$, and then gave a characterization for them: "If a curve α is an inclined curve then $\sum_{i=1}^{n-2} {H_i}^2 = \text{constant}$" [11].

Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean space $\mathbb{E}^3$, [8]. After them, Önder et al. defined a new kind of slant helix in Euclidean 4-space $\mathbb{E}^4$ which they called $\boldsymbol{B}_2-$slant helix and they gave characterizations of this slant helices in $\mathbb{E}^4$ [10].

As a set, the set of quaternions $\mathbb{Q}$ coincide with $\mathbb{E}^4$, a 4-dimensional vector space over the real numbers. Considering this feature of quaternions, the Serret-Frenet formulae of a curve in 3-dimensional real Euclidean space $\mathbb{E}^3$ were given by Bharathi and Nagaraj with the help of spatial quaternions. By means of these formulae, the Serret-Frenet formulae of the quaternionic curves were obtained, [1]. Many studies have been made after this work. One of them was made by Karadağ and Sivridağ who defined inclined curves and harmonic curvatures of quaternion

*This work is presented in the 1st International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2012)

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Manuscript received August 2012.

valued functions, [9]. And, Gök et al.'s defined a new kind of quaternionic slant helix called $\boldsymbol{B}_2-$-slant helix in $\mathbb{E}^4$, [4].

It is well known that the Frenet Frame can be established only for differentiable curves. But at some points, curvature of the curve may vanish, in the other words the second derivative of the curve may be equal to zero. In this situation, we need an alternative frame for the curve. For this reason, Bishop defined an alternative frame for the curves in 3-dimensional Euclidean space, [2]. The Frenet frame is completely local but is indeterminable where the curve is locally straight. The other coordinate frame, the parallel transport frame, is defined everywhere but depends on a numerical integration over the entire curve, [7]. The advantages of the parallel transport frame (also called Bishop frame) and the comparable Bishop frame with the Frenet frame in Euclidean 3-space were given by Bishop [2] and Hanson [7].

Bükçü and Karacan defined slant helix according to Bishop frame in Euclidean 3-space, [3]. Then, Gökçelik et al.'s gave the relations between the parallel transport frame and Frenet frame of a curve in 4 - dimensional Euclidean space. Then, they characterized curves whose position vectors lie in their normal, rectifying and osculating planes in $\mathbb{E}^4$, [5].

In this paper, firstly, we have given a new kind of slant helix with respect to parallel transport frame which we call spatial quaternionic slant helix and some necessary and sufficient conditions for the spatial quaternionic slant helix in $\mathbb{E}^3$. We have introduced a new definition of harmonic curvature functions in terms of $\boldsymbol{M}_3$ according to parallel transport frame and defined a new kind of slant helix which we call quaternionic $\boldsymbol{M}_3-$slant helix by using the new harmonic curvature functions in $\mathbb{E}^4$.

2. Preliminaries

A real quaternion is defined as $q = d + a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ where

$$\begin{array}{ll} i) & \boldsymbol{e}_i \times \boldsymbol{e}_i = -\boldsymbol{e}_4, \quad \boldsymbol{e}_4 = +1, \quad (1 \leq i \leq 3) \\ ii) & \mathbf{e}_i \times \boldsymbol{e}_j = \boldsymbol{e}_k = -\boldsymbol{e}_j \times \boldsymbol{e}_i \quad (1 \leq i, j \leq 3). \end{array}$$

Here, a, b, c, d are components of the quaternion $q \in \mathbb{Q}$. Also, if we take $S_q = d$ and $\boldsymbol{V}_q =$ $= a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, a quaternion can be expressed as $q = S_q + \boldsymbol{V}_q$. If we get two quaternions p and q, their quaternionic product is defined as follows;

$$p \times q = S_p S_q - \langle \boldsymbol{V}_p, \boldsymbol{V}_q \rangle + S_p \boldsymbol{V}_q + S_q \boldsymbol{V}_p + \boldsymbol{V}_p \wedge \boldsymbol{V}_q.$$

A feature of quaternions is that the product of two quaternions is non-commutative. $\widehat{q}$ denotes the conjugate of the quaternion q and it is defined as $\hat{q} = S_q - \boldsymbol{V}_q$. In the set of quaternions, the function h which is real valued, symmetric and bilinear is defined as

$$\begin{array}{l} h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}, \\ (p, q) \to h(p, q) = \frac{1}{2}(p \times \widehat{q} + q \times \widehat{p}) \end{array}$$

and this function is called quaternionic inner product. In addition the norm of a quaternion is

$$N(q)^2 = h(q, q) = q \times \widehat{q} = d^2 + a^2 + b^2 + c^2 \ .$$

If $N(q) = 1$, q is entitled unit quaternion. Also if $q + \widehat{q} = 0$ for $q \in \mathbb{Q}$, q is called a spatial quaternion. The set of all spatial quaternions is isomorphic to 3-dimensional real vector space $\mathbb{R}^3$. So, the quaternionic product of two quaternions p and q can be written as $p \times q = -\langle p, q \rangle + p \wedge q$.

We say that a normal vector field $\boldsymbol{m}$ along a curve is relatively parallel if its derivative is tangential, [2]. We use $\boldsymbol{t}(s)$ and two relatively parallel vector fields $\boldsymbol{m}_1(s)$ and $\boldsymbol{m}_2(s)$ to construct an alternative frame. This frame is called parallel transport frame along the curve α.

The derivative formulae for the parallel transport frame can be given in the following matrix form:

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{m}_1{}' \\ \boldsymbol{m}_2{}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{pmatrix}.$$

There are a lot of methods used to represent rotations like orthonormal matrices, Euler angles and quaternions. Quaternions are the most useful method to represent rotations. Unit quaternions play an important role during the transition between quaternions and rotations as they have the remarkable property of capturing all of the geometry and group structure of rotations in the simplest possible way. Every unit quaternion represents a rotation in the Euclidean space. Using a quaternion $q = (q_1, q_2, q_3, q_4)$, we can generate a rotation matrix with

$$R = \begin{pmatrix} {q_4}^2 + {q_2}^2 - {q_3}^2 - {q_1}^2 & -2q_1q_4 + 2q_2q_3 & 2q_4q_3 + 2q_2q_1 \\ 2q_2q_3 + 2q_4q_1 & {q_4}^2 - {q_2}^2 + {q_3}^2 - {q_1}^2 & 2q_3q_1 - 2q_2q_4 \\ 2q_2q_1 - 2q_3q_4 & 2q_2q_4 + 2q_3q_1 & {q_4}^2 - {q_2}^2 - {q_3}^2 + {q_1}^2 \end{pmatrix} \tag{1}$$

for the given rotation in the Euclidean space, and we can represent these rotations for the standard coordinate axes with the unit quaternions :

$$q_y = (0, \sin\frac{\phi}{2}, 0, \cos\frac{\phi}{2}), q_x = (\sin\frac{\theta}{2}, 0, 0, \cos\frac{\theta}{2}), q_z = (0, 0, \sin\frac{\psi}{2}, \cos\frac{\psi}{2}), \tag{2}$$

respectively, [12].

Finite rotations are described by 3×3 rotational transformation matrices with respect to standard basis in 3-dimensional Euclidean space $\mathbb{E}^3$. The 3-dimensional special orthogonal group $SO(3)$ is formed by these matrices. The groups of unit real quaternions are isomorphic to the topological 3-sphere S^3, which is also the topological space of the Lie group $SU(2)$ in ordinary 3-dimensional Euclidean space $\mathbb{E}^3$. The relationship between the Euclidean projective space RP^3 and $SO(3)$ is given as $RP^3 = SO(3) \cong S^3/\{\pm 1\}$.

3. Spatial quaternionic slant helix according to parallel transport frame

Let $\{\alpha \in \mathbb{Q} \,|\, \alpha + \hat{\alpha} = 0\}$ be the space of spatial quaternions in 3-dimensional Euclidean space $\mathbb{E}^3$ and the spatial quaternionic curve α be given by

$$\begin{aligned} &\alpha : I \subset \mathbb{R} \to \mathbb{E}^3 \\ &\qquad s \to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i, \qquad (1 \le i \le 3) \end{aligned}$$

where $I = [0, 1] \subset \mathbb{R}$ and $s \in I$ is arc length parameter. $\{\boldsymbol{t}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$ denotes the Frenet frame and $k(s), r(s)$ are curvatures at the point $\alpha(s)$ of the curve α for $\forall s \in I$, [1]. Then the following theorem can be given.

Theorem 3.1. *Let $\{\boldsymbol{t}, \boldsymbol{n}_1, \boldsymbol{n}_2\}$ be Frenet frame of spatial quaternionic curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ at the point $\alpha(s)$. Then, the Frenet formulae of α are given by*

$$\begin{aligned} \boldsymbol{t}' &= k\boldsymbol{n}_1, \\ \boldsymbol{n}_1{}' &= -k\boldsymbol{t} + r\boldsymbol{n}_2, \\ \boldsymbol{n}_2{}' &= -r\boldsymbol{n}_1, \end{aligned} \tag{3}$$

where $\{k(s), r(s)\}$ denote curvatures of the curve α, [1].

Theorem 3.2. *Let the spatial quaternionic curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be given by arc length parameterization by parameter s. For $\forall s \in I$, $\{\boldsymbol{t}(s), \boldsymbol{m}_1(s), \boldsymbol{m}_2(s)\}$ denotes parallel transport frame of the curve α and $\{k_1, k_2\}$ are the curvatures with respect to this frame. The parallel transport formulae along the curve α are*

$$\begin{aligned} \boldsymbol{t}'(s) &= k_1(s)\boldsymbol{m}_1(s) + k_2(s)\boldsymbol{m}_2(s), \\ \boldsymbol{m}_1{}'(s) &= -k_1(s)\boldsymbol{t}(s), \\ \boldsymbol{m}_2{}'(s) &= -k_2(s)\boldsymbol{t}(s), \end{aligned}$$

where

$$\begin{aligned} k(s) &= \sqrt{{k_1}^2 + {k_2}^2}, \\ \psi(s) &= \arctan(\tfrac{k_2}{k_1}), \\ r(s) &= \tfrac{d\psi}{ds}. \end{aligned}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, ψ with $\psi = \int r(s)ds$, [7].

Definition 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed spatial quaternionic curve and $\boldsymbol{U}$ is a unit spatial quaternion with fixed direction. For $\forall s \in I$, if*

$$h(\boldsymbol{m}_1(s), \boldsymbol{U}) = \cos\varphi, \quad \varphi = constant$$

holds, α is called a spatial quaternionic slant helix.

Theorem 3.3. *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed spatial quaternionic curve with nonzero natural curvatures $\{k_1, k_2\}$. Then α is a spatial quaternionic slant helix if and only if $\frac{k_1}{k_2}$ is constant.*

Proof. Let α be spatial quaternionic slant helix in $\mathbb{E}^3$ with nonzero natural curvatures $\{k_1, k_2\}$. So, from the Definition 3.1, we get

$$h\left(\boldsymbol{m}_1, \boldsymbol{U}\right) = constant,$$

where $\boldsymbol{U}$ is a unit spatial quaternion, called the axis of spatial quaternionic slant helix. By differentiation the last equation, we get

$$h(\boldsymbol{m}_1{}', \boldsymbol{U}) = h(-k_1\boldsymbol{t}, \boldsymbol{U}) = -k_1 h(\boldsymbol{t}, \boldsymbol{U}) = 0.$$

Since, $k_1 \neq 0$, we can easily write

$$h(\boldsymbol{t}, \boldsymbol{U}) = 0.$$

If we take again derivative of the last equation, we can find as follows

$$\begin{aligned} h(\boldsymbol{t}', \boldsymbol{U}) &= h(k_1\boldsymbol{m}_1 + k_2\boldsymbol{m}_2, \boldsymbol{U}), \\ &= k_1 h(\boldsymbol{m}_1, \boldsymbol{U}) + k_2 h(\boldsymbol{m}_2, \boldsymbol{U}), \\ &= k_1 cos\varphi + k_2 sin\varphi = 0. \end{aligned}$$

Therefore we obtain that $\frac{k_1}{k_2} = -\tan\varphi = constant$.

Conversely, suppose that $\frac{k_1}{k_2} = -\tan\varphi$. Then we can write $\boldsymbol{U} \in Sp\left\{\boldsymbol{m}_1, \boldsymbol{m}_2\right\}$, i.e.,

$$\boldsymbol{U} = \boldsymbol{m}_1 \cos\varphi + \boldsymbol{m}_2 \sin\varphi.$$

Differentiating the last equality,

$$\boldsymbol{U}' = (k_1 \cos\varphi + k_2 \sin\varphi)\boldsymbol{t} = 0.$$

So $\boldsymbol{U}$ is a constant quaternion. Thus, the proof is done. □

Theorem 3.4. *Let $\alpha = \alpha(s)$ be unit speed spatial quaternionic in $\mathbb{E}^3$. Then α is a spatial quaternionic slant helix if*

$$\det(\boldsymbol{m}_1', \boldsymbol{m}_2'', \boldsymbol{m}_3''') = 0.$$

Proof. Let α be a spatial quaternionic slant helix. From Theorem 3.3. suppose that $\frac{k_1}{k_2}$ be constant. From the parallel transport formulae, we have equalities as

$$\begin{aligned}
&-\boldsymbol{m}_1{}' = k_1\boldsymbol{t}\\
&-\boldsymbol{m}''{}_1 = k_1{}'\boldsymbol{t} + k_1^2\boldsymbol{m}_1 + k_1k_2\boldsymbol{m}_2\\
&-\boldsymbol{m}'''{}_1 = \left(k_1{}'' - k_1^3 - k_1{}^2k_2\right)t + \left(3k_1k_1{}'\right)\boldsymbol{m}_1 + \left(2k_1{}'k_2 + k_1k_2{}'\right)\boldsymbol{m}_2.
\end{aligned}$$

So we get

$$\begin{aligned}
\det(\boldsymbol{m}_1{}', \boldsymbol{m}_2{}'', \boldsymbol{m}_3{}''') &= \begin{bmatrix} k_1 & 0 & 0 \\ k_1{}' & k_1^2 & k_1k_2 \\ k_1{}'' - k_1^3 - k_1{}^2k_2 & 3k_1k_1{}' & 2k_1{}'k_2 + k_1k_2{}' \end{bmatrix} = \\
&= -k_1{}^3k_2{}^2\left(\frac{k_1}{k_2}\right)',
\end{aligned}$$

where $k_2 \neq 0$. Since $\frac{k_1}{k_2}$ is constant, we have $\det(\boldsymbol{m}_1{}', \boldsymbol{m}_2{}'', \boldsymbol{m}_3{}''') = 0$.

Conversely, suppose that $\det(\boldsymbol{m}_1{}', \boldsymbol{m}_2{}'', \boldsymbol{m}_3{}''') = -k_1{}^3k_2{}^2\left(\frac{k_1}{k_2}\right)' = 0$. Then it is clear that $\frac{k_1}{k_2} = constant$. So, α is a spatial quaternionic slant helix. □

4. Quaternionic M_3–slant helix according to parallel transport frame

Let $\mathbb{Q}$ denote the set of real quaternions and we choose $I = [0, 1] \subset \mathbb{R}$. β is called a quaternionic curve if it is given by

$$\begin{aligned}
&\beta : I \subset \mathbb{R} \to \mathbb{Q},\\
&s \to \beta(s) = \sum_{i=1}^{4} \beta_i(s)\boldsymbol{e}_i, \qquad (1 \leq i \leq 4), \quad \boldsymbol{e}_4 = 1
\end{aligned}$$

for $\forall s \in I$. For the arc length parameter s, the Frenet vectors at the point $\beta(s)$ is $\{\boldsymbol{T}(s), \boldsymbol{N}_1(s), \boldsymbol{N}_2(s), \boldsymbol{N}_3(s)\}$ and $\{\kappa,\ k,\ (r - \kappa)\}$ denote the Frenet curvatures of the quaternionic curve β. Then the following theorem can be given.

Theorem 4.1. *Let $\{\boldsymbol{T}(s), \boldsymbol{N}_1(s), \boldsymbol{N}_2(s), \boldsymbol{N}_3(s)\}$ be Frenet frame of the curve $\beta : I \subset \mathbb{R} \to \mathbb{Q}$ at the point $\beta(s)$. Frenet formulae of the quaternionic curve β are expressed as*

$$\begin{array}{lll}
\boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}_1(s), & \kappa(s) = \|\boldsymbol{T}'(s)\|, & \boldsymbol{N}_1(s) = \boldsymbol{t}(s) \times \boldsymbol{T}(s) \\
\boldsymbol{N}_1{}'(s) = -\kappa(s)\boldsymbol{T}(s) + k(s)\boldsymbol{N}_2(s)\ , & & \boldsymbol{N}_2(s) = \boldsymbol{n}_1(s) \times \boldsymbol{T}(s) \\
\boldsymbol{N}_2{}'(s) = -k(s)\boldsymbol{N}_1(s) + (r(s) - \kappa(s))\,\boldsymbol{N}_3(s), & & \boldsymbol{N}_3(s) = \boldsymbol{n}_2(s) \times \boldsymbol{T}(s) \\
\boldsymbol{N}_3{}'(s) = -\left(r(s) - \kappa(s)\right)\boldsymbol{N}_2(s)\ . & &
\end{array} \tag{4}$$

Here, the unit tangent vector $\boldsymbol{T}$ of the quaternionic curve β is given by the relation $\boldsymbol{t}(s) = = \boldsymbol{N}_1(s) \times \hat{\boldsymbol{T}}(s)$. So, the torsion of the quaternionic curve β is the principal curvature of the spatial quaternionic curve α. In addition, the third curvature of β is $(r(s) - \kappa(s))$ where $r(s)$ is the torsion of the spatial curve α and $\kappa(s)$ is the principal curvature of β, [1].

Theorem 4.2. *Let $\beta : I \subset R \to Q$ be a quaternionic curve with arc length parameter s and $\{\boldsymbol{T}, \boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3\}$ denotes Frenet frame of the quaternionic curve. Also, $\{\boldsymbol{T}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3\}$*

denotes parallel transport frame of β. The matrix form of the parallel transport formulae of the quaternionic curve β is

$$\begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{M_1}' \\ \boldsymbol{M_2}' \\ \boldsymbol{M_3}' \end{pmatrix} = \begin{pmatrix} 0 & {k_1}^* & {k_2}^* & {k_3}^* \\ -{k_1}^* & 0 & 0 & 0 \\ -{k_2}^* & 0 & 0 & 0 \\ -{k_3}^* & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{M_1} \\ \boldsymbol{M_2} \\ \boldsymbol{M_3} \end{pmatrix}, \tag{5}$$

where $\{{k_1}^*, {k_2}^*, {k_3}^*\}$ are nonzero curvatures of the quaternionic curve β according to parallel transport frame. The following equations hold,

$$\begin{aligned} {k_1}^* &= \kappa(\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi)\ , \\ {k_2}^* &= -\kappa\sin\psi\cos\theta\ , \\ {k_3}^* &= \kappa(\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi)\ . \end{aligned}$$

Moreover, the relation between the angles and curvatures is

$$\theta' = -\frac{(r-\kappa)}{\sqrt{\kappa^2+k^2}}, \qquad \phi' = \frac{\sqrt{(r-\kappa)^2-(\theta')^2}}{\cos\theta}, \qquad \psi' = -k - \tan\theta\sqrt{(r-\kappa)^2-(\theta')^2}$$

and the Frenet curvatures may be expressed as

$$\kappa(s) = \sqrt{({k_1}^*)^2 + ({k_2}^*)^2 + ({k_3}^*)^2}\ , k = -\psi' + \theta'\tan\psi\tan\theta\ , r - \kappa = -\frac{\theta'}{\cos\psi}\ , \phi'\cos\theta + \theta'\tan\psi = 0\ .$$

Proof. By combining the results of the quaternionic rotation matrix in (1) and unit quaternions in the equation (2), the relation between the Frenet frame and the parallel transport frame at the point $\beta(s)$ of the quaternionic curve in 4-dimensional Euclidean space can be given by

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \\ \boldsymbol{N}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi & -\sin\psi\cos\theta & \cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ 0 & \cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi & \cos\psi\cos\theta & -\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ 0 & -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \\ \boldsymbol{M}_3 \end{pmatrix}, \tag{6}$$

where ϕ, θ and ψ are Euler angles. If we make the necessary arrangement in the equations given by (6), we obtain that

$$\begin{aligned} &\boldsymbol{T} = \boldsymbol{T}(s), \\ &\boldsymbol{M}_1 = (\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi)\boldsymbol{N}_1 + (\cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi)\boldsymbol{N}_2 - \\ &\qquad - \cos\theta\sin\phi\boldsymbol{N}_3, \\ &\boldsymbol{M}_2 = -\sin\psi\cos\theta\ \boldsymbol{N}_1 + \cos\psi\cos\theta N_2 + \sin\theta N_3, \\ &\boldsymbol{M}_3 = (\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi)\boldsymbol{N}_1 + (-\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi)\boldsymbol{N}_2 + \\ &\qquad + \cos\theta\cos\phi\boldsymbol{N}_3. \end{aligned} \tag{7}$$

Considering the equation (4) and differentiating the equations (7), we have

$$\begin{aligned} \boldsymbol{M}_1' = &-\kappa(\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi)\,\boldsymbol{T} + \\ &+ (-k(\cos\psi\sin\theta\sin\phi + \sin\psi\sin\phi) + \psi'(-\sin\psi\cos\phi - \cos\psi\sin\theta\sin\phi) + \\ &+ \phi'(-\cos\psi\sin\phi - \sin\psi\sin\theta\cos\phi) - \theta'\sin\psi\cos\theta\sin\phi)\boldsymbol{N}_1 + \\ &+ (k(\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi) + \\ &+ (r-\kappa)\cos\theta\sin\phi + \psi'(-\sin\psi\sin\theta\sin\phi + \cos\psi\sin\phi) + \\ &+ \phi'(\cos\psi\sin\theta\cos\phi + \sin\psi\cos\phi) + \theta'\cos\psi\cos\theta\sin\phi)\boldsymbol{N}_2 + \\ &+ ((r-\kappa)(\cos\psi\sin\theta\sin\phi + \sin\psi\sin\phi) + \theta'\sin\theta\sin\phi - \phi'\cos\theta\cos\phi)\boldsymbol{N}_3. \end{aligned}$$

In a similar way, we get ${M_2}'$ and ${M_3}'$ as

$$\begin{aligned}\boldsymbol{M}_2{}' = {} & \kappa \sin\psi \cos\theta \boldsymbol{T} + \\ & + (-k\cos\psi\cos\theta - \psi'\cos\psi\cos\theta + \theta'\sin\psi\sin\theta)\boldsymbol{N}_1 + \\ & + [-k\sin\psi\cos\theta - (r-\kappa)\sin\theta - \psi'\sin\psi\cos\theta - \theta'\cos\psi\sin\theta]\boldsymbol{N}_2 + \\ & + [(r-\kappa)\cos\psi\cos\theta + \theta'\cos\theta]\boldsymbol{N}_3,\end{aligned}$$

$$\begin{aligned}\boldsymbol{M}_3{}' = {} & -\kappa(\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi)\boldsymbol{T} + \\ & + (-k(-\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) + \psi'(-\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi) + \\ & + \phi'(\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi) + \theta'\sin\psi\cos\theta\cos\phi)\boldsymbol{N}_1 + \\ & + (-(r-\kappa)\cos\theta\cos\phi + k(\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi) + \phi'(\cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi) + \\ & + \psi'(\sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi) - \theta'\cos\psi\cos\theta\cos\phi)\boldsymbol{N}_2 + \\ & + ((r-\kappa)(-\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) - \theta'\cos\phi\sin\theta - \phi'\cos\theta\sin\phi)\boldsymbol{N}_3.\end{aligned}$$

In addition, by considering the equation (7) and using quaternionic inner product with $\boldsymbol{T}'$, we have

$$\begin{aligned} {k_1}^* &= h(\boldsymbol{T}', \boldsymbol{M}_1) = \kappa(\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi)\,, \\ {k_2}^* &= h(\boldsymbol{T}', \boldsymbol{M}_2) = -\kappa\sin\psi\cos\theta\,, \\ {k_3}^* &= h(\boldsymbol{T}', \boldsymbol{M}_3) = \kappa(\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi). \end{aligned} \tag{8}$$

From the equation (8), it is obvious that

$$\kappa(s) = \sqrt{({k_1}^*)^2 + ({k_2}^*)^2 + ({k_3}^*)^2}\ .$$

Moreover, since $\boldsymbol{M}_1, \boldsymbol{M}_2$ and $\boldsymbol{M}_3$ are relatively parallel vector fields, the normal components of the $\boldsymbol{M}_1{}', \boldsymbol{M}_2{}'$ and $\boldsymbol{M}_3{}'$ must be zero. So, $h(\boldsymbol{M}_1{}', \boldsymbol{M}_2) = 0$ and $h(\boldsymbol{M}_1{}', \boldsymbol{M}_3) = 0$. Thus, the following statements hold;

$$k = -\psi' + \theta'\tan\psi\tan\theta\ , \quad r - \kappa = -\frac{\theta'}{\cos\psi}\ , \quad \phi'\cos\theta + \theta'\tan\psi = 0\ .$$

If we choose $\theta' = -\frac{(r-\kappa)}{\sqrt{\kappa^2+k^2}}$, we get $\cos\psi = \frac{1}{\sqrt{\kappa^2+k^2}}$ and it is obvious that

$$\phi' = \frac{\sqrt{(r-\kappa)^2 - (\theta')^2}}{\cos\theta}, \quad \psi' = -k - \tan\theta\sqrt{(r-\kappa)^2 - (\theta')^2}\ .$$

□

Definition 4.1. *Let $\beta : I \to \mathbb{Q}$ be a quaternionic curve with an arc length parameter s. $\boldsymbol{X}$ is a unit quaternion which has constant components and $\{\boldsymbol{T}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3\}$ denotes the parallel transport frame at the point $\beta(s)$. If*

$$h(\boldsymbol{M}_3(s), \boldsymbol{X}) = \cos\varphi, \quad \varphi = constant$$

for $\forall s \in I$, β is called quaternionic $\boldsymbol{M}_3-$slant helix according to parallel transport frame, where φ is a constant angle between the last vector field $\boldsymbol{M}_3$ with a fixed direction $\boldsymbol{X}$.

Definition 4.2. *Let $\beta = \beta(s)$ be a quaternionic curve parameterized by arc length parameter s and $\{{k_1}^*, {k_2}^*, {k_3}^*\}$ be nonzero curvatures according to parallel transport frame. In that case harmonic curvature functions in terms of $\boldsymbol{M}_3$ are defined by*

$$\begin{aligned} & H_i : I \subset \mathbb{R} \to \mathbb{R} \\ & H_1 = 0, \quad H_2 = \tfrac{{k_2}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_2}^*}{{k_1}^{*\prime}{k_2}^* - {k_2}^{*\prime}{k_1}^*}, \quad H_3 = \tfrac{{k_1}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_1}^*}{{k_2}^{*\prime}{k_1}^* - {k_1}^{*\prime}{k_2}^*}. \end{aligned}$$

Theorem 4.3. *Let $\beta : I \to \mathbb{Q}$ be a unit speed quaternionic curve and $\{\boldsymbol{T}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3\}$ denotes the parallel transport frame of the curve β and $\{{k_1}^*, {k_2}^*, {k_3}^*\}$ are nonzero curvatures according to parallel transport frame. If the quaternionic curve β is a $\boldsymbol{M}_3-$slant helix in $\mathbb{Q}$, then*

$$\begin{aligned} h(\boldsymbol{T}(s), \boldsymbol{X}) &= H_1(s)\cos\varphi, \\ h(\boldsymbol{M}_1(s), \boldsymbol{X}) &= H_2(s)\cos\varphi, \\ h(\boldsymbol{M}_2(s), \boldsymbol{X}) &= H_3(s)\cos\varphi, \\ h(\boldsymbol{M}_3(s), \boldsymbol{X}) &= \cos\varphi, \end{aligned}$$

where φ is a constant angle between $\boldsymbol{M}_3$ with a fixed direction $\boldsymbol{X}$.

Proof. Let β be a quaternionic $\mathbf{M}_3-$slant helix in $\mathbb{Q}$. Then $h(\boldsymbol{M}_3(s), \boldsymbol{X}) = \cos\varphi$. Considering parallel transport formulae and differentiating the above equation with respect to s, we obtain that

$$\begin{aligned} h({\boldsymbol{M}_3}'(s), \boldsymbol{X}) &= 0, \\ -{k_3}^* h(\boldsymbol{T}(s), \boldsymbol{X}) &= 0, \quad {k_3}^* \neq 0, \\ h(\boldsymbol{T}(s), \boldsymbol{X}) &= 0. \end{aligned} \tag{9}$$

If we take derivative of the equation (9)

$$\begin{aligned} &h(\boldsymbol{T}'(s), \boldsymbol{X}) = 0, \\ &h({k_1}^*\boldsymbol{M}_1 + {k_2}^*\boldsymbol{M}_2 + {k_3}^*\boldsymbol{M}_3, \boldsymbol{X}) = 0. \end{aligned}$$

Since the quaternionic inner product is linear, from the last equation we acquire

$${k_1}^* h(\boldsymbol{M}_1(s), \boldsymbol{X}) + {k_2}^* h(\boldsymbol{M}_2(s), \boldsymbol{X}) + {k_3}^* h(\boldsymbol{M}_3(s), \boldsymbol{X}) = 0. \tag{10}$$

From the equation (10), we obtain that

$$h(\boldsymbol{M}_2(s), \boldsymbol{X}) = -\frac{{k_1}^*}{{k_2}^*} h(\boldsymbol{M}_1(s), \boldsymbol{X}) - \frac{{k_3}^*}{{k_2}^*} h(\boldsymbol{M}_3(s), \boldsymbol{X}).$$

Again differentiating at the last equation, it is easy to obtain

$$h({\boldsymbol{M}_2}'(s), \boldsymbol{X}) = \frac{{k_2}^{*\prime}{k_1}^* - {k_1}^{*\prime}{k_2}^*}{{k_2}^{*2}} h(\boldsymbol{M}_1(s), \boldsymbol{X}) + \frac{{k_2}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_2}^*}{{k_2}^{*2}} h(\boldsymbol{M}_3(s), \boldsymbol{X}).$$

Considering the equation (5) and (9)

$$h(\boldsymbol{M}_1(s), \boldsymbol{X}) = \frac{{k_2}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_2}^*}{{k_1}^{*\prime}{k_2}^* - {k_2}^{*\prime}{k_1}^*} h(\boldsymbol{M}_3(s), \boldsymbol{X}). \tag{11}$$

Similarly using the equation (10), we find that

$$h(\boldsymbol{M}_1(s), \boldsymbol{X}) = -\frac{{k_2}^*}{{k_1}^*} h(\boldsymbol{M}_2(s), \boldsymbol{X}) - \frac{{k_3}^*}{{k_1}^*} h(\boldsymbol{M}_3(s), \boldsymbol{X}).$$

If we take derivative of the last equation, we have

$$h({\boldsymbol{M}_1}'(s), \boldsymbol{X}) = \frac{{k_1}^{*\prime}{k_2}^* - {k_2}^{*\prime}{k_1}^*}{{k_1}^{*2}} h(\boldsymbol{M}_2(s), \boldsymbol{X}) + \frac{{k_1}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_1}^*}{{k_1}^{*2}} h(\boldsymbol{M}_3(s), \boldsymbol{X}),$$

$$h(\boldsymbol{M}_2(s), \boldsymbol{X}) = \frac{{k_1}^{*\prime}{k_3}^* - {k_3}^{*\prime}{k_1}^*}{{k_2}^{*\prime}{k_1}^* - {k_1}^{*\prime}{k_2}^*} h(\boldsymbol{M}_3(s), \boldsymbol{X}). \tag{12}$$

From the equations (9), (11) and (12) and the Definition 4.2, we obtain that

$$\begin{aligned} h(\boldsymbol{T}(s), \boldsymbol{X}) &= H_1(s) h(\boldsymbol{M}_3(s), \boldsymbol{X}), \\ h(\boldsymbol{M}_1(s), \boldsymbol{X}) &= H_2(s) h(\boldsymbol{M}_3(s), \boldsymbol{X}), \\ h(\boldsymbol{M}_2(s), \boldsymbol{X}) &= H_3(s) h(\boldsymbol{M}_3(s), \boldsymbol{X}), \\ h(\boldsymbol{M}_3(s), \boldsymbol{X}) &= \cos\varphi. \end{aligned}$$

This proves the theorem. $\square$

Theorem 4.4. *Let the quaternionic curve $\beta : I \subset \mathbb{R} \to \mathbb{Q}$ be given by arc length parameterization and $\{\boldsymbol{T}(s), \boldsymbol{M}_1(s), \boldsymbol{M}_2(s), \boldsymbol{M}_3(s)\}$ denotes parallel transport frame of the curve β at the point $\beta(s)$. If $\boldsymbol{X}$ is the axis of the quaternionic curve $\beta(s)$ while $\beta(s)$ is a quaternionic $\boldsymbol{M}_3-$slant helix, then $\boldsymbol{X}$ can be written in the following forms:*

$$\boldsymbol{X} = (H_1 \boldsymbol{T}(s) + H_2 \boldsymbol{M}_1(s) + H_3 \boldsymbol{M}_2(s) + M_3(s))h(\boldsymbol{M}_3(s), \boldsymbol{X}),$$

where $H_i(s)$, $(i = 1, 2, 3)$ are the harmonic curvature functions.

Proof. Suppose that $\boldsymbol{X}$ is the axis of the quaternionic $\boldsymbol{M}_3-$slant helix β. We know that $\boldsymbol{X} \in Sp\{\boldsymbol{T}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3\}$. So, we can write

$$\boldsymbol{X} = \lambda_1 \boldsymbol{T} + \lambda_2 \boldsymbol{M}_1 + \lambda_3 \boldsymbol{M}_2 + \lambda_4 \boldsymbol{M}_3, \qquad \lambda_i \in \mathbb{R}, \quad 1 \leq i \leq 4.$$

As β is quaternionic $\mathbf{M}_3-$slant helix, from Theorem 4.3 we have

$$\begin{aligned} \lambda_1 &= h(\boldsymbol{T}, \boldsymbol{X}) = H_1 \cos\varphi = H_1 h(\boldsymbol{M}_3, \boldsymbol{X}) = 0 \ , \\ \lambda_2 &= h(\boldsymbol{M}_1, \boldsymbol{X}) = H_2 \cos\varphi = H_2 h(\boldsymbol{M}_3, \boldsymbol{X}) \ , \\ \lambda_3 &= h(\boldsymbol{M}_2, \boldsymbol{X}) = H_3 \cos\varphi = H_3 h(\boldsymbol{M}_3, \boldsymbol{X}) \ , \\ \lambda_4 &= h(\boldsymbol{M}_3, \boldsymbol{X}) \ = \cos\varphi. \end{aligned}$$

Therefore, we find that

$$\boldsymbol{X} = (H_1 \boldsymbol{T}(s) + H_2 \boldsymbol{M}_1(s) + H_3 \boldsymbol{M}_2(s) + \boldsymbol{M}_3(s))h(\boldsymbol{M}_3(s), \boldsymbol{X}).$$

This proves the theorem. □

References

[1] Bharathi, K., Nagaraj, M., (1987), Quaternion valued function of a real variable serret-frenet formulae, Indian J. Pure Appl. Math., 18(6), pp.507-511.

[2] Bishop, L.R., (1975), There is more than one way to frame a curve, Amer. Math. Monthly, 82(3), pp.246-251.

[3] Bükçü, B., Karacan, M.K., (2009), The slant helices according to bishop frame, Int. J. Comput. Math. Sci., 3(2), pp.63-66.

[4] Gök, İ., Okuyucu, O.Z., Kahraman, F., Hacısalihoğlu, H.H., (2011), On the quaternionic $\boldsymbol{B}_2-$ slant helices in the euclidean space $\mathbb{E}^4$, Adv. Appl. Clifford Algebr., 21(4), pp.707-719.

[5] Gökçelik, F., Bozkurt, Z., Gök, İ., Ekmekçi, F.N., Yaylı, Y., Parallel transport frame in 4-dimensional euclidean space $\mathbb{E}^4$, arXiv.org > math > arXiv:1207.2999.

[6] Hamilton, W.R., (1899), Elements of Quaternions, I, II and III, Chelsea, New York.

[7] Hanson A.J., (2006), Visualizing Quaternions, Morgan Kaufmann, Elseiver.

[8] Izumiya, S., Takeuchi, N., (2004), New special curves and developable surfaces, Turk. J. Math., 28, pp.153-163.

[9] Karadağ, M., Sivridağ, A.İ., (1997), Tek değişkenli kuaterniyon değerli fonksiyonlar ve eğilim çizgileri, Erciyes Üniversitesi, Fen Bilimleri Dergisi, 13(1-2), pp.23-36.

[10] Önder, M., Kazaz, M., Kocayiğit, H., Kılıç, O., (2008), $\boldsymbol{B}_2-$slant helix in euclidean 4-space $\mathbb{E}^4$, Int. J. Cont. Math. Sci., 3(29), pp.1433-1440.

[11] Özdamar, E., Hacısalihoğlu, H.H., (1975), A characterization of inclined curves in euclidean n-space, Communication de la faculté des sciences de L'Université d'Ankara, séries A1, 24A, pp.15-22.

[12] Shoemake, K., (1985), Animating rotation with quaternion curves, ACM Siggraph, 19(3), pp.245-254.

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