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EXPLICIT PARALLEL RESOLVENT METHODS FOR SYSTEM OF GENERAL VARIATIONAL INCLUSIONS

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ABSTRACT. In this paper, we introduce and consider a new system of extended general variational inclusions involving eight different operators. Using the resolvent operator techniques, we show that the new system of extended general variational inclusions is equivalent to the fixed point problem. We prove the strong convergence of some new explicit iterative parallel algorithms using resolvent methods under certain conditions. Our results improve and extend the recent results by Noor and Noor [11].

Keywords: systems of general variational inclusions, explicit iteration algorithms, strong convergence, Hilbert spaces.

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1. Introduction

Inspired and motivated by current research in this area, we introduce and consider a new system of extended general variational inclusions involving eight different nonlinear operators. This class of system includes the system of general variational inclusions involving seven operators introduced by Noor and Noor [11]. Using the resolvent technique, we have shown that the new system of extended general variational inclusions are equivalent to fixed point problems. This alternative equivalent formulation is used to suggest and analyze some new explicit iterative algorithms for solving this system of variational inclusions. We would like to emphasize that these explicit iterative algorithms are distinctly different from the known methods of Noor and Noor [11] and Noor et al. [13,14]. We suggest and analyze some new explicit iterative parallel algorithms for solving this system. We also prove the strong convergence of the proposed iteration algorithms under suitable conditions. Our results represent a refinement and improvement of the recent results of Noor and Noor [11]. The interested readers are encouraged to find new, novel and innovative applications of variational inequalities and optimization problem in pure and applied sciences. The numerical implementation of the new proposed methods in this paper is another direction for further research.

Let K be a nonempty closed and convex set in a real Hilbert space H, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let T_1, T_2, A, B, g_1, g, h, $h : H \to H$ be eight nonlinear operators.

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We consider the problem of finding $x^*, y^* \in H$ such that

$$\begin{cases} 0 \in \rho T_1(y^*) + \rho A(g_1(x^*)) - g(y^*) + g_1(x^*), & \rho > 0, \\ 0 \in \eta T_2(x^*) + \eta B(h_1(y^*)) + h_1(y^*) - h(x^*), & \eta > 0, \end{cases} \tag{1}$$

which is called the system of general variational inclusions involving eight different operators.

Let us discuss some special cases of the system of general variational inclusions (1).

(i) If $B = A$, then (1) is equivalent to finding $x^*, y^* \in H$ such that

$$\begin{cases} 0 \in \rho T_1(y^*) + \rho A(g_1(x^*)) - g(y^*) + g_1(x^*), & \rho > 0, \\ 0 \in \eta T_2(x^*) + \eta A(h_1(y^*)) + h_1(y^*) - h(x^*), & \eta > 0, \end{cases}$$

which is called the system of general variational inclusions involving seven different operators studied by Noor and Noor [11].

(ii) If $A(\cdot) = \partial\phi_1(\cdot)$, $B(\cdot) = \partial\phi_2(\cdot)$, the subdifferential of proper, convex and lower-semicontinuous functions, then (1) is equivalent to finding $x^*, y^* \in H$ such that $\forall x \in H$,

$$\begin{cases} \rho T_1(y^*) + g_1(x^*) - g(y^*), g(x) - g_1(x^*) \geq \rho\phi_1(g_1(x^*)) - \rho\phi_1(g(x)), & \rho > 0, \\ \eta T_2(x^*) + h_1(y^*) - h(x^*), h(x) - h_1(y^*) \geq \eta\phi_2(h_1(y^*)) - \eta\phi_2(h(x)), & \eta > 0, \end{cases} \tag{2}$$

which is called the system of mixed general variational inequalities involving different operators introduced by Noor [8-10].

(iii) If ϕ is an indicator function of a closed and convex set K in H, and $g_1 = g = h_1 = h = I$, then (2) is equivalent to finding $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \text{ for all } x \in H \text{ and for } \rho > 0, \\ \langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \text{ for all } x \in H \text{ and for } \eta > 0, \end{cases}$$

which is called the system of nonlinear variational inequalities involving two different nonlinear operators studied by Huang and Noor [4].

This shows that the system of extended general variational inclusions involving eight different operators (1) is more general and includes several classes of variational inclusions or variational inequalities as special cases. For the applications, formulation, numerical methods and other aspects of variational inequalities, see [1-18].

Definition 1.1. *Let $\mu > 0$ be a constant. A mapping $T : H \to H$ is called μ-Lipschitizian iff for all $x, y \in H$ one has*

$$\|Tx - Ty\| \leq \mu\|x - y\|.$$

Definition 1.2. *Let $r > 0$ be a constant. A mapping $T : H \to H$ is called r-strongly monotonic iff for all $x, y \in H$, one has*

$$\langle Tx - Ty, x - y \rangle \geq r||x - y||^2.$$

Definition 1.3. *Let $\gamma > 0, r > 0$ be constants. A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive iff for all $x, y \in H$, one has*

$$\langle Tx - Ty, x - y \rangle \geq -\gamma||Tx - Ty||^2 + r||x - y||^2.$$

Clearly a r-strongly monotonic mapping is a relaxed (γ, r)-cocoercive mapping, but the converse is not true.

2. Fixed point iteration algorithms

We need the following well known concepts and results.

Definition 2.1. [1] *If A is a maximal monotone operator on H, then for any constant $\rho > 0$, the resolvent operator associated with A is defined by*

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \textit{for all } u \in H, \tag{3}$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|.$$

Now we are in position to prove that any solution of systems of extended general variational inclusion (1) is exactly a solution of some fixed point problems.

Lemma 2.1. *If the operators A, B are maximal monotone, then $(x^*, y^*) \in H$ is a solution of (1), if and only if, $(x^*, y^*) \in H$ is a solution of the following fixed point problem:*

$$\begin{cases} g_1(x^*) = J_A[g(y^*) - \rho T_1(y^*)], \\ h_1(y^*) = J_B[h(x^*) - \eta T_2(x^*)]. \end{cases} \tag{4}$$

Proof. $(x^*, y^*) \in H$ is a solution of (1),

$$\Longleftrightarrow \begin{cases} g(y^*) - \rho T_1(y^*) \in (I + \rho A)(g_1(x^*)), \\ h(x^*) - \eta T_2(x^*) \in (I + \eta B)(h_1(y^*)), \end{cases}$$

$$\Longleftrightarrow \begin{cases} g_1(x^*) = J_A[g(y^*) - \rho T_1(y^*)], \\ h_1(y^*) = J_B[h(x^*) - \eta T_2(x^*)]. \end{cases}$$

The desired result. □

Lemma 2.1 is used to suggest a new explicit fixed point iteration algorithm for solving the system of extended general variational inclusions (1) since we can rewrite (4) in the following form.

$$\begin{cases} x^* = (1 - \alpha_n)x^* + \alpha_n(x^* - g_1(x^*)) + \alpha_n J_A[g(y^*) - \rho T_1(y^*)], \\ y^* = (y^* - h_1(y^*)) + J_B[h(x^*) - \eta T_2(x^*)], \end{cases}$$

where $\alpha_n \in (0, 1]$ for all $n \geq 0$ satisfies some certain conditions.

This equivalent formulation enables us to suggest the following explicit fixed point algorithm for solving (1), which is called parallel algorithm.

Algorithm 2.1. *For arbitrarily chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n(x_n - g_1(x_n)) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)], \\ y_{n+1} &= (y_n - h_1(y_n)) + J_B[h(x_n) - \eta T_2(x_n)], \end{aligned}$$

where $\alpha_n \in (0, 1]$ for all $n \geq 0$ satisfies some certain conditions.

For $g_1 = g$ and $h_1 = h$, then Algorithm 2.1 reduces to the following algorithm for (1).

Algorithm 2.2. *For arbitrarily chosen initial points* $x_0, y_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g(x_n)) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)],$$
$$y_{n+1} = (y_n - h(y_n)) + J_B[h(x_n) - \eta T_2(x_n)],$$

where $\alpha_n \in (0,1]$ *for all* $n \geq 0$ *satisfies some certain conditions.*

For $g_1 = g$, $h_1 = h$, and $A = B$, then Algorithm 2.1 reduces to the following algorithm for (1).

Algorithm 2.3. *For arbitrarily chosen initial points* $x_0, y_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g(x_n)) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)],$$
$$y_{n+1} = y_n - h(y_n) + J_A[h(x_n) - \eta T_2(x_n)],$$

where $\alpha_n \in (0,1]$ *for all* $n \geq 0$ *satisfies some certain conditions.*

For $g_1 = g$, $h_1 = h$, and $\alpha_n \equiv 1$ for all $n \geq 0$, then Algorithm 2.1 reduces to the following algorithm for (1).

Algorithm 2.4. *For arbitrarily chosen initial points* $x_0, y_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = x_n - g(x_n) + J_A[g(y_n) - \rho T_1(y_n)],$$
$$y_{n+1} = y_n - h(y_n) + J_B[h(x_n) - \eta T_2(x_n)].$$

Let us recall the Algorithms studied by Noor and Noor [11] for the special case as $A = B$.

Algorithm 2.5. $[11, Algorithm\, 3.1]$ *For arbitrarily chosen initial points* $x_0, y_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_{n+1} - g_1(x_{n+1})) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)], \tag{5}$$
$$y_{n+1} = y_{n+1} - h(y_{n+1}) + J_A[h_1(x_{n+1}) - \eta T_2(x_{n+1})], \tag{6}$$

where $\alpha_n \in (0,1]$ *for all* $n \geq 0$ *satisfies some certain conditions.*

For $g_1 = g$ and $h_1 = h$, then Algorithm 2.5 reduces to the following algorithm.

Algorithm 2.6. $[11, Algorithm\, 3.2]$ *For arbitrarily chosen initial points* $x_0, y_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_{n+1} - g(x_{n+1})) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)],$$
$$y_{n+1} = y_{n+1} - h(y_{n+1}) + J_A[h(x_{n+1}) - \eta T_2(x_{n+1})],$$

where $\alpha_n \in (0,1]$ *for all* $n \geq 0$ *satisfies some certain conditions.*

Remark 2.1. *We would like to emphasize that Algorithms of [11] are actually* **implicit** *fixed point iteration algorithms, since at each iteration step, we need to solve nonlinear equations to find the values of* x_{n+1} *and* y_{n+1}. *For example, assume that* x_n *and* y_n *are given in* **Algorithm 2.5**, *in order to get the new iterative points, we have to solve the nonlinear equations (5) and (6) for the values of* x_{n+1} *and* y_{n+1}. *That is, we need to solve* x_{n+1} *and* y_{n+1} *in sequence at every iterative step. This is a real hard work if* g *or* h *is not invertible, even much more difficult than the original problem (1). Moreover, our Algorithms are much easier in implement of computation than Algorithms in [11] and our computational workload is less than those of [11].*

From the above discussion, it is clear that Algorithm 2.1 is quite general and includes several new and previously known algorithms for solving systems of general variational inclusions (1).

Remark 2.2. *We now consider some numerical comparisons. From (5), we have*

It is easy to rewrite (5) into

$$(1-\alpha_n)x_{n+1} = (1-\alpha_n)x_n - \alpha_n g_1(x_{n+1}) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)],$$

which can be written as follows if $\alpha_n \in (0,1)$,

$$(I + \frac{\alpha_n}{1-\alpha_n} g_1)(x_{n+1}) = x_n + \frac{\alpha_n}{1-\alpha_n} J_A[g(y_n) - \rho T_1(y_n)], \tag{7}$$

where I is the identity matrix in R^n. In (7), if $(I + \frac{\alpha_n}{1-\alpha_n} g_1)$ is not invertible, then x_{n+1} can not be calculated easily. Even if $(I + \frac{\alpha_n}{1-\alpha_n} g_1)$ is invertible, we have to find the inverse of $(I + \frac{\alpha_n}{1-\alpha_n} g_1)$, which is itself difficult problem. In the mean time, (6) can be written as

$$h(y_{n+1}) = J_A[h_1(x_{n+1}) - \eta T_2(x_{n+1})]. \tag{8}$$

In (8), if h is not invertible, then y_{n+1} can not be solved easily. Even if h is invertible, we have to find the inverse of h. To have some numerical comparisons, let us consider the simplest case. Let h be invertible, then from (8). Then

$$y_{n+1} = h^{-1}(J_A[h_1(x_{n+1}) - \eta T_2(x_{n+1})]).$$

It is easy to learn from [14] that the computational workload of computing the inverse of h is $O(N^3)$, while the computational workload of our explicit algorithms in **Algorithm 2.1-2.4** *is $O(N)$ in the finite dimensional space R^N. It shows that the workload of* **Algorithm 2.5** *is much more than that of our algorithms.*

3. Main results

In this section, we study Algorithms 2.2 and we will show that our explicit algorithms work well with strong convergence.

Lemma 3.1. *If A is maximal monotone, J_A is the resolvent operator defined in Definition 3.1, and T is relaxed (γ, r)-cocoercive and μ-Lipschitzian continuous, then for any $w_1, w_2 \in K$, for all $\rho > 0$,*

$$||J_A[w_1 - \rho T w_1] - J_A[w_2 - \rho T w_2]|| \leq \theta_T ||w_1 - w_2||,$$

where $\theta_T = \sqrt{1 - 2\rho(r - \gamma\mu^2) + \rho^2\mu^2} \in (0,1)$.

Proof. Since T is relaxed (γ, r)-cocoercive and μ-Lipschitzian continuous, then from (4), we obtain

$$\begin{aligned}
& ||J_A[w_1 - \rho T w_1] - J_A[w_2 - \rho T w_2]||^2 \\
\leq\ & ||[w_1 - \rho T w_1] - [w_2 - \rho T w_2]||^2 \\
=\ & ||(w_1 - w_2) - \rho(T w_1 - T w_2)||^2 \\
=\ & ||w_1 - w_2||^2 - 2\rho\langle T w_1 - T w_2, w_1 - w_2\rangle + \rho^2 ||T w_1 - T w_2||^2 \\
\leq\ & ||w_1 - w_2||^2 - 2\rho[-\gamma||T w_1 - T w_2||^2 + r||w_1 - w_2||^2] + \rho^2||T w_1 - T w_2||^2 \\
=\ & (1 - 2\rho r)||w_1 - w_2||^2 + (2\rho\gamma + \rho^2)||T w_1 - T w_2||^2 \\
\leq\ & (1 - 2\rho r)||w_1 - w_2||^2 + (2\rho\gamma + \rho^2)\mu^2||w_1 - w_2||^2 \\
=\ & \theta_T^2 ||w_1 - w_2||^2,
\end{aligned}$$

from which, we have

$$\|J_A[w_1 - \rho T w_1] - J_A[w_2 \rho T w_2]\| \leq \theta_T \|w_1 - w_2\|,$$

the required result. $\square$

We now investigate the strong convergence of Algorithm 2.2 and this is the main motivation of our next result.

Theorem 3.1. *Let (x^*, y^*) be the solution of (1). Let $A, B : H \to H$ be maximal monotone, and let $T_1 : H \to H$ be relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian continuous, $T_2 : H \to H$ be relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian continuous, $g : H \to H$ be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitzian continuous, and $h : H \to H$ be relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitzian continuous. Then for arbitrary chosen initial points $x_0, y_0 \in H$, x_n and y_n obtained from Algorithm 2.2 converge strongly to x^* and y^* respectively if the following conditions are satisfied:*

$$|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}| < \frac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2(2 - k_1 - k_2)(k_1 + k_2)}}{\mu_1^2}, \tag{9}$$

$$r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{(2 - k_1 - k_2)(k_1 + k_2)}, \tag{10}$$

$$|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}| < \frac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2(2 - k_1 - k_2)(k_1 + k_2)}}{\mu_2^2}, \tag{11}$$

$$r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{(2 - k_1 - k_2)(k_1 + k_2)}, \tag{12}$$

$$\theta_g = k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2} < 1, \tag{13}$$

$$\theta_h = k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2} < 1, \tag{14}$$

$$\theta_{T_1} = \sqrt{1 - 2\rho(r_1 - \gamma_1\mu_1^2) + \rho^2\mu_1^2}, \tag{15}$$

$$\theta_{T_2} = \sqrt{1 - 2\eta(r_2 - \gamma_2\mu_2^2) + \eta^2\mu_2^2} \tag{16}$$

$$\theta_1 = \theta_g + \theta_h + \theta_{T_1}, \tag{17}$$

$$\theta_2 = \theta_g + \theta_h + \theta_{T_2}, \tag{18}$$

$$0 < \theta_2 < \inf \alpha_n \leq 1, \quad \alpha_n \in (0, 1]. \tag{19}$$

Proof. Let x^*, y^* be the solution of (1).

Since $g : H \to H$ is relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitzian continuous, by setting $w_1 = y_n$, $w_2 = y^*$ and $\rho = 1$, then by Lemma 3.1, we obtain

$$||(y_n - y^*) - (g(y_n) - g(y^*))|| \leq \theta_g ||y_n - y^*||, \tag{20}$$

where θ_g is defined by (13). Similarly,

$$||(x_n - x^*) - (g(x_n) - g(x^*))|| \leq \theta_g ||x_n - x^*||. \tag{21}$$

Since $h : H \to H$ is relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitzian continuous, by setting $w_1 = x_n$, $w_2 = x^*$ and $\rho = 1$, then by Lemma 3.1, we obtain

$$||(x_n - x^*) - (h(x_n) - h(x^*))|| \leq \theta_h ||x_n - x^*||, \tag{22}$$

where θ_h is defined by (14). Similarly,

$$||(y_n - y^*) - (h(y_n) - h(y^*))|| \leq \theta_h ||y_n - y^*||. \tag{23}$$

Since $T_1 : H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian continuous, by setting $w_1 = y_n$, $w_2 = y^*$, then by Lemma 3.1, we have

$$||(y_n - y^*) - \rho(T_1(y_n) - T_1(y^*))|| \le \theta_{T_1}||y_n - y^*||, \tag{24}$$

where θ_{T_1} is defined by (15).

Since $T_2 : H \to H$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian continuous, by setting $w_1 = x_n$, $w_2 = x^*$ and $\rho = \eta$, then by Lemma 3.1, we have

$$||(x_n - x^*) - \eta(T_2(x_n) - T_2(x^*))|| \le \theta_{T_2}||x_n - x^*||, \tag{25}$$

where θ_{T_2} is defined by (16).

Consequently by Algorithm 2.2, it follows from (6), $g_1 = g$ and $h_1 = h$ that

$$\begin{aligned}
& ||x_{n+1} - x^*|| \\
= \ & ||(1-\alpha_n)x_n + \alpha_n(x_n - g(x_n)) + \alpha_n J_A[g(y_n) - \rho T_1(y_n)] \\
& -[(1-\alpha_n)x^* + \alpha_n(x^* - g(x^*)) + \alpha_n J_A[g(y^*) - \rho T_1(y^*)]]|| \\
\le \ & (1-\alpha_n)||x_n - x^*|| + \alpha_n||(x_n - x^*) - (g(x_n) - g(x^*))|| \\
& +\alpha_n||J_A[g(y_n) - \rho T_1(y_n)] - J_A[g(y^*) - \rho T_1(y^*)]|| \\
\le \ & (1-\alpha_n)||x_n - x^*|| + \alpha_n||(x_n - x^*) - (g(x_n) - g(x^*))|| \\
& +\alpha_n||[g(y_n) - \rho T_1(y_n)] - [g(y^*) - \rho T_1(y^*)]|| \\
\le \ & (1-\alpha_n)||x_n - x^*|| + \alpha_n||(x_n - x^*) - (g(x_n) - g(x^*))|| \\
& +\alpha_n||(y_n - y^*) - (g(y_n) - g(y^*))|| + \alpha_n||(y_n - y^*) - \rho(T_1(y_n) - T_1(y^*))|| \\
\le \ & (1-\alpha_n)||x_n - x^*|| + \alpha_n\theta_g||x_n - x^*|| + \alpha_n\theta_g||y_n - y^*|| + \alpha_n\theta_{T_1}||y_n - y^*||,
\end{aligned} \tag{26}$$

where (26) is from (20),(21),(24). Meanwhile,

$$\begin{aligned}
& ||y_{n+1} - y^*|| \\
= \ & ||y_n - h(y_n) + J_B[h(x_n) - \eta T_2(x_n)] \\
& -[y^* - h(y^*) + J_B[h(x^*) - \eta T_2(x^*)]]|| \\
= \ & ||[(y_n - y^*) - (h(y_n) - h(y^*))] + (J_B[h(x_n) - \eta T_2(x_n)] - J_B[h(x^*) - \eta T_2(x^*)])|| \\
\le \ & ||(y_n - y^*) - (h(y_n) - h(y^*))|| + ||J_B[h(x_n) - \eta T_2(x_n)] - J_B[h(x^*) - \eta T_2(x^*)]|| \\
\le \ & ||(y_n - y^*) - (h(y_n) - h(y^*))|| + ||[h(x_n) - \eta T_2(x_n)] - [h(x^*) - \eta T_2(x^*)]|| \\
\le \ & ||(y_n - y^*) - (h(y_n) - h(y^*))|| \\
& +||(x_n - x^*) - (h(x_n) - h(x^*))|| + ||(x_n - x^*) - \eta(T_2(x_n) - T_2(x^*))|| \\
\le \ & \theta_h||y_n - y^*|| + \theta_h||x_n - x^*|| + \theta_{T_2}||x_n - x^*||.
\end{aligned} \tag{27}$$

where (27) is from (22),(23),(25).

Therefore, it follows from (26) and (27) that

$$\begin{aligned}
& [||x_{n+1} - x^*|| + ||y_{n+1} - y^*||] \\
\le \ & [(1-\alpha_n)||x_n - x^*|| + \alpha_n\theta_g||x_n - x^*|| + \alpha_n\theta_g||y_n - y^*|| + \alpha_n\theta_{T_1}||y_n - y^*||] \\
& +[\theta_h||y_n - y^*|| + \theta_h||x_n - x^*|| + \theta_{T_2}||x_n - x^*||] \\
= \ & [(1-\alpha_n) + \alpha_n\theta_g + \theta_h + \theta_{T_2}] \cdot ||x_n - x^*|| + [\alpha_n\theta_g + \alpha_n\theta_{T_1} + \theta_h] \cdot ||y_n - y^*||.
\end{aligned} \tag{28}$$

Since $\theta_1 = \theta_g + \theta_h + \theta_{T_1}$, $\quad \theta_2 = \theta_g + \theta_h + \theta_{T_2}$, then by the conditions (9)-(16), it is easy to check that $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$.

Since from the condition (19), $0 < \theta_2 < \inf \alpha_n \leq 1$, then there must exist a sufficiently small positive constant $a > 0$ such that $0 < \theta_2 + a \leq \alpha_n \leq 1$ for all $n \geq 0$. Note that $\theta_2 = \theta_g + \theta_h + \theta_{T_2} \in (0,1)$, and $\alpha_n \in (0,1]$, thus in (28),

$$\begin{aligned} & (1-\alpha_n) + \alpha_n\theta_g + \theta_h + \theta_{T_2} \leq \\ \leq \ & (1-\alpha_n) + \theta_2 \leq (1-a) < 1. \end{aligned}$$

On the other hand, it follows from $\alpha_n \in (0,1]$ that

$$\alpha_n\theta_g + \alpha_n\theta_{T_1} + \theta_h \leq \theta_g + \theta_{T_1} + \theta_h = \theta_1 < 1.$$

Set $\theta = \max\{(1-a), \theta_1\}$. Thus $0 < \theta < 1$. Therefore, it follows from (28) that

$$\begin{aligned} & [||x_{n+1} - x^*|| + ||y_{n+1} - y^*||] \leq \\ \leq \ & \theta[||x_n - x^*|| + ||y_n - y^*||]. \end{aligned}$$

Hence, we conclude that

$$\lim_{n\to\infty} [||x_n - x^*|| + ||y_n - y^*||] = 0,$$

and

$$\lim_{n\to\infty} ||x_n - x^*|| = \lim_{n\to\infty} ||y_n - y^*||] = 0.$$

$\square$

If $\alpha_n \equiv 1$ for all $n \geq 0$, then we have the following result as a special case of Theorem 4.1.

Theorem 3.2. *Let (x^*, y^*) be the solution of (1). Let $A, B : H \to H$ be maximal monotone, and let $T_1 : H \to H$ be relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian continuous, $T_2 : H \to H$ be relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian continuous, $g : H \to H$ be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitzian continuous, and $h : H \to H$ be relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitzian continuous. Then for arbitrary chosen initial points $x_0, y_0 \in H$, x_n and y_n obtained from Algorithm 2.4 converge strongly to x^* and y^* respectively if the conditions (9)-(18) are satisfied.*

Proof. Observe that $\alpha_n \equiv 1$ for all $n \geq 0$ and $\theta_2 < 1$. This implies that the condition (19) is satisfied because $\theta_2 < 1 \equiv \alpha_n$. Hence, the desired result can be obtained by Theorem 3.1. Indeed, since $\alpha_n \equiv 1$ for all $n \geq 0$, then from (28),

$$\begin{aligned} & [||x_{n+1} - x^*|| + ||y_{n+1} - y^*||] \leq \\ \leq \ & [(1-\alpha_n) + \alpha_n\theta_g + \theta_h + \theta_{T_2}] \cdot ||x_n - x^*|| + [\alpha_n\theta_g + \alpha_n\theta_{T_1} + \theta_h] \cdot ||y_n - y^*|| = \\ = \ & [\theta_g + \theta_h + \theta_{T_2}] \cdot ||x_n - x^*|| + [\theta_g + \theta_{T_1} + \theta_h] \cdot ||y_n - y^*||. \end{aligned}$$

Observe that $\theta_1 = \theta_g + \theta_h + \theta_{T_1} < 1$ and $\theta_2 = \theta_g + \theta_h + \theta_{T_2} < 1$ by the conditions (9)-(16). Set $\theta = \max\{\theta_1, \theta_2\}$, then it is clear that $\theta \in (0,1)$. Therefore,

$$[||x_{n+1} - x^*|| + ||y_{n+1} - y^*||] \leq \theta \cdot [||x_n - x^*|| + ||y_n - y^*||],$$

which hence implies that

$$\lim_{n\to\infty} [||x_n - x^*|| + ||y_n - y^*||] = 0,$$

and

$$\lim_{n\to\infty} ||x_n - x^*|| = \lim_{n\to\infty} ||y_n - y^*||] = 0.$$

$\square$

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References

[1] Bnouhachem, A., Noor, M.A., Khalfuoui, M., Sheng, Z., (2013), A resolvent algorithm of generalized system for relaxed coercive mixed variational inequalities, Appl. Comput. Math., 12(2), pp.125-132.

[2] Brezis, H., (1973), Operateurs Maximaux Monotone et Semigroups de Contractions dan Espaces de Hilbert, North-Holland, Amsterdam.

[3] Glowinski, R., Lions,J.L., Trémolières,R., (1981), Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam.

[4] Huang, Z.Y., Noor, M.A., (2007), An explicit projection method for a system of nonlinear variational inequalities with different (γ, r)-cocoercive mappings, Appl. Math. Comput., 190, pp.356-361.

[5] Huang, Z.Y., Noor, M.A., (2012), Studies on common solutions of a variational inequality and a fixed point problem, J. Optim. Theory Appl., 154(2), pp.525-535.

[6] Huang, Z.Y., Noor, M.A., Al-Said, E., (2010), On an open question of Takahashi for nonexpansive mappings and inverse strongly monotone mappings, J. Optim. Theory Appl., 147(1), pp.194-204.

[7] Lions, J.L., Stampacchia, G., (1967), Variational inequalities, Commun. Pure Appl. Math., 20, pp.493-512.

[8] Noor, M.A., (2009), On a system of general mixed variational inequalities, Optim. Lett., 3, pp.437-457.

[9] Noor, M.A., (2008), On iterative methods for solving a system of mixed variational inequalities, Appl. Anal., 87, pp.99-108.

[10] Noor, M.A., Noor, K.I., (2009), Projection algorithms for solving system of general variational inequalities, Nonlinear Anal., 70, pp.2700-2706.

[11] Noor, M.A., Noor, K.I., (2011), Resolvent methods for solving system of general variational inclusions, J. Optim. Theory Appl., 148, pp.422-430.

[12] Noor, M. A., Noor, K. I., (2013), Some new classes of quasi split feasibility problems, Appl. Math. Inform. Sci., 7(4), pp.1544-1552.

[13] Noor, M. A., Noor, K. I., Al-Said, E., (2011), Resolvent iterative methods for solving system of extended general variational inclusions, J. Inequal. Appl., Article 10: 371241.

[14] Noor, M. A., Noor, K. I., Al-Said, E., (2011), On a system of general variational inclusions, AMO-Advan. Model. Optim., 13, pp.221-231.

[15] Noor, M.A., Noor, K.I., Al-Said, E., (2011), Iterative projection methods for general nonconvex variational inequalities, Appl. Comput. Math., 10(2), pp.309-320.

[16] Noor, M. A., Noor, K.I., Khan, A.G., (2013), Some iterative schemes for solving extended general quasi variational inequalities, Appl. Math. Inform. Sci., 7(3), pp.917-925.

[17] Ortega, J.M., Rheinboldt, W.C., (2000), Iterative solution of nonlinear equations in several variables, Philadelphia, SIAM.

[18] Stampacchia, G., (1964), Formes bilineaires coercivities sur les ensembles convexes, C.R. Acad. Sci. Paris, 258, pp.4413-4416.

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